DIFFERENCE BETWEEN CHAOTIC ORDER AND USUAL ORDER IN GENERALIZED FURUTA INEQUALITIES

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Abstract. Recently, Furuta has shown several inequalities with forms of grand Furuta inequality and clarified difference between chaotic order and usual order of positive operators. Here the chaotic order $A \gg B$ for positive invertible operators A and B is defined by $\log A \geq \log B$. In this note, we present the following inequalities which are based on the Furuta inequality for chaotic order and can be regarded as interpolations of recent Furuta's results. Among others, we obtain the following:
Let $A,\ B$ be positive invertible operators satisfying $A \gg B$.
(1) If $0 \leq \delta \leq \beta \leq p$ and $r \geq 0$, then

$$B^{\delta} \geq A^{-r} \sharp_{\frac{\delta+r}{\beta+r}} B^{\beta} \geq A^{-r} \sharp_{\frac{\delta+r}{\beta+r}} (A^{-r} \sharp_{\frac{\beta+r}{p+r}} B^{p}).$$

(2) If $|\delta| \leq p \leq \beta \leq 2p$ and $r \geq t \geq |\delta|$ for some $\delta \in \mathbb{R}$, then

$$A^{-t} \sharp_{\frac{\delta+t}{p+t}} B^{p} \geq A^{-r} \sharp_{\frac{\delta+r}{\beta+r}} (A^{-t} \natural_{\frac{\beta+t}{p+t}} B^{p}) \geq A^{-r} \sharp_{\frac{\delta+r}{\beta+r}} B^{\beta}.$$

1. Introduction. Throughout this note, A and B are positive operators on a Hilbert space. For convenience, we denote $A \geq 0$ (resp. $A > 0$) if A is a positive (resp. positive invertible) operator. The α-geometric mean of A and B introduced by Kubo-Ando [18] is given by

$$A \sharp_{\alpha} B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}} \quad \text{for} \ \ 0 \leq \alpha \leq 1.$$

It gives us a fruiteful expression for the Furuta inequality [8](cf.[9]) as follows ([2],[12],[13],[14]):

Furuta inequality: *If $A \geq B \geq 0$, then*

$$(F) \qquad A^{-r} \sharp_{\frac{1+r}{p+r}} B^{p} \leq A \quad \text{and} \quad B \leq B^{-r} \sharp_{\frac{1+r}{p+r}} A^{p}$$

holds for $r \geq 0$ and $p \geq 1$.

In succession, Furuta [10] gave the following inequality as a generalization of the Furuta inequality. We called it the grand Furuta inequality in [4],[5] and [15]. It is established to interpolate the Furuta inequality and the Ando-Hiai inequality [1] equivalent to the main result of log majorization. We here cite it in terms of operator mean, in which the binary operation $\natural_s$ is defined by the same formula as $\sharp_{\alpha}$, i.e., $A \natural_s B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{s}A^{\frac{1}{2}}$ for any real numbers s.

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Grand Furuta inequality: *If $A \geq B \geq 0$ and A is invertible, then for each $1 \leq p$ and $0 \leq t \leq 1$,*

$$A^{-r+t} \sharp_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq A \quad \text{and} \quad B \leq B^{-r+t} \sharp_{\frac{1-t+r}{(p-t)s+r}} (B^t \natural_s A^p)$$

holds for $t \leq r$ and $1 \leq s$.

The best possibility of the power $\frac{1-t+r}{(p-t)s+r}$ is shown in [19]. Recently, Furuta [11] has shown the following inequalities Type I-(c), Type I-(u) and Type II-(c), Type II-(u) as varieties of the grand Furuta inequality, where $A \gg B$ means $\log A \geq \log B$ for $A,\ B > 0$.

$$\text{(TypeI} - \text{(c))} \qquad A \gg B \Longleftrightarrow I \geq A^{-r} \sharp_{\frac{r}{\delta+r}} B^{\delta} \geq A^{-r} \sharp_{\frac{r}{\delta+r}} (A^{-t} \sharp_{\frac{\delta+t}{p+t}} B^p)$$

for $r \geq 0,\ t \geq 0$ and $0 \leq \delta \leq p$.

$$\text{(TypeI} - \text{(u))} \qquad A \geq B \Longleftrightarrow A \geq A^{-r} \sharp_{\frac{1+r}{\delta+r}} B^{\delta} \geq A^{-r} \sharp_{\frac{1+r}{\delta+r}} (A^{-t} \sharp_{\frac{\delta+t}{p+t}} B^p)$$

for $r \geq 0,\ t \geq 0$ and $1 \leq \delta \leq p$.

$$\text{(TypeII} - \text{(c))} \qquad A \gg B \Longleftrightarrow I \geq A^{-r} \sharp_{\frac{r}{\beta+r}} (A^{-t} \natural_{\frac{\beta+t}{p+t}} B^p) \geq A^{-r} \sharp_{\frac{r}{\beta+r}} B^{\beta}$$

for $r \geq t \geq 0$ and $0 \leq p \leq \beta \leq 2p$.

$$\text{(TypeII} - \text{(u))} \qquad A \geq B \Longleftrightarrow A \geq A^{-r} \sharp_{\frac{1+r}{\beta+r}} (A^{-t} \natural_{\frac{\beta+t}{p+t}} B^p) \geq A^{-r} \sharp_{\frac{1+r}{\beta+r}} B^{\beta}$$

for $r \geq t \geq 1$ and $1 \leq p \leq \beta \leq 2p+1$.

We should note that, roughly speaking, difference between (c) and (u) in above reflects on that of the indices $\frac{r}{\gamma+r}$ and $\frac{1+r}{\gamma+r}$, which appear in Furuta inequality (F) and chaotic Furuta inequality (CF) stated in the next section, respectively. Motivated by these inequalities, we discuss several varieties of the grand Furuta inequality in this note. For this, we investigate them from a different view from Furuta's one. Consequently we obtain some generalizations of them. In the proofs, our previous results are used, which are cited in the next section for the sake of convenience.

2. Prerequisite results.

We had arranged the inequalities (F) in one line by using an operator mean [12].

Satellite theorem of the Furuta inequality: *If $A \geq B \geq 0$, then*

$$\text{(SF)} \qquad A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \leq B \leq A \leq B^{-r} \sharp_{\frac{1+r}{p+r}} A^p$$

holds for all $r \geq 0$ and $1 \leq p$.

For $A,\ B > 0$, we denote by $A \gg B$ if $\log A \geq \log B$ and call it the chaotic order ([3],[16],[17]). The next characterization of the chaotic order in [3] is useful and the starting point of our following discussions.

Chaotic Furuta inequality: *If $A \gg B$, then*

$$\text{(CF)} \qquad A^{-r} \sharp_{\frac{r}{p+r}} B^p \leq I \leq B^{-r} \sharp_{\frac{r}{p+r}} A^p$$

for any $r \geq 0$ and $p \geq 0$.

The next (SCF) will clarify the difference between $A \gg B$ and $A \geq B$ by comparing with (SF).

Satellite theorem of chaotic Furuta inequality: *If $A \gg B$, then*

(SCF) $$A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \leq B \quad \text{and} \quad A \leq B^{-r} \sharp_{\frac{1+r}{p+r}} A^p$$

holds for any $r \geq 0$ and $p \geq 1$.
In comparison with (SF), we may represent (SCF) as follows:

$$A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \leq B \ll A \leq B^{-r} \sharp_{\frac{1+r}{p+r}} A^p$$

More generally, we had given characterizations of chaotic order in [16](cf.[17]) as follows:

Theorem A. *If $A \gg B$, then the following* (SFC1) *and* (SFC2) *hold.*

(SCF1) $$A^{-r} \sharp_{\frac{\delta+r}{p+r}} B^p \leq B^\delta \quad \text{and} \quad A^\delta \leq B^{-r} \sharp_{\frac{\delta+r}{p+r}} A^p \quad \text{for} \quad r \geq 0 \quad \text{and} \quad 0 \leq \delta \leq p.$$

(SCF2) $$A^{-r} \sharp_{\frac{\alpha+r}{p+r}} B^p \leq A^\alpha \text{ and } B^\alpha \leq B^{-r} \sharp_{\frac{\alpha+r}{p+r}} A^p \quad \text{for} \quad -r \leq \alpha \leq 0 \quad \text{and} \quad 0 \leq p.$$

Replacing s in (GF) with $\frac{\beta-t}{p-t}$ for $1 \leq p \leq \beta$, we can state the grand Furuta inequality (GF) by the satellite form as follows [15]:

If $A \geq B > 0$, then for $r \geq 0$, $0 \leq t \leq 1$, $0 \leq t < p \leq \beta$, $0 \leq \delta \leq 1$ and $\delta \leq \beta$

$$A^{-r} \sharp_{\frac{\delta+r}{\beta+r}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\delta}{\beta}} \leq B^\delta \leq A^\delta \leq (B^t \natural_{\frac{\beta-t}{p-t}} A^p)^{\frac{\delta}{\beta}} \leq B^{-r} \sharp_{\frac{\delta+r}{\beta+r}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p).$$

The middle part of this inequality shown in [4], [5] is essential. It is now improved as follows [6]:

Theorem B. *If $A \geq B > 0$, then for $0 \leq t \leq 1 \leq p \leq \beta$,*

$$H(\beta) = (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}},$$

is a decreasing function with $\beta \geq p$ and $H(\beta) \leq B^p$. In particular

$$(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}} \leq B \leq A.$$

3. Monotone operator function related to the Furuta inequality. In this section, we give an operator function $H_{A,B}(t)$ and investigate its behavior under the chaotic order and the usual order. Very recently, the following theorem is obtained in [7] (cf.[17]).

Theorem C. *If $A \gg B$ and $0 \leq p \leq \beta \leq 2p$, then*

$$H_{A,B}(t) = A^{-t} \natural_{\frac{\beta+t}{p+t}} B^p$$

is an increasing function for $t \geq 0$, that is, $H_{A,B}(r) \geq H_{A,B}(t)$ for $r \geq t \geq 0$.

Since $\lim_{\alpha\to 0}\frac{A^\alpha-1}{\alpha}=\log A$, we may regard $A\gg B$ as $A^0\ge B^0$. Then we investigate Theorem C in a wider flame:

Theorem 1. *Let $\alpha\ge 0$, $0\le p\le\beta\le 2p+\alpha$ and $A^\alpha\ge B^\alpha>0$. Then*

$$H_{A,B}(t)=A^{-t}\ \natural_{\frac{\beta+t}{p+t}}\ B^p$$

is an increasing function for $t\ge\alpha$ and $H_{A,B}(\alpha)\ge B^\beta$.

Proof. Assume that $r\ge t\ge\alpha$. Then we have $A^t\le B^{-p}\ \sharp_{\frac{t+p}{r+p}}\ A^r$ by (SCF1) and $\frac{\beta-p}{p+t}\le\frac{p+\alpha}{p+t}\le 1$, so that

$$\begin{aligned}
H_{A,B}(t) &= A^{-t}\ \natural_{\frac{\beta+t}{p+t}}\ B^p = B^p\ \natural_{\frac{p-\beta}{p+t}}\ A^{-t}\\
&= B^p(B^{-p}\ \sharp_{\frac{\beta-p}{p+t}}\ A^t)B^p \le B^p(B^{-p}\ \sharp_{\frac{\beta-p}{p+t}}\ (B^{-p}\ \sharp_{\frac{t+p}{r+p}}\ A^r))B^p\\
&= B^p(B^{-p}\ \sharp_{\frac{\beta-p}{r+p}}\ A^r)B^p = B^{\frac{p}{2}}(I\ \sharp_{\frac{\beta-p}{r+p}}\ B^{\frac{p}{2}}A^rB^{\frac{p}{2}})B^{\frac{p}{2}}\\
&= B^{\frac{p}{2}}(B^{\frac{p}{2}}A^rB^{\frac{p}{2}})^{\frac{\beta-p}{r+p}}B^{\frac{p}{2}} = B^{\frac{p}{2}}(B^{-\frac{p}{2}}A^{-r}B^{-\frac{p}{2}})^{\frac{p-\beta}{r+p}}B^{\frac{p}{2}}\\
&= B^p\ \natural_{\frac{p-\beta}{r+p}}\ A^{-r} = A^{-r}\ \natural_{\frac{\beta+r}{p+r}}\ B^p = H_{A,B}(r).
\end{aligned}$$

The latter $H_{A,B}(\alpha)\ge B^\beta$ is obtained as follows: Since $H_{A,B}(0)=B^\beta$, we have only to see the case $\alpha>0$.

$$\begin{aligned}
H_{A,B}(\alpha) &= A^{-\alpha}\ \natural_{\frac{\beta+\alpha}{p+\alpha}}\ B^p = B^p\ \natural_{\frac{p-\beta}{p+\alpha}}\ A^{-\alpha} = B^p(B^{-p}\ \sharp_{\frac{\beta-p}{p+\alpha}}\ A^\alpha)B^p\\
&\ge B^p(B^{-p}\ \sharp_{\frac{\beta-p}{p+\alpha}}\ B^\alpha)B^p = B^\beta.
\end{aligned}$$

Next we give a variant of our result in [3], which will be used in the next section.

Theorem 2. *Let $A,\ B>0$ and δ be fixed. If $A\gg B$, then*

$$F_{A,B}(t,p)=A^{-t}\ \sharp_{\frac{\delta+t}{p+t}}\ B^p$$

is decreasing for both p and t with $p,\ t\ge|\delta|$. Incidentally, if $\delta\le 0$ (resp. $\delta\ge 0$), then $F_{A,B}(t,p)\le A^\delta$ (resp. $F_{A,B}(t,p)\le B^\delta$).

Proof. For $r\ge t\ge|\delta|$, it follows from (SCF2) that

$$\begin{aligned}
F_{A,B}(t,p) &= A^{-t}\ \sharp_{\frac{\delta+t}{p+t}}\ B^p = B^p\ \sharp_{\frac{p-\delta}{p+t}}\ A^{-t}\\
&\ge B^p\ \sharp_{\frac{p-\delta}{p+t}}\ (A^{-r}\ \sharp_{\frac{-t+r}{p+r}}\ B^p) = B^p\ \sharp_{\frac{p-\delta}{p+t}}\ (B^p\ \sharp_{\frac{p+t}{p+r}}\ A^{-r})\\
&= B^p\ \sharp_{\frac{p-\delta}{p+r}}\ A^{-r} = A^{-r}\ \sharp_{\frac{\delta+r}{p+r}}\ B^p = F_{A,B}(r,p)
\end{aligned}$$

Next, for $\beta\ge p\ge|\delta|$, it follows from (SCF1) that

$$F_{A,B}(t,\beta)=A^{-t}\ \sharp_{\frac{\delta+t}{\beta+t}}\ B^\beta = A^{-t}\ \sharp_{\frac{\delta+t}{p+t}}\ (A^{-t}\ \sharp_{\frac{p+t}{\beta+t}}\ B^\beta)\le A^{-t}\ \sharp_{\frac{\delta+t}{p+t}}\ B^p = F_{A,B}(t,p).$$

If $\delta \le 0$, we have $F_{A,B}(t,p) = A^{-t} \sharp_{\frac{\delta+t}{p+t}} B^p \le A^\delta$ by (SCF2) and if $\delta \ge 0$, then $F_{A,B}(t,p) \le B^\delta$ by (SCF1).

4. A variety of the Furuta inequality. In this section, we investigate Furuta's inequalities in [11]. For this, we discuss them in a more general setting. Consequently we point out that Theorem A is an essence of the inequalities type I and the inequalities type II can be explained by Theorems 1, 2 and 3.

Theorem 3. *Let $A \gg B$ for $A,\ B > 0$ and $\delta \in \mathbb{R}$, then the following statements hold:*
(1) *If $0 \le \delta \le \beta \le p$ and $r \ge 0,\ t \ge 0$, then*

$$B^\delta \ge A^{-r} \sharp_{\frac{\delta+r}{\beta+r}} B^\beta \ge A^{-r} \sharp_{\frac{\delta+r}{\beta+r}} (A^{-t} \sharp_{\frac{\beta+t}{p+t}} B^p).$$

(2) *If $|\delta| \le p \le \beta \le 2p + |\delta|$ and $r \ge t \ge |\delta|$, then*

$$A^{-t} \sharp_{\frac{\delta+t}{p+t}} B^p \ge A^{-r} \sharp_{\frac{\delta+r}{\beta+r}} (A^{-t} \natural_{\frac{\beta+t}{p+t}} B^p) \ge A^{-r} \sharp_{\frac{\delta+r}{\beta+r}} B^\beta.$$

Proof. (1) It is proved by applying Theorem A twice. As a matter of fact, it follows from $B^\delta \ge A^{-r} \sharp_{\frac{\delta+r}{\beta+r}} B^\beta$ and $B^\beta \ge A^{-t} \sharp_{\frac{\beta+t}{p+t}} B^p$.
(2) Since $B^\beta \le H_{A,B}(t) \le H_{A,B}(r)$ by Theorem 1, we have

$$\begin{aligned} & A^{-r} \sharp_{\frac{\delta+r}{\beta+r}} B^\beta \le A^{-r} \sharp_{\frac{\delta+r}{\beta+r}} H_{A,B}(t) (= A^{-r} \sharp_{\frac{\delta+r}{\beta+r}} (A^{-t} \natural_{\frac{\beta+t}{p+t}} B^p)) \\ \le\ & A^{-r} \sharp_{\frac{\delta+r}{\beta+r}} H_{A,B}(r) = A^{-r} \sharp_{\frac{\delta+r}{\beta+r}} (A^{-r} \natural_{\frac{\beta+r}{p+r}} B^p) \\ =\ & A^{-r} \sharp_{\frac{\delta+r}{p+r}} B^p = F_{A,B}(r,p) \le F_{A,B}(t,p) = A^{-t} \sharp_{\frac{\delta+t}{p+t}} B^p, \end{aligned}$$

where the last inequality is ensured by Theorem 2.

By Theorem 3, we can interpolate Furuta's results from (c) to (u) of each type. As a matter of fact, $\delta = 0$ and $\delta = 1$ correspond to (c) and (u) respectively.

The following inequality is also one of inequalities due to Furuta [11]:
For $r \ge 0$ and $0 \le t \le 1 \le p \le \beta \le 2p - t$,

$$(*) \qquad A \ge B \Longleftrightarrow A \ge A^{-r} \sharp_{\frac{1+r}{\beta+r}} B^\beta \ge A^{-r} \sharp_{\frac{1+r}{\beta+r}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p).$$

It is a direct application of the Löwner-Heinz inequality because the assumption implies $1 \le \frac{\beta-t}{p-t} \le 2$ and so

$$A^t \natural_{\frac{\beta-t}{p-t}} B^p = B^p(B^{-p} \sharp_{\frac{\beta-p}{p-t}} A^{-t})B^p \le B^p(B^{-p} \sharp_{\frac{\beta-p}{p-t}} B^{-t})B^p \le B^\beta,$$

which is our first step to prove Theorem B incidentally. We finally point out that the above inequality $(*)$ by Furuta has the following improvement.

Theorem 4. *If $A \ge B > 0$ and $r \ge 0,\ 0 \le t \le 1 \le p \le \beta$, then*

$$A^{-r} \sharp_{\frac{1+r}{\beta+r}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \le A^{-r} \sharp_{\frac{1+r}{p+r}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}} \le A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \le B.$$

Proof. Put $C = (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}}$. Then we have $A \ge B \ge C$ and $C^p \le B^p$ by Theorem B. Therefore (SCF1) implies that

$$A^{-r} \sharp_{\frac{p+r}{\beta+r}} C^\beta \le C^p,$$

and so

$$\begin{aligned} A^{-r} \sharp_{\frac{1+r}{\beta+r}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) &= A^{-r} \sharp_{\frac{1+r}{\beta+r}} C^\beta = A^{-r} \sharp_{\frac{1+r}{p+r}} (A^{-r} \sharp_{\frac{p+r}{\beta+r}} C^\beta) \\ &\le A^{-r} \sharp_{\frac{1+r}{p+r}} C^p \le A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \le B \end{aligned}$$

by $C^p \le B^p$ and (SF).

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