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L-UP AND MIRROR ALGEBRAS

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Abstract. In this paper we consider several families of abstract algebras including the well-known BCK-algebras and several larger classes including the class of d-algebras which is a generalization of BCK-algebras. For these algebras it is usually difficult and often impossible to obtain a complementation operation and the associated "de Morgan's laws". In this paper we construct a "mirror image" of a given algebra which when adjoined to the original algebra permit a natural complementation to take place. The class of BCK-algebras is not closed under this operation but the class of d-algebras is, thus explaining why it may be better to work with this class rather than the class of BCK-algebras. Other classes of interest in this setting are also discussed.

1. Introduction.

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([3, 4]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [1, 2] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. The present authors ([7]) introduced the notion of d-algebras which is another useful generalization of BCK-algebras, and then they investigated several relations between d-algebras and BCK-algebras as well as some other interesting relations between d-algebras and oriented digraphs. Recently, Y. B. Jun, E. H. Roh and H. S. Kim ([5]) introduced a new notion, called an BH-algebra, which is a generalization of $BCH/BCI/BCK$-algebras, and defined the notions of ideals and boundedness in BH-algebras, and showed that there is a maximal ideal in bounded BH-algebras. Furthermore, they constructed the quotient BH-algebras via translation ideals and obtained the fundamental theorem of homomorphisms for BH-algebras as a consequence. The present authors ([8]) gave an analytic method for constructing proper examples of a great variety of non-associative algebras of the BCK-type and generalizations of these. In this paper we consider several families of abstract algebras including the well-known BCK-algebras and several larger classes including the class of d-algebras which is a generalization of BCK-algebras. For these algebras it is usually difficult and often impossible to obtain a complementation operation and the associated "de Morgan's laws". In this paper we construct a "mirror image" of a given algebra which when adjoined to the original algebra permit a natural complementation to take place. The class of BCK-algebras is not closed under this operation but the class of d-algebras is, thus explaining why it may be better to work with this class rather than the class of BCK-algebras. Other classes of interest in this setting are also discussed.

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2. Up algebras.

Suppose that $(X;*,0)$ is an algebra of type (2,0) with T a subset of the following axioms:
(I) $x*x=0$,
(II) $0*x=0$,
(III) $x*y=0$ and $y*x=0$ imply $x=y$
(IV) $x*0=x$,
(V) $(x*y)*z=(x*z)*y$,
(VI) $(x*(x*y))*y=0$,
(VII) $((x*y)*(x*z))*(z*y)=0$,
(VIII) $x*y=0 \Rightarrow x*(y*x)=x$,
for any x,y,z in X.

In such a case we shall refer to $(X;*,0)$ as a T-*algebra.* Using this device, we observe that we can deal simultaneously with statements concerning different classes of algebras. Indeed, note that included are:
(1) d-algebra, when $T_1=\{$(I), (II), (III)$\}$,
(2) BH-algebra, when $T_2=\{$(I), (II), (IV)$\}$,
(3) $d-BH$-algebra, when $T_3=T_1\cup T_2$,
(4) BCH-algebra, when $T_4=\{$(I), (III), (V)$\}$,
(5) BCI-algebra, when $T_5=\{$(I), (III), (VI), (VII)$\}$,
(6) BCK-algebra, when $T_6=\{$(I), (II), (III), (VI), (VII)$\}$.
The axioms for BCK-algebras are known to be independent ([6]). The following examples demonstrate further differences among classes of T_i-algebras for $i=1,\cdots,6$.

Example 2.1. Let $X:=\{0,1,2,3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	3
3	3	3	1	0

It is easy to verify that $(X;*,0)$ is a $d-BH$-algebra, but not a BCH-algebra, since $(2*3)*2=1\neq 0=(2*2)*3$.

Example 2.2. Let $X=\{0,1,2,3\}$ be a set with the following tables:

$*_1$	0	1	2	3
0	0	3	0	2
1	1	0	0	0
2	2	2	0	3
3	3	3	1	0

$*_2$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	3
3	2	3	1	0

Then $(X;*_1,0)$ is a BH-algebra, but not a d-algebra. At the same time, $(X;*_2,0)$ is a d-algebra, but not a BH-algebra.

We introduce the following notations:

$$(x \wedge y)_L = x * (x * y)$$

and

$$(x \wedge y)_R = y * (y * x)$$

noting that in many situations, e.g., in Boolean algebras, $(x \wedge y)_L = (x \wedge y)_R = x \wedge y$ when $x * y = x - y$ is the difference of sets. However, the relation $(x \wedge y)_L = (x \wedge y)_R$ does not hold in general, as follows from the example below:

Example 2.3. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	0
3	3	3	1	0

Then $(X; *, 0)$ is a d-algebra, and $(3 \wedge 2)_L = 3$, but $(3 \wedge 2)_R = 2$.

Given a T-algebra $(X; *, 0)$, it is said to be a *left* (resp., *right*) L-up algebra if there is defined an operation $(x \vee y)_L$ such that $(x \wedge (x \vee y)_L)_L = x$ (resp., $((x \vee y)_L \wedge y)_L = y$) for any $x, y \in X$. An L-up algebra is both a left L-up algebra and a right L-up algebra. Similarly, $(X; *, 0)$ is said to be a *left* (resp., *right*) *R-up algebra* if there is defined an operation $(x \vee y)_R$ such that $(x \wedge (x \vee y)_R)_R = x$ (resp., $((x \vee y)_R \wedge y)_R = y$) for any $x, y \in X$. An *R-up algebra* is both a left R-up algebra and a right R-up algebra. An algebra $(X; *, 0)$ is a *dual L-up algebra* if $((x \vee y)_L \wedge x))_L = x$ and $(y \wedge (x \vee y)_L)_L = y$, for any $x, y \in X$. An algebra $(X; *, 0)$ is said to be a *dual R-up algebra* if $((x \vee y)_R)_R = x$ and $(y \wedge (x \vee y)_R)_R = y$, for any $x, y \in X$. We observe several possibilities at work. First, note that $(x \wedge y)_L = x * (x * y) = (y \wedge x)_R$ in all cases. Now suppose that $(x \vee y)_L$ or $(x \vee y)_R$ have been obtained in some way. Then we define "conjugate symmetries" as follows:

$$
\begin{aligned}
(x \overset{0}{\vee} y)_L &:= (x \vee y)_L, & (x \overset{0}{\vee} y)_R &:= (x \vee y)_R;\\
(x \overset{1}{\vee} y)_L &:= (y \vee x)_L, & (x \overset{1}{\vee} y)_R &:= (y \vee x)_R;\\
(x \overset{2}{\vee} y)_L &:= (x \vee y)_R, & (x \overset{2}{\vee} y)_R &:= (x \vee y)_L;\\
(x \overset{3}{\vee} y)_L &:= (y \vee x)_R, & (x \overset{3}{\vee} y)_R &:= (y \vee x)_L;
\end{aligned}
$$

We construct a table for computation of conjugate symmetries as $(x \overset{12}{\vee} y)_L = (y \overset{1}{\vee} x)_L = (y \vee x)_R = (x \overset{2}{\vee} y)_L$, $(x \overset{12}{\vee} y)_R = (y \overset{2}{\vee} x)_R = (y \vee x)_L = (x \overset{3}{\vee} y)_R$, i.e., $\overset{1}{\vee} \cdot \overset{2}{\vee} = \overset{12}{\vee} = \overset{3}{\vee}$ in this "multiplication" to obtain the Klein 4-group as follows:

$\cdot$	$\overset{0}{\vee}$	$\overset{1}{\vee}$	$\overset{2}{\vee}$	$\overset{3}{\vee}$
$\overset{0}{\vee}$	$\overset{0}{\vee}$	$\overset{1}{\vee}$	$\overset{2}{\vee}$	$\overset{3}{\vee}$
$\overset{1}{\vee}$	$\overset{1}{\vee}$	$\overset{0}{\vee}$	$\overset{3}{\vee}$	$\overset{2}{\vee}$
$\overset{2}{\vee}$	$\overset{2}{\vee}$	$\overset{3}{\vee}$	$\overset{0}{\vee}$	$\overset{1}{\vee}$
$\overset{3}{\vee}$	$\overset{3}{\vee}$	$\overset{2}{\vee}$	$\overset{1}{\vee}$	$\overset{0}{\vee}$

Suppose now that we start with an L-up algebra, i.e.,

$$x = (x \wedge (x \vee y)_L)_L, \qquad y = ((x \vee y)_L \wedge y)_L$$

for all $x, y \in X$. If we introduce $\overset{1}{\vee}$, then we obtain:

$$x = (x \wedge (x \overset{1}{\vee} y)_L)_L, \qquad y = ((y \overset{1}{\vee} x)_L \wedge y)_L$$

and interchanging the rules of x and y,

$$x = ((x \overset{1}{\vee} y)_L \wedge x)_L, \qquad y = (y \wedge (x \overset{1}{\vee} y)_L)_L,$$

produces a dual L-up algebra. If we introduce $\overset{2}{\vee}$, then we obtain:

$$x = (x \wedge (x \overset{2}{\vee} y)_R)_L, \qquad y = ((x \overset{2}{\vee} y)_R \wedge y)_L$$

and thus

$$x = ((x \overset{2}{\vee} y)_R \wedge x)_R, \qquad y = (y \wedge (x \overset{2}{\vee} y)_R)_R,$$

which yields a dual R-up algebra. Finally, via the introduction of $\overset{3}{\vee}$ we obtain:

$$x = (x \wedge (x \overset{3}{\vee} y)_R)_L, \qquad y = ((x \overset{3}{\vee} y)_R \wedge y)_L,$$

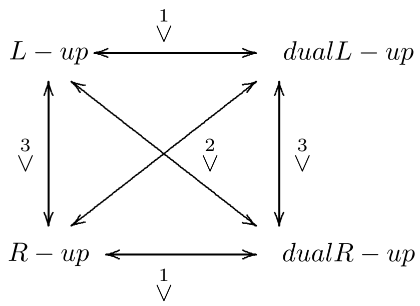
i.e.,

$$x = ((y \overset{3}{\vee} x)_R \wedge x)_R, \qquad y = (y \wedge (y \overset{3}{\vee} x)_R)_R,$$

and interchanging the roles of x and y we obtain:

$$x = (x \wedge (x \overset{3}{\vee} y)_R)_R, \qquad y = ((x \overset{3}{\vee} y)_R \wedge y)_R,$$

which are precisely the conditions for an R-up algebra. Thus, we may construct a "symmetry diagram":



This does not mean that an L-up algebra is necessarily an R-up algebra or one of the other types of algebras. On the other hand, theorems and statements for L-up algebras have corresponding statements for R-up, dual L-up and dual R-up algebras via the scheme outlined above.

Proposition 2.4. *Every bounded implicative BCK-algebra is an L-up algebra.*

Proof. Since any bounded implicative BCK-algebra is a Boolean algebra (see [6, pp. 34]), $x \wedge y = inf\{x, y\}$ and $x \vee y = sup\{x, y\}$. Hence $x \wedge (x \vee y) = inf\{x, sup\{x, y\}\} = x$ and $(x \vee y) \wedge y = inf\{sup\{x, y\}, y\} = y$. □

Example 2.5. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	0
3	3	3	1	0

Then $(X; *, 0)$ is a BCK-algebra. If we define an $\wedge_L$-table and an $\vee_L$-table as follows:

$\wedge_L$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
3	0	1	2	3

$\vee_L$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

then it is an L-up algebra.

Example 2.6. Consider the following BH-algebra, which is not a BCK/BCI-algebra.

$*$	0	1	2	3
0	0	0	1	1
1	1	0	3	0
2	2	2	0	0
3	3	2	1	0

If we define an $\wedge_L$-table and an $\vee_L$-table as follows:

$\wedge_L$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

$\vee_L$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

then $(X; *, 0)$ is an L-up algebra.

3. Mirror Algebras

Suppose $(X; *, 0)$ is a T-algebra. Let $M(X) := X \times \{0, 1\}$ and define a binary operation "$*$" on $M(X)$ as follows:

(m_1). $(x,0)*(y,0) := (x*y,0)$,
(m_2). $(x,1)*(y,1) := (y*x,0)$,
(m_3). $(x,0)*(y,1) := ((x\wedge y)_L,0) = (x*(x*y),0)$,
(m_4). $(x,1)*(y,0) := \begin{cases} (y,1) & \text{when } x*y=0, \\ (x,1) & \text{when } x*y\neq 0. \end{cases}$

Then we say that $M(X) := (M(X);*,(0,0))_L$ is a *left mirror algebra* of the T-algebra X. Similarly, if we define

$$(x,0)*(y,1) := ((x\wedge y)_R,0) = (y*(y*x),0)$$

then $M(X) := (M(X);*,(0,0))_R$ is a *right mirror algebra* of the T-algebra X.

Example 3.1. Let $X := \{0,1,2\}$ be a set with the following table:

$*$	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

Then we construct the mirror algebra $M(X)$ of X as follows:

$*$	0	a	b	c	d	e
0	0	0	0	0	0	0
a	a	0	c	b	e	d
b	b	0	0	b	b	0
c	c	0	c	0	c	d
d	d	0	d	0	0	d
e	e	0	e	b	e	0

where $0 := (0,0), a := (0,1), b := (1,0), c := (1,1), d := (2,0)$ and $e := (2,1)$.

Proposition 3.2. *If $(X;*,0)$ is a d-algebra then its mirror algebra $M(X)$ is also a d-algebra.*

Proof. Since $(x,1)*(y,0) \in \{(x,1),(y,1)\}$, $(x,1)*(y,0) = (0,0) = (y,0)*(x,1)$ is impossible. Hence $(x,i)*(y,j) = (y,j)*(x,i) = (0,0)$ means $i=j$ and thus $x*y = y*x = 0$ so that $x=y$ as well. Hence, the condition (III) for d-algebras holds. Other conditions are easy to be checked, and omit the proof. It follows that $(M(X);*,(0,0))_L$ is a d-algebra. □

Similar argument can be used to demonstrate that $(M(X);*,(0,0))_R$ is also a d-algebra. We can easily prove the following proposition.

Proposition 3.3. *If $(X;*,0)$ is a $d-BH$-algebra then its mirror algebra $M(X)$ is also a $d-BH$-algebra.*

Remark. The mirror algebra $M(X)$ of a BCK-algebra $(X;*,0)$ need not be a BCK-algebra. Consider a BCK-algebra with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	2
3	3	3	3	0

Since $[(3,1)*((3,1)*(2,1))]*(2,1) = (2*3,0) = (2,0) \neq (0,0)$, $M(X)$ is not a BCK-algebra. Moreover, the mirror algebra $M(X)$ of a BCH-algebra $(X;*,0)$ need not be a BCH-algebra also. Consider a BCH-algebra $(X;*,0)$ which is not a BCK/BCI-algebra as follows:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	3	3
2	2	0	0	2
3	3	0	0	0

Since $((1,0)*(3,0))*(2,1) = (3,0) \neq (0,0) = ((1,0)*(2,1))*(3,0)$, its mirror algebra $M(X)$ is not a BCH-algebra.

Theorem 3.4. *Let $(X;*,0)$ be an algebra satisfying at least the conditions (I), (II), (IV), (V) and (VIII). Then its mirror algebra $M(X)$ is a left L-up algebra.*

Proof. Given elements $(x,i),(y,j) \in M(X)$, it is enough to show that $((x,i) \wedge ((x,i) \vee (y,j))_L)_L = (x,i)$, where $i,j \in \{0,1\}$. We consider 4 cases. Case(1). $i=j=0$. We assume that the conditions (I) and (IV) hold. If $x*(y*x)=0$ then $((x,0) \wedge ((x,0) \vee (y,0)_L)_L = ((x,0) \wedge (y*x,0))_L = (x,0)*((x,0)*(y*x,0)) = (x,0)*(x*(y*x),0) = (x,0)*(0,0) = (x,0)$. If $x*(y*x) \neq 0$, then $((x,0) \wedge ((x,0) \vee (y,0))_L)L_{=}((x,0) \wedge (x,0))_L = (x,0)*((x,0)*(x,0)) = (x,0)*(x*x,0) = (x,0)*(0,0) = (x,0)$. Case(2). $i=j=1$. We assume the conditions (I), (II), (IV) and (V) hold. Then, by routine computation,

$$((x,1) \wedge ((x,1) \vee (y,1))_L)_L = \begin{cases} ((x*(x*y))*x, 1) & \text{if } x*[(x*(x*y))*x] = 0, \\ (x,1) & \text{otherwise.} \end{cases}$$

We know that

$$\begin{aligned} x*[(x*(x*y))*x] &= x*[(x*x)*(x*y)] && \text{[by (V)]} \\ &= x*(0*(x*y)) && \text{[by (I)]} \\ &= x*0 && \text{[by (II)]} \\ &= x. && \text{[by (IV)]} \end{aligned}$$

Hence $((x,1) \wedge ((x,1) \vee (y,1)_L)_L = (x,1)$ in any cases. Case (3). $i=1$ and $j=0$. Assume the conditions (I), (II) and (V) hold. Then

$$\begin{aligned} x*[(x \wedge y)_L * x] &= x*[(x*(x*y))*x \\ &= x*((x*x)*(x*y)) \\ &= x*(0*(x*y)) \\ &= x. \end{aligned}$$

Hence

$$\begin{aligned}((x,1)\wedge((x,1)\vee(y,0))_L)_L &= (x,1)*((x\wedge y)_L * x, 0)\\ &= \begin{cases}((x\wedge y)_L * x, 1) & \text{if } x*[(x\wedge y)_L * x]=0,\\ (x,1) & \text{otherwise}\end{cases}\\ &= \begin{cases}((0\wedge y)_L * 0, 1) & \text{if } x=0,\\ (x,1) & \text{otherwise}\end{cases}\\ &= (x,1).\end{aligned}$$

Case (4). $i = 0$ and $j = 1$. Assume the conditions (I), (IV) and (VIII) hold. If $x * y = 0$, then by (VIII) $x = x * (y * x)$, and hence $x * [x * (x * (y * x))] = x * (x * x) = x * 0 = x$. It follows that

$$\begin{aligned}((x,0)\wedge((x,0)\vee(y,1))_L)_L &= \begin{cases}((x,0)\wedge(y*x,0) & \text{if } x*y=0,\\ (x,1)\wedge(0,1) & \text{otherwise}\end{cases}\\ &= \begin{cases}(x*[x*(x*(y*x))],0) & \text{if } x*y=0,\\ (x*(x*(x*0)),0) & \text{otherwise}\end{cases}\\ &= (x,0),\end{aligned}$$

proving the theorem. □

Since every implicative BCK-algebra satisfies all conditions described in Theorem 3.4, we give the following corollary:

Corollary 3.5. *If $(X;*,0)$ is an implicative BCK-algebra, then its mirror algebra $M(X)$ is a left L-up algebra.*

References

1. Qing Ping Hu and Xin Li, *On BCH-algebras*, Math. Seminar Notes **11** (1983), 313-320.
2. Qing Ping Hu and Xin Li, *On proper BCH-algebras*, Math. Japonica **30** (1985), 659-661.
3. K. Iséki and S. Tanaka, *An introduction to theory of BCK-algebras*, Math. Japonica **23** (1978), 1-26.
4. K. Iséki, *On BCI-algebras*, Math. Seminar Notes **8** (1980), 125-130.
5. Y. B. Jun, E. H. Roh and H. S. Kim, *On BH-algebras*, Sci. Mathematicae **1** (1998), 347-354.
6. J. Meng and Y. B. Jun, *BCK*-algebras, Kyung Moon Sa Co., Seoul (1994).
7. J. Neggers and H. S. Kim, *On d-algebras*, Math. Slovaca **49** (1999), 19-26.
8. J. Neggers and H. S. Kim, *On analytic T-algebras*, Sci. Math. Japonica **53** (2001), 25-31.

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