MORE ON CAUCHY NETS IN APARTNESS SPACES

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Abstract. This note extends the work of an earlier paper. In particular, we give a necessary and sufficient condition for an apartness space to have the property that convergence implies Cauchyness.

The constructive theory of apartness[1] (point–set and set–set) has been developed in a series of papers over the past three years [17, 5, 6, 14, 16, 9]. In this latest paper in the series, we present a streamlined system of five axioms for a set–set apartness structure; *we regard this system, which consists of fewer, and in at least one case simpler, axioms than the one we used in* [7] *as the definitive one for our apartness theory.* We then derive some fundamental properties of Cauchy nets (using a slightly weaker notion of Cauchyness than in [7]). In particular, we show that every convergent net in an apartness space X is a Cauchy net if and only if X has a certain weak separation property.

We work throughout within the Bishop–style constructive framework, in which 'constructive' means 'developed using intuitionistic logic' [1, 2, 4, 15].

Our starting point is a set X equipped with a (set–set) **apartness relation** $\bowtie$, applicable to subsets of X and satisfying the following axioms.

B1 $X \bowtie \emptyset$.

B2 $S \bowtie T \Rightarrow S \cap T = \emptyset$.

B3 $R \bowtie (S \cup T) \Leftrightarrow R \bowtie S \wedge R \bowtie T$.

B4 $S \bowtie T \Rightarrow T \bowtie S$.

B5 $x \bowtie S \Rightarrow \exists T(x \bowtie T \wedge \forall y(y \bowtie S \vee y \in T))$.

Note that for a point x of S we write $x \bowtie S$ as shorthand for $\{x\} \bowtie S$. Also, we define an inequality on X by

$$x \neq y \Leftrightarrow \{x\} \bowtie \{y\}.$$

This has the minimal properties that one would expect of an inequality relation: namely,

$$\begin{aligned} x \neq y &\Rightarrow \neg(x = y), \\ x \neq y &\Rightarrow y \neq x. \end{aligned}$$

There are three notions of complement applicable to a subset S of the apartness space X :

- the **logical complement**

$$\neg S = \{x \in X : x \notin S\},$$

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[1] The motivation for this theory lay in the classical theory of nearness and proximity; see [10, 12, 13].

- the **complement**

$$\sim S = \{x \in X : \forall s \in S \, (x \neq s)\},$$

- and the **apartness complement**

$$-S = \{x \in X : x \bowtie S\}.$$

We have

$$-S \subset \sim S \subset \neg S.$$

The canonical example of an apartness spaces is a uniform space $(X, \mathcal{U})$, for which, in addition to the usual classical properties of the uniform structure $\mathcal{U}$ (see [3], Chapter 2), we postulate one property that automatically holds under classical logic:

$$\forall U \in \mathcal{U} \; \exists V \in \mathcal{U} \;\; (X = U \cup \sim V).$$

This property enables us to make membership decisions that are vital for many proofs in uniform–space theory; see, for example, [14]. The apartness of subsets S, T of X is then defined by

$$S \bowtie T \Leftrightarrow \exists U \in \mathcal{U} \, (S \times T \subset \sim U).$$

The apartness complements in X form a base for a topology, the **apartness topology**, on X. The open sets in this topology are called **nearly open sets.**

We say that an apartness space X is **Hausdorff** if

$$x \neq y \Rightarrow \exists U, V \;\; (x \in -U \wedge y \in -V \wedge U \subset \sim V)$$

—that is, if the apartness topology is Hausdorff in a natural sense.

By a **directed set** we mean a nonempty set D with a preorder[2] $\succeq$ such that for all $m, n \in D$ there exists $p \in D$ with $p \succeq m$ and $p \succeq n$. A **net** in a set X is a mapping $n \mapsto x_n$ of D into X; we denote such a net by $(x_n)_{n \in D}$. It is shown in [8] that an apartness space is Hausdorff if and only if it has the **strong unique limits property:** *If $(x_n)_{n \in D}$ is a net in X that converges to a limit x, and if $x \neq y \in X$, then $(x_n)_{n \in D}$ is eventually bounded away from y.*

A mapping $f : X \to Y$ between apartness spaces is said to be

- **continuous** if $f(x) \bowtie f(A)$ implies that $x \bowtie A$;
- **strongly continuous** if $f(A) \bowtie f(B)$ implies that $A \bowtie B$.

The strongly continuous maps are precisely the morphisms in the category of apartness spaces, and are clearly continuous.

Proposition 1 *Let $f : X \to Y$ be a continuous mapping between apartness spaces, and let $(x_n)_{n \in D}$ be a net in X that converges to a limit $x \in X$. Then the net $(f(x_n))_{n \in D}$ converges to $f(x)$.*

[2]The classical theory of nets requires a partial order. If we used a partial order in our constructive theory, we would run into difficulties which the classical theory avoids by applications of the axiom of choice, which entails the law of excluded middle [11].

PROOF. Let T be a subset of Y such that $f(x) \in -T$. By axiom B5, there exists $B \subset Y$ such that $f(x) \in -B$ and $Y = -T \cup B$. By continuity, $x \in -f^{-1}(B)$. Choose n_0 such that $x_n \in -f^{-1}(B)$ for all $n \succeq n_0$. For such n it is clear that $f(x_n) \notin B$ and hence that $f(x_n) \in -T$. Q.E.D

A net $s = (x_n)_{n\in D}$ in an apartness space X is a **Cauchy net** if for all subsets A, B of D with $s(A) \bowtie s(B)$, there exists n_0 such that if $n \in A$ for some $n \succeq n_0$, then

$$B \subset \neg \{n : n \succeq n_0\}.$$

Minor modifications of the proof given in [7] enable us to show that in a metric space X, a sequence is Cauchy in this sense if and only if it satisfies the usual metric Cauchy property. It is simple to prove that if $f : X \to Y$ is a strongly continuous map between apartness spaces and $(x_n)_{n\in D}$ is a Cauchy net in X, then $(f(x_n))_{n\in D}$ is a Cauchy net in Y.

A consequence of the strong axiom system we used in [7] was that every convergent net in an apartness space is a Cauchy net. Our new, streamlined axiom system leads to a much more informative result.

Theorem 2 *The following are equivalent conditions on an apartness space* $(X, \bowtie)$.

(i) *Every convergent net in* X *is a Cauchy net.*

(ii) X *is* ***weakly symmetrically separated***,[3] *in the sense that*

$$S \bowtie T \Rightarrow \forall x \in X \exists U \subset X \left(x \bowtie U \wedge \neg (S - U \neq \emptyset \wedge T - U \neq \emptyset)\right).$$

PROOF. Assuming (i), let $S \bowtie T$ and $x \in X$. Let

$$D = \{(\xi, U) : x \in -U \wedge \xi \in -U\},$$

with equality defined by

$$(\xi, U) = (\xi', U') \Leftrightarrow (\xi = \xi' \wedge -U = -U'),$$

and for each $n = (\xi, U)$ in D define $x_n = \xi$. It is easy to see that D is a directed set under the **reverse inclusion preorder** defined by

$$(\xi, U) \succeq (\xi', U') \Leftrightarrow -U \subset -U',$$

so that $\mathcal{N}_x = (x_n)_{n\in D}$ is a net.[4] Now define

$$A = \{n \in D : s(n) \in S\},$$
$$B = \{n \in D : s(n) \in T\}.$$

Since $s(A) \subset S$ and $s(B) \subset T$, we have $s(A) \bowtie s(B)$. It follows from (i) that there exists $n_0 = (y_0, U_0) \in D$ such that

$$\exists n \in A \, (n \succeq n_0) \Rightarrow B \subset \neg \{n \in D : n \succeq n_0\}. \tag{1}$$

Using axiom B5, choose $V_0 \subset X$ such that $x \in -V_0$ and $X = -U_0 \cup V_0$; in turn, choose $W_0 \subset X$ such that $x \in -W_0$ and $X = -V_0 \cup W_0$. Now suppose that there exist $y \in S - W_0$

[3] Classically, an apartness space is always weakly symmetrically separated.

[4] By using the reverse inclusion preorder in this way, we are able to avoid the full axiom of choice.

and $z \in T - W_0$. Then $(y, W_0) \in D$ and $s(y, W_0) = y \in S$, so $(y, W_0) \in A$; since also $-W_0 \subset -V_0 \subset -U_0$, we see that $(y, W_0) \succeq (y_0, U_0) = n_0$. Thus the antecedent of (1) holds with $n = (y, W_0)$; whence

$$B \subset \neg \{n \in D : n \succeq n_0\}.$$

On the other hand, either $z \in -U_0$ or $z \in V_0$. In the former case, $s(z, U_0) = z \in T$, so $(z, U_0) \in B$; whence $\neg ((z, U_0) \succeq n_0 = (y_0, U_0))$, which is absurd. Thus $z \in V_0$. But $z \in -W_0 \subset -V_0$, so we have a contradiction. We conclude that

$$\neg (S - W_0 \neq \emptyset \wedge T - W_0 \neq \emptyset).$$

Thus (ii) holds.

Now assume (ii), let $s = (x_n)_{n \in D}$ be a net converging to an element x in X, and let A, B be subsets of D such that $s(A) \bowtie s(B)$. By (ii), there exists $U \subset X$ such that

$$x \in -U \wedge \neg (s(A) - U \neq \emptyset \wedge s(B) - U \neq \emptyset).$$

Choose n_0 in D such that $x_n \in -U$ for all $n \succeq n_0$. Suppose that for some $n \succeq n_0$ we have $n \in A$. Then $x_n \in s(A) - U$, so $s(B) - U = \emptyset$; whence $B \subset \neg\{n : n \succeq n_0\}$. Q.E.D

An apartness space X is said to have the **nested neighbourhoods property** if

$$x \in -U \Rightarrow \exists V \, (x \in -V \wedge \neg V \bowtie U).$$

In that case, X is Hausdorff. For if $x \neq y$ in X then $x \bowtie \{y\}$, so there exists $U \subset X$ with $x \in -U$ and $\neg U \bowtie \{y\}$. By B4, $y \bowtie \neg U$; applying the nested neighbourhoods property again, we obtain $V \subset X$ such that $y \in -V$ and $\neg V \bowtie \neg U$. Then

$$-U \subset \neg U \subset -\neg V \subset \sim \neg V \subset \sim -V.$$

A (perforce directed) subset E of a directed set D is said to be **cofinal** if for each $n \in D$ there exists $m \in E$ with $m \succeq n$. By a **subnet** of a net $s = (x_n)_{n \in D}$ we mean a net $(x_n)_{n \in E}$ where E is a cofinal subset of D.

Theorem 3 *Let $s = (x_n)_{n \in D}$ be a Cauchy net that contains a subnet converging to a limit x in an apartness space X with the nested neighbourhoods property. Then s converges to x.*

PROOF. Let $(x_i)_{i \in I}$ be a subnet of s converging to x in X, and let $x \in -U$. Since X has the nested neighbourhoods property, there exists $V \subset X$ such that $x \in -V$ and $\neg V \bowtie U$; again using the nested neighbourhoods property, we can find $W \subset X$ such that $x \in -W$ and $\neg W \bowtie V$. Let

$$A = \{n \in D : x_n \in -W\},$$
$$B = \{n \in D : x_n \in V\}.$$

Since $-W \bowtie V$, we have $s(A) \bowtie s(B)$; whence there exists $n_0 \in D$ such that if $n \succeq n_0$ for some $n \in A$, then

$$B \subset \neg \{n : n \succeq n_0\}. \tag{2}$$

But there exists $i_0 \in I$ such that $x_i \in -W$ for all $i \in I$ with $i \succeq i_0$. Choose $i_1 \in I$ such that $i_1 \succeq n_0$. Since I is directed, there exists $i \in I$ such that $i \succeq i_0$ and $i \succeq i_1$; whence $x_i \in -W$

—so $i \in A$ —and $i \succeq n_0$. Thus (2) holds. It follows that if $n \succeq n_0$, then $x_n \notin V$ and so $x_n \in -U$. Thus s converges to x. Q.E.D

We end with a natural description of the closure operation in the apartness topology. Note that, by definition, the **closure** of a subset A of X consists of those points $x \in X$ of which every neighbourhood in the apartness topology intersects A; equivalently, this is the set of all $x \in X$ such that for each $U \subset X$, if $x \in -U$, then $-U$ intersects A.

Theorem 4 *The closure of a subset A in the apartness topology on an apartness space X consists of all points of X that are limits of nets in A.*

PROOF. If $(x_n)_{n \in D}$ is a net in A converging to an element x of X, then for each U with $x \in -U$ there exists $n \in D$ such that $x_n \in -U$. Hence $x \in \overline{A}$.

Conversely, if $x \in \overline{A}$, then $A - U$ is nonempty for each $U \subset X$ with $x \in -U$. Let

$$D = \{(y, U) : x \in -U \wedge y \in A - U\}.$$

Then D is directed by the reverse inclusion preorder $\succeq$ defined in the proof of Theorem 2. Let $(y_n)_{n \in D}$ be the net in A defined by the mapping $(y, U) \mapsto y$, and let $U \subset X$ be such that $x \in -U$. Since $x \in \overline{A}$, there exists $y \in A - U$; let $n_0 = (y, U)$. For each $n = (z, V) \succeq n_0$ we have $x \in -V \subset -U$ and $z \in A - V$; whence $y_n \in -V \subset -U$. Thus $(y_n)_{n \in D}$ converges to x. Q.E.D

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