MICROSCOPIC SUBSETS OF A BANACH SPACE AND CHARACTERIZATIONS OF THE DROP PROPERTY

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Abstract. We extend the notion of *microscopic sets* introduced for subsets of the real line, to the case of subsets of a Banach space X. Then we introduce the *Microscopic Drop Property* for closed and convex subsets of X, and we eventually show that, when X is reflexive, it is equivalent to the usual Drop Property defined by D.N. Kutzarova.

1. Introduction Let $(X, \|\cdot\|)$ be a real Banach space. The Drop Property appeared at the end of the 80's as a property of the norm of X, ([7], [9] and [10]).
Later, D. N. Kutzarova [4] proposed a definition of Drop Property for more general sets.

Definition 1 ([4]) Let C be closed and convex. C has the *Drop Property* iff for any closed F with $F \cap C = \emptyset$ there exists $x_0 \in F$ such that $D(x_0, C) \cap F = \{x_0\}$, where the set $D(x_0, C) := \text{co}\{x_0, C\}$ is called the *drop* generated by x_0 and C.
We denote by $\mathbf{DP}(X)$ the hyperspace of sets having the Drop Property and by $\mathbf{DPB}(X)$ the hyperspace of bounded sets having the Drop Property.

Among the properties of $\mathbf{DP}(X)$ the following is of particular interest:

Theorem 1 ([5], Theorem 3) *If C has the Drop Property, then it is either compact or it has non empty interior.*

Indeed this yields that if A, B are disjoint closed convex sets, and $A \in \mathbf{DP}(X)$ then they can be separated at least in the large sense.
We can extend **Definition 1** in the following way.

Definition 2 Let $\mathcal{K}$ be a non-empty class of subsets of X, and let C be a closed and convex subset of X. We shall say that $C \in (\mathcal{K}) - \mathbf{DP}(X)$ if, for every closed F with $F \cap C = \emptyset$, there exists $x_0 \in F$ such that $D(x_0, C) \cap F \in \mathcal{K}$.

In [6] the Drop Property has been characterized in the following way:

Theorem 2 ([6], Theorem 4) *Let $C \subset X$. The following are equivalent:*
i) C has the Drop Property;
ii) for every closed set F such that $F \cap C = \emptyset$ there exists $x_0 \in F$ such that $D(x_0, C) \cap F$ is compact;
iii) for every closed set F such that $F \cap C = \emptyset$ and for every $\varepsilon > 0$ there exists $x_\varepsilon \in F$ such that $\alpha(D(x_\varepsilon, C) \cap F) < \varepsilon$ (where α is the Kuratowski measure of non-compactness).

Theorem 2 above states that

$$\mathbf{DP}(X) = (\mathbf{P_k}(X)) - \mathbf{DP}(X)$$

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where $\mathbf{P_k}(X)$ denotes the class of compact subsets of X : it therefore suggests that the Drop Property can be characterized by the existence of vertices $x \in F$ such that $D(x, C) \cap F$ is "small" in some sense.
In this paper we shall introduce a new class of small sets in a Banach space X, the *microscopic sets,* and we shall compare the induced Drop Property with the classical one.

2. Preliminaries We denote by $F(C)$ the set

$$F(C) = \{x^* \in X^* \setminus \{0\}, \text{ which are bounded above } C\}.$$

For every $x^* \in F(C)$ and $\delta > 0$ the *slice* $S(x^*, C, \delta)$ is defined as

$$S(x^*, C, \delta) = \{x \in C : x^*(x) \geq M - \delta\}$$

where $M = \sup\{x^*(x) : x \in C\}$.
According to [5] we remind the *property* (α):
C has the *property* (α) iff $\lim_{\delta \to 0} \alpha(S(x^*, C, \delta)) = 0$, for every $x^* \in F(C)$ and $\delta > 0$.

Property (α) is a useful characterization of sets having the Drop Property:

Theorem 3 ([5], Theorem 7, Theorem 8) *Let X be a reflexive Banach space. If C is bounded noncompact, or unbounded, the following conditions are equivalent:*
i) C has the Drop Property;
ii) C has non empty interior and has the property (α).

Definition 3 Given a closed bounded convex set C, a sequence $(x_n)_n$ in $X \setminus C$ such that $x_{n+1} \in D(x_n, C)$, for all $n \in I\!N$, is called a *stream.* C is called the *basis* of the stream.
A stream $(y_n)_n$ is said a *dyadic stream* if it can be represented by the following inductive formula $y_1 = \dfrac{x + x_1}{2}$ and $y_n = \dfrac{y_{n-1} + x_n}{2}$, for $n \geq 2$, where $x \in X \setminus C$ and $(x_n) \subset C$.
By induction one can prove that $y_n = \dfrac{1}{2^n} + \sum_{i=1}^{n} \dfrac{1}{2^{n-i+1}} x_i$, for $n \in I\!N$.

With this concept, in [3] another characterization of the Drop Property is stated:

Theorem 4 ([3], Theorem 2) *A closed bounded convex set C in a Banach space X has the Drop Property iff every stream in $X \setminus C$ has a norm converging subsequence.*

Moreover we can observe that even in the unbounded case one implication of the previous result holds true, namely, if C has the Drop Property, every stream has a norm converging subsequence.

3. Microscopic sets in infinite dimensional setting In [1] the following class of small subsets of $I\!R$ as been introduced:

Definition 4 [1] $M \subset I\!R$ is called *microscopic* iff for every $\varepsilon > 0$ one can find a sequence $(I_n)_n$ of intervals such that $M \subset \bigcup_{n=1}^{\infty} I_n$ and $\lambda(I_n) \leq \varepsilon^n$, for all $n \in I\!N$.

This concept extends to subset M of X in at least two different ways:

Definition 5 $M \subset X$ is *microscopic* iff for every $\varepsilon > 0$ one can find a sequence $(x_n)_n \in X$ such that $M \subset \bigcup_{n=1}^{\infty} (x_n + \varepsilon^n X_1)$, where X_1 is the closed unit ball of X. We denote with $\mathbf{P_m}(X)$ the class of microscopic subsets of X.
$M \subset X$ is *scalarly microscopic* iff for each functional $x^* \in X^*$, $x^*(M)$ is microscopic in $I\!R$. We denote with $\mathbf{P_{sm}}(X)$ the class of scalarly microscopic subsets of X.

Since clearly

$$\widehat{X} \subset \mathbf{P_m}(X) \subset \mathbf{P_{sm}}(X),$$

where $\widehat{X}$ denoted the class of singletons of X; we immediately have

$$\mathbf{DP}(X) \subset (\mathbf{P_m}(X)) - \mathbf{DP}(X) \subset (\mathbf{P_{sm}}(X)) - \mathbf{DP}(X).$$

But we shall show that

$$(\mathbf{P_{sm}}(X)) - \mathbf{DP}(X) = \mathbf{DP}(X)$$

and therefore we bound ourselves to $\mathcal{K} = \mathbf{P_{sm}}(X)$.
Using the properties of microscopic subset of $I\!R$ proved in [1] one can easily obtain:

Proposition 1 *The following hold:*
i) every subset of a scalarly microscopic set is scalarly microscopic;
ii) every countable union of scalarly microscopic sets is scalarly microscopic;
iii) given a scalarly microscopic set M and $x \in I\!R$, $x + M$ is scalarly microscopic;
iv) given a scalarly microscopic set M and $\alpha \in I\!R$, αM is scalarly microscopic;
v) X is not scalarly microscopic;
vi) every countable set is scalarly microscopic.

4. Streams and polygonals For any pair $x, y \in X$ define the *interval* $[x, y]$ to be the set

$$[x, y] = \text{co}\{x, y\}.$$

In [2] it is proven that, if $x \neq y$, $[x, y] \notin \mathbf{P_{sm}}(X)$.

Proposition 2 (No cross properties) *Given a stream $(x_n)_n \in X \backslash C$ such that $x_n \neq x_k$ for $n \neq k$, the following hold:*
i) for all $p, k \in I\!N$ with $k \neq p \neq k+1$, $x_p \notin [x_k, x_{k+1}]$;
ii) for all $k, p \in I\!N$ such that $k \neq p+1 \neq k+1 \neq p$, $[x_k, x_{k+1}] \cap [x_p, x_{p+1}] = \emptyset$.

Proof. *i)* Observe first that if $x \in D(y, C)$ and $y \in D(x, C)$ then $x = y$.
Suppose, by contradiction, that $x_p \in [x_k, x_{k+1}]$. Then from Lemma 4.3.3. of [8], one easily yields that $p > k+1$.
On the other side Proposition 4.2.8. of [8] implies that $x_{k+2} \in D(x_p, C)$ and therefore necessarily $p = k+2$. But if $x_{k+2} \in [x_k, x_{k+1}]$, there are $z \in C$, $s, t \in]0, 1[$ such that

$$tx_{k+1} + (1-t)z = sx_{k+1} + (1-s)x_k$$

and necessarily $s \neq t$ (otherwise $x_k = z \in C$). Suppose for instance $t > s$; then

$$\frac{t-s}{1-s}x_{k+1} + \frac{1-t}{1-s}z = x_k$$

namely $x_k \in D(x_{k+1}, C)$, and therefore $x_k = x_{k+1}$: contradiction.
ii) Suppose

$$[x_k, x_{k+1}] \cap [x_p, x_{p+1}] \neq \emptyset.$$

Without loss of generality we can suppose $p > k+1$.
Then by Proposition 4.2.8. of [8], again, $x_{k+2} \in D(y, C)$. If $y \in D(x_{k+2}, C)$ then $y = x_{k+2}$, and thus $x_{k+2} \in [x_k, x_{k+1}]$ which contradicts i).
Otherwise if $y \notin D(x_{k+2}, C)$, then, since $p \geq k+2$, $y \notin D(x_p, C)$: but by definition of stream, $x_{p+1} \in D(x_p, C)$ and by convexity of the drop $[x_p, x_{p+1}] \in D(x_p, C)$ and, as $y \in [x_p, x_{p+1}]$, $y \in D(x_p, C)$: contradiction. ■

Now we can introduce a new concept:

Definition 6 Given a sequence $(x_n)_n \subset X$ we define the *induced polygonal* as the set

$$P((x_n)_n) = \bigcup_{n=1}^{\infty} [x_n, x_{n+1}].$$

Lemma 1 *Let (y_n) be a dyadic stream with basis C such that there exists $\delta > 0$ for which*

$$\mathrm{d}(y_n, \mathrm{span}\{y_1, ..., y_{n-1}\}) > \delta.$$

Then the induced polygonal $P((y_n)_n)$ is closed.

Proof. Let $(z_n)_n \in P$ be convergent to $z \in X$. We shall show that $z \in P$.
For $k \in I\!N$, let $I\!N_k = \{p \text{ such that } z_p \in [y_k, y_{k+1}]\}$: if $I\!N_{k_0}$ is infinite for some $k_0 \in I\!N$, then, a subsequence of $(z_n)_n$ is contained in $[y_{k_0}, y_{k_0+1}]$, whence $z \in [y_{k_0}, y_{k_0+1}] \subset P$.
Alternatively, assume that $I\!N_k$ is finite, for every $k \in I\!N$: we shall show that this will lead us to a contradiction.
We first note that without loss of generality we can assume that the z_n's are not corners of the polygonal.
To simplify the indices we shall now define a new sequence $(\zeta_n)_n$, such that

$$\zeta_n \in]y_n, y_{n+1}[\qquad \text{for each } n, \tag{1}$$

in the following way: for those segments $[y_k, y_{k+1}]$ containing some z_n we pick just one of them as ζ_k; for all the other segments we define ζ_k to be the middle point.
We prove now that $(\zeta_n)_n$ has no converging subsequences; since originally we have chosen $(z_n)_n$ converging to z this will lead to a contradiction.
By (1) for every n there exists $\alpha_n \in]0, 1[$ such that

$$\zeta_n = \alpha_n y_n + (1 - \alpha_n) y_{n+1},$$

and by definition of dyadic stream

$$\zeta_n = 2\alpha_n y_{n+1} - \alpha_n x_n + (1 - \alpha_n) y_{n+1} = (1 + \alpha_n) y_{n+1} - \alpha_n x_n,$$

for some $x_n \in C$.
Note that, for every $n \in I\!N$, we find $\mathrm{d}(x_n, \mathrm{span}\{x_1, ..., x_{n-1}\}) > 2\delta$. Then

$$\zeta_n = \frac{1+\alpha_n}{2^{n+1}} x + \sum_{i=1}^{n-1} \frac{1+\alpha_n}{2^{n+2-i}} x_i + \frac{1-3\alpha_n}{2^2} x_n + \frac{1+\alpha_n}{2} x_{n+1}.$$

So one can show that

$$\|\zeta_{n+p} - \zeta_n\| \geq \delta - \frac{1}{2^n}\|x\|,$$

and thus $(\zeta_n)_n$ cannot have converging subsequences. ■

5. The Microscopic Drop Property As announced in the previous section, we shall consider the *Microscopic Drop Property* as follows:

Definition 7 C has the *Microscopic Drop Property* iff for each closed F with $F \cap C = \emptyset$ there exists $x \in F$ such that $D(x, C) \cap F \in \mathbf{P_{sm}(X)}$.
For the sake of simplicity we shall denote by $\mathbf{MDP}(X)$ the class of sets having the Microscopic Drop Property, namely $\mathbf{MDP}(X) := (\mathbf{P_{sm}(X)})\text{-}\mathbf{DP}(X)$.

Remark 1 Each compact set in X has the Microscopic Drop Property.

Proposition 3 *If C has the Microscopic Drop Property then for each stream $(x_n)_n \subset X \setminus C$ the generated polygonal is not closed.*

Proof. Suppose that there exists a stream $(x_n)_n \in X \setminus C$ such that the polygonal $P = P((x_n)_n)$ is closed. Observe first that P and C are disjoint.
Let y be a point of P. Then there exists $\overline{n} \in \mathbb{N}$ such that $y \in [x_{\overline{n}}, x_{\overline{n}+1}]$.
By Proposition 4.2.8. of [8], $x_{\overline{n}+2} \in D(y, C)$ and then, by convexity $[y, x_{\overline{n}+2}] \subset D(y, C)$.
So $D(y, C) \cap P$ contains a non microscopic set therefore it is non microscopic. ■

Now we want to extend **Theorem 1** to bounded sets C having the Microscopic Drop Property.

Theorem 5 *Let C be bounded in X. If C has the Microscopic Drop Property then it is either compact or it has non empty interior.*

Proof. The case of C finite dimensional can be proven as in Theorem 3 in [5].
Assume then that C is infinite-dimensional, non compact and has empty interior.
Then as in Theorem 4.4.2. of [8] there exists a dyadic stream that fulfills the assumpion of **Lemma 1.** Hence the induced polygonal is closed, which, from **Proposition 3**, contradicts the Microscopic Drop Property. ■

We shall now investigate the relationship between the Microscopic Drop Property and the property (α).

Theorem 6 *If C has the Microscopic Drop Property, then it has the property (α).*

Proof. Suppose that C does not fulfill the property (α).
Then there exists a linear functional $x_0^* \in X^*$ such that

$$\inf_{\nu>0} \alpha(S(x_0^*, C; \nu)) > 0.$$

Now as in [5], Proposition 1, we can consider a stream $(x_n)_n \subset X \setminus C$ such that

$$\mathrm{d}(x_n, \mathrm{span}\{x_1, ..., x_{n-1}\}) > \frac{\delta}{2}, \tag{2}$$

with $0 < \delta < \frac{1}{2} \inf_{\nu>0} \alpha(S(x_0^*, C, \nu))$. Let $P = P((x_n)_n)$. From **Lemma 1** P is closed, which contradicts **Proposition 3**. ■

Theorem 7 *Let C be unbounded in X. If C has the Microscopic Drop Property then it has non empty interior.*

Proof. In the case of finite dimensional C, if $C^\circ \neq \emptyset$, the property (α) does not hold, in contradiction with **Theorem 6.**
Assume then that C is infinite dimensional. For $\delta > 0$ fixed, with the same argument of Theorem 4.4.2. in [8], we can construct a sequence $(x_n)_n \in C$ such that

$$\mathrm{d}(x_n, \mathrm{span}\{x_1, ..., x_{n-1}\}) > \delta \qquad \forall n \in I\!N.$$

If $C^\circ \neq \emptyset$, there exists $x \in X \setminus C$ be such that the dyadic stream $(y_n)_n$, generated by $(x_n)_n$ with initial point x, is disjoint from C; furthermore

$$\mathrm{d}(y_n, \mathrm{span}\{y_1 ..., y_{n+1}\}) > \frac{\delta}{2}.$$

Therefore, from **Lemma 1**, the polygonal $P = P((y_n)_n)$ is closed and disjoint from C. Then by **Proposition 3**, the assumption follows. ∎

Theorem 8 *Let X be reflexive, and let C be non compact. Then the following are equivalent:*
i) C has the Drop Property;
ii) C has non empty interior and has the property (α);
iii) C has the Microscopic Drop Property.

Proof. ¿From Theorem 7 and 8 of [5] i) and ii) are equivalent.
As already pointed out the implication $i) \Rightarrow iii)$ is trivial.
The implication $iii) \Rightarrow ii)$ is an immediate consequence of **Theorems 5, 6** and **7**.

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