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FOLDING THEORY APPLIED TO SOME TYPES OF POSITIVE IMPLICATIVE HYPER BCK-IDEALS IN HYPER BCK-ALGEBRAS

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ABSTRACT. The foldness of $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideals and $PI(\ll, \ll, \ll)_{BCK}$-ideals is considered. The fuzzy version of such notions is also discussed.

1. INTRODUCTION

The study of BCK-algebras was initiated by K. Iséki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Since then a great deal of literature has been produced on the theory of BCK-algebras, In particular, emphasis seems to have been put on the ideal theory of BCK-algebras. The hyperstructure theory (called also multialgebras) is introduced in 1934 by F. Marty [9] at the 8th congress of Scandinavian Mathematicians. In [8], Y. B. Jun et al. applied the hyperstructures to BCK-algebras, and introduced the concept of a hyper BCK-algebra which is a generalization of a BCK-algebra, and investigated some related properties. They also introduced the notion of a hyper BCK-ideal and a weak hyper BCK-ideal, and gave relations between hyper BCK-ideals and weak hyper BCK-ideals. Y. B. Jun et al. [7] gave a condition for a hyper BCK-algebra to be a BCK-algebra, and introduced the notion of a strong hyper BCK-ideal, a weak hyper BCK-ideal and a reflexive hyper BCK-ideal. They showed that every strong hyper BCK-ideal is a hypersubalgebra, a weak hyper BCK-ideal and a hyper BCK-ideal; and every reflexive hyper BCK-ideal is a strong hyper BCK-ideal. In [4], Y. B. Jun and X. L. Xin introduced the notion of an implicative hyper BCK-ideal. They gave the relations among hyper BCK-ideals, implicative hyper BCK-ideals and positive implicative hyper BCK-ideals. They stated some characterizations of implicative hyper BCK-ideals. And they also introduced the notion of implicative hyper BCK-algebras and investigated the relation between implicative hyper BCK-ideals and implicative hyper BCK-algebras. In [5], Y. B. Jun and X. L. Xin introduced the notion of a positive implicative hyper BCK-ideal, and investigated some related properties. Y. B. Jun and W. H. Shim [1] discussed several types of positive implicative hyper BCK-ideals in hyper BCK-algberas, and investigated their relations. In this paper we consider the foldness of $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideals and $PI(\ll, \ll, \ll)_{BCK}$-ideals in hyper BCK-algebras, and discuss their fuzzy version.

2. PRELIMINARIES

We include some elementary aspects of hyper BCK-algebras that are necessary for this paper, and for more details we refer to [3] and [8]. Let H be a nonempty set endowed with a hyper operation "$\circ$", that is, $\circ$ is a function from $H \times H$ to $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$. For

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two subsets A and B of H, denote by $A \circ B$ the set $\bigcup_{a\in A, b\in B} a \circ b$. We shall use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$, or $\{x\} \circ \{y\}$.

By a *hyper BCK-algebra* we mean a nonempty set H endowed with a hyper operation "$\circ$" and a constant 0 satisfying the following axioms:

(K1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(K2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(K3) $x \circ H \ll \{x\}$,
(K4) $x \ll y$ and $y \ll x$ imply $x = y$,

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$.

In any hyper BCK-algebra H, the following hold (see [3] and [8]):

(p1) $0 \ll x$,
(p2) $A \subseteq B$ implies $A \ll B$,
(p3) $x \circ 0 = \{x\}$ and $A \circ 0 = A$

for all $x, y, z \in H$ and for all nonempty subsets A, B and C of H. In what follows let H denote a hyper BCK-algebra unless otherwise specified.

Definition 2.1. [8] A nonempty subset A of H is called a *hyper BCK-ideal* of H if it satisfies the following conditions:

(I1) $0 \in A$,
(I2) $\forall x, y \in H$ $(x \circ y \ll A,\ y \in A \ \Rightarrow\ x \in A)$.

Definition 2.2. [8] A nonempty subset A of H is called a *weak hyper BCK-ideal* of H if it satisfies (I1) and

(I3) $\forall x, y \in H$ $(x \circ y \subseteq A,\ y \in A \ \Rightarrow\ x \in A)$.

Definition 2.3. [1] A nonempty subset A of H is called a *$PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal* of H if it satisfies (I1) and

(I4) $\forall x, y, z \in H$ $((x \circ y) \circ z \ll A,\ y \circ z \subseteq A \ \Rightarrow\ x \circ z \subseteq A)$.

We place a bar over a symbol to denote a fuzzy set so $\bar{A}$, $\bar{B}$, $\cdots$ all represent fuzzy sets in H. We write $\bar{A}(x)$, a number in $[0, 1]$, for the membership function of $\bar{A}$ evaluated at $x \in H$. An α-cut of $\bar{A}$, written $\bar{A}[\alpha]$, is defined as

$$\{x \in H \mid \bar{A}(x) \geq \alpha\} \text{ for } 0 < \alpha \leq 1.$$

We separately specify $\bar{A}[0]$ as the closure of the union of all the $\bar{A}[\alpha]$ for $0 < \alpha \leq 1$.

Definition 2.4. [6] A fuzzy set $\bar{A}$ in H is called a *fuzzy hyper BCK-ideal* of H if it satisfies:

(F1) $\forall x, y \in H$ $(x \ll y \ \Rightarrow\ \bar{A}(x) \geq \bar{A}(y))$
(F2) $\forall x, y \in H$ $\left(\bar{A}(x) \geq \min\left\{\inf_{a\in x\circ y} \bar{A}(a),\ \bar{A}(y)\right\}\right)$.

Proposition 2.5. [6] *A fuzzy set $\bar{A}$ in H is a fuzzy hyper BCK-ideal of H if and only if the level set $\bar{A}[\alpha]$, $\alpha \in \mathrm{Im}(\bar{A})$, of $\bar{A}$ is a hyper BCK-ideal of H.*

Definition 2.6. [2] A fuzzy set $\bar{A}$ in H is called a *fuzzy $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal* of H if it satisfies (F1) and

(F3) $\forall x, y, z \in H$ $\left(\inf_{a\in x\circ z} \bar{A}(a) \geq \min\left\{\inf_{b\in (x\circ y)\circ z} \bar{A}(b),\ \inf_{c\in y\circ z} \bar{A}(c)\right\}\right)$.

3. Folding Theory of Some Types of Positive Implicative Hyper BCK-ideals

For any $x, y \in H$ and any natural number n, denote

$$x \circ y^n = (\cdots((x \circ \underbrace{y) \circ y) \cdots) \circ y}_{n\text{-times}}$$

Definition 3.1. Let k, m, and n be natural numbers. A nonempty subset A of H is called a $(k, m; n)$*-fold* $PI(\ll, \subseteq, \subseteq)_{BCK}$*-ideal* of H if it satisfies (I1) and

(I5) $\forall x, y, z \in H\ ((x \circ y) \circ z^k \ll A,\ y \circ z^m \subseteq A\ \Rightarrow\ x \circ z^n \subseteq A)$.

Example 3.2. Let $H = \{0, a, b\}$ be a hyper BCK-algebra with the following Cayley table:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

Then $A = \{0, a\}$ is a $(k, m; n)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of H for natural numbers k, m and n.

Example 3.3. Let $H = \{0, a, b\}$ be a hyper BCK-algebra with the following Cayley table:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{a\}$	$\{0, a\}$

Then $A = \{0, b\}$ is a $(k, m; n)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of H for natural numbers k, m, and $n > 2$. But it is not a $(2, 3; 1)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of H since $(b \circ a) \circ a^2 = \{0\} \ll A$ and $a \circ a^3 = \{0\} \subseteq A$, but $b \circ a^1 = \{a\} \nsubseteq A$.

Theorem 3.4. *Every $(k, m; n)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal is a hyper BCK-ideal for natural numbers k, m, and n.*

Proof. Let A be a $(k, m; n)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of H. Let $x, y \in H$ be such that $x \circ y \ll A$ and $y \in A$. Putting $z = 0$ in (I5), we get $(x \circ y) \circ 0^k = x \circ y \ll A$ and $y \circ 0^m = \{y\} \subseteq A$. It follows from (I5) that $\{x\} = x \circ 0^n \subseteq A$, i.e., $x \in A$. Hence A is a hyper BCK-ideal of H. □

The converse of Theorem 3.4 may not be true. In fact, consider the hyper BCK-algebra H in Example 3.3. Then $A := \{0\}$ is a hyper BCK-ideal of H. But it is not a $(k, m; 1)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of H for $k \geq 2$ because $(b \circ 0) \circ a^k = \{0\} \ll A$ and $0 \circ a^m = \{0\} \subseteq A$, but $b \circ a^1 = \{a\} \nsubseteq A$.

Theorem 3.5. *For natural number m, let A be a $(m, m; m)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of H. Then, for $w \in H$, the set*

$$A_w := \{x \in H \mid x \circ w^m \subseteq A\}$$

is a weak hyper BCK-ideal of H.

Proof. Obviously $0 \in A_w$. Let $x, y \in H$ be such that $x \circ y \subseteq A_w$ and $y \in A_w$. Then $(x \circ y) \circ w^m \subseteq A$ and $y \circ w^m \subseteq A$, which implies that $(x \circ y) \circ w^m \ll A$ and $y \circ w^m \subseteq A$. It follows from (I5) that $x \circ w^m \subseteq A$ or equivalently $x \in A_w$. Therefore A_w is a weak hyper BCK-ideal of H. □

Lemma 3.6. [3] *Let A be a subset of H. If I is a hyper BCK-ideal of H such that $A \ll I$, then A is contained in I.*

Theorem 3.7. *Let A be a hyper BCK-ideal of H. If*

$$A_w := \{x \in X \mid x \circ w^m \subseteq A\}$$

is a weak hyper BCK-ideal of H for all $w \in H$, then A is a $(m, m; m)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of H.

Proof. Let $x, y, z \in H$ be such that $(x \circ y) \circ z^m \ll A$ and $y \circ z^m \subseteq A$. Then $(x \circ y) \circ z^m \subseteq A$ by Lemma 3.6 and $y \in A_z$. Thus for each $t \in x \circ y$, we have $t \circ z^m \subseteq A$ or equivalently $t \in A_z$. Hence $x \circ y \subseteq A_z$. Since A_z is a weak hyper BCK-ideal of H, we get $x \in A_z$, i.e., $x \circ z^m \subseteq A$. Therefore A is a $(m, m; m)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of H. □

Definition 3.8. Let k, m, and n be natural numbers. A nonempty subset A of H is called a *$(k, m; n)$-fold $PI(\ll, \ll, \ll)_{BCK}$-ideal* of H if it satisfies (I1) and

(I6) $\forall x, y, z \in H \ ((x \circ y) \circ z^k \ll A,\ y \circ z^m \ll A \ \Rightarrow\ x \circ z^n \ll A)$.

The following example shows that there is a $(k, m; n)$-fold $PI(\ll, \ll, \ll)_{BCK}$-ideal which is not a hyper BCK-ideal.

Example 3.9. Let $H = \{0, a, b\}$ be a hyper BCK-algebra with the following Cayley table:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

Then $A = \{0, b\}$ is a $(k, m; n)$-fold $PI(\ll, \ll, \ll)_{BCK}$-ideal of H for natural numbers k, m and n, but not a hyper BCK-ideal of H since $a \circ b = \{0, a\} \ll A$ and $b \in A$, but $a \notin A$.

Definition 3.10. [1] A nonempty subset A of H is said to be *closed* if for every $x, y \in H$, $x \ll y$ and $y \in A$ imply $x \in A$.

Example 3.11. In Example 3.9, $A := \{0, a\}$ is closed, which is a $(k, m; n)$-fold $PI(\ll, \ll, \ll)_{BCK}$-ideal of H.

Theorem 3.12. *Every closed $(k, m; n)$-fold $PI(\ll, \ll, \ll)_{BCK}$-ideal is a hyper BCK-ideal.*

Proof. Let A be a closed $(k, m; n)$-fold $PI(\ll, \ll, \ll)_{BCK}$-ideal of H and let $x, y \in H$ be such that $x \circ y \ll A$ and $y \in A$. Taking $z = 0$ in (I6) and using (p3), we have $(x \circ y) \circ 0^k = x \circ y \ll A$ and $y \circ 0^m = \{y\} \ll A$. It follows from (I6) and (p3) that $\{x\} = x \circ 0^n \ll A$ so that there exists $u \in A$ such that $x \ll u$. Since A is closed, we get $x \in A$. Hence A is a hyper BCK-ideal of H. □

4. Fuzzification of Folding Theory Applied to Some Types of Positive Implicative Hyper BCK-ideals

Definition 4.1. Let k, m, and n be natural numbers. A fuzzy set $\bar{A}$ in H is called a *$(k, m; n)$-fold fuzzy positive implicative ideal* of H if it satisfies (F1) and

(F4) $\forall x, y, z \in H \ \left(\inf_{a \in x \circ z^k} \bar{A}(a) \geq \min\left\{ \inf_{b \in (x \circ y) \circ z^m} \bar{A}(b),\ \inf_{c \in y \circ z^n} \bar{A}(c) \right\} \right)$

Note that $(1, 1; 1)$-fold fuzzy positive implicative ideal is a fuzzy $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal.

Example 4.2. Let $H = \{0, a, b\}$ be a hyper BCK-algebra in Example 3.2. Define a fuzzy set $\bar{A}$ in H by $\bar{A}(0) = \bar{A}(a) = 0.7$ and $\bar{A}(b) = 0.07$. It is easily checked that, for natural numbers k, m, and n, $\bar{A}$ is a $(k, m; n)$-fold fuzzy positive implicative ideal of H.

Theorem 4.3. *Every $(k, m; n)$-fold fuzzy positive implicative ideal is a fuzzy hyper BCK-ideal, where k, m and n are natural numbers.*

Proof. Let $\bar{A}$ be a $(k, m; n)$-fold fuzzy positive implicative ideal of H and let $x, y \in H$. Taking $z = 0$ in (F4) and using (p3), we have

$$\begin{aligned}\bar{A}(x) &= \inf_{a \in x \circ 0^k} \bar{A}(a) \\ &\geq \min\Big\{\inf_{b \in (x \circ y) \circ 0^m} \bar{A}(b),\ \inf_{c \in y \circ 0^n} \bar{A}(c)\Big\} \\ &= \min\Big\{\inf_{b \in x \circ y} \bar{A}(b),\ \bar{A}(y)\Big\}.\end{aligned}$$

Hence $\bar{A}$ is a fuzzy hyper BCK-ideal of H. □

The converse of Theorem 4.3 may not be true as seen in the following example.

Example 4.4. Let $H = \{0, a, b, c\}$ be a hyper BCK-algebra with the following Cayley table:

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0\}$	$\{0\}$
c	$\{c\}$	$\{c\}$	$\{b\}$	$\{0\}$

Define a fuzzy set $\bar{A}$ in H by $\bar{A}(0) = \bar{A}(a) = 0.6$ and $\bar{A}(b) = \bar{A}(c) = 0.07$. Then $\bar{A}$ is a fuzzy hyper BCK-ideal. But it is not a $(1, 2; 3)$-fold fuzzy positive implicative ideal of H, since

$$\inf_{u \in c \circ b} \bar{A}(u) = 0.07 \ngeq 0.6 = \min\Big\{\inf_{v \in (c \circ a) \circ b^2} \bar{A}(v),\ \inf_{w \in a \circ b^3} \bar{A}(w)\Big\}.$$

Example 4.5. Let $H = \{0, a, b\}$ be a hyper BCK-algebra in Example 3.3. Define a fuzzy set $\bar{A}$ in H by $\bar{A}(0) = 0.5$ and $\bar{A}(a) = \bar{A}(b) = 0.3$. Then $\bar{A}$ is a fuzzy hyper BCK-ideal, but not a $(1, m; n)$-fold fuzzy positive implicative ideal of H for natural numbers $m \geq 2$ and n, since

$$\inf_{u \in b \circ a} \bar{A}(u) = 0.3 \ngeq 0.5 = \min\Big\{\inf_{v \in (b \circ 0) \circ a^m} \bar{A}(v),\ \inf_{w \in 0 \circ a^n} \bar{A}(w)\Big\}.$$

Theorem 4.6. *If $\bar{A}$ is a $(k, m; n)$-fold fuzzy positive implicative ideal of H, then the α-cut $\bar{A}[\alpha]$ of $\bar{A}$ is an $(m, n; k)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of H, where $\alpha \in \mathrm{Im}(\bar{A})$.*

Proof. Let $\bar{A}$ be a $(k, m; n)$-fold fuzzy positive implicative ideal of H and let $\alpha \in \mathrm{Im}(\bar{A})$. Both (p1) and (F1) induce the inequality $\bar{A}(0) \geq \bar{A}(x)$ for all $x \in H$, and so $0 \in \bar{A}[\alpha]$. Let $x, y, z \in H$ be such that $(x \circ y) \circ z^m \ll \bar{A}[\alpha]$ and $y \circ z^n \subseteq \bar{A}[\alpha]$. Then for every $a \in (x \circ y) \circ z^m$, there exists $a' \in \bar{A}[\alpha]$ such that $a \ll a'$, and therefore $\bar{A}(a) \geq \bar{A}(a')$ by (F1). Hence $\bar{A}(a) \geq \alpha$ for all $a \in (x \circ y) \circ z^m$. It follows from (F4) that, for every $b \in x \circ z^k$,

$$\bar{A}(b) \geq \inf_{c \in x \circ z^k} \bar{A}(c) \geq \min\Big\{\inf_{u \in (x \circ y) \circ z^m} \bar{A}(u),\ \inf_{v \in y \circ z^n} \bar{A}(v)\Big\} \geq \alpha$$

so that $b \in \bar{A}[\alpha]$, that is, $x \circ z^k \subseteq \bar{A}[\alpha]$. Consequently, $\bar{A}[\alpha]$ is a $(m, n; k)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of H. □

We now consider the converse of Theorem 4.6.

Theorem 4.7. *Let $\bar{A}$ be a fuzzy set in H such that $\bar{A}[\alpha]$, $\alpha \in \mathrm{Im}(\bar{A})$, is an $(m, n; k)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of H. Then $\bar{A}$ is a $(k, m; n)$-fold fuzzy positive implicative ideal of H.*

Proof. Assume that $\bar{A}[\alpha]$, $\alpha \in \mathrm{Im}(\bar{A})$, is an $(m,n;k)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of H. Then $\bar{A}[\alpha]$ is a hyper BCK-ideal of H by Theorem 3.4. It follows from Proposition 2.5 that $\bar{A}$ is a fuzzy hyper BCK-ideal of H, and so the condition (F1) is valid. Now let $\alpha = \min\Big\{\inf_{u\in(x\circ y)\circ z^m} \bar{A}(u),\ \inf_{v\in y\circ z^n} \bar{A}(v)\Big\}$. Then $\bar{A}(u') \geq \inf_{u\in(x\circ y)\circ z^m} \bar{A}(u) \geq \alpha$ and $\bar{A}(v') \geq \inf_{v\in y\circ z^n} \bar{A}(v) \geq \alpha$ for all $u' \in (x\circ y)\circ z^m$ and $v' \in y\circ z^n$. Hence $u', v' \in \bar{A}[\alpha]$, which implies that $(x\circ y)\circ z^m \subseteq \bar{A}[\alpha]$, hence $(x\circ y)\circ z^m \ll \bar{A}[\alpha]$, and $y\circ z^n \subseteq \bar{A}[\alpha]$. Since $\bar{A}[\alpha]$ is a $(m,n;k)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of H, it follows from (I5) that $x\circ z^k \subseteq \bar{A}[\alpha]$ so that $\bar{A}(d) \geq \alpha$ for all $d \in x\circ z^k$. Consequently,

$$\inf_{d\in x\circ z^k} \bar{A}(d) \geq \alpha = \min\Big\{\inf_{u\in(x\circ y)\circ z^m} \bar{A}(u),\ \inf_{v\in y\circ z^n} \bar{A}(v)\Big\}.$$

Thus $\bar{A}$ is a $(k,m;n)$-fold fuzzy positive implicative ideal of H. □

Theorem 4.8. *If $\bar{A}$ is a $(k,m;n)$-fold fuzzy positive implicative ideal of H, then $\bar{A}[\alpha]$, $\alpha \in \mathrm{Im}(\bar{A})$, is an $(m,n;k)$-fold $PI(\ll, \ll, \ll)_{BCK}$-ideal of H.*

Proof. Let $\bar{A}$ be a $(k,m;n)$-fold fuzzy positive implicative ideal of H and let $\alpha \in \mathrm{Im}(\bar{A})$. Both (p1) and (F1) induce the inequality $\bar{A}(0) \geq \bar{A}(x)$ for all $x \in H$, and so $0 \in \bar{A}[\alpha]$. Let $x, y, z \in H$ be such that $(x\circ y)\circ z^m \ll \bar{A}[\alpha]$ and $y\circ z^n \ll \bar{A}[\alpha]$. Then for every $a \in (x\circ y)\circ z^m$ and $b \in y\circ z^n$, there exists $a', b' \in \bar{A}[\alpha]$ such that $a \ll a'$ and $b \ll b'$. It follows that $\bar{A}(a) \geq \bar{A}(a') \geq \alpha$ and $\bar{A}(b) \geq \bar{A}(b') \geq \alpha$ for all $a \in (x\circ y)\circ z^m$ and $b \in y\circ z^n$. Hence $\inf_{a\in(x\circ y)\circ z^m} \bar{A}(a) \geq \alpha$ and $\inf_{b\in y\circ z^n} \bar{A}(b) \geq \alpha$. Using (F4), we get for every $c \in x\circ z^k$,

$$\bar{A}(c) \geq \inf_{u\in x\circ z^k} \bar{A}(u) \geq \min\Big\{\inf_{a\in(x\circ y)\circ z^m} \bar{A}(a),\ \inf_{b\in y\circ z^n} \bar{A}(b)\Big\} \geq \alpha,$$

and thus $c \in \bar{A}[\alpha]$. This shows that $x\circ z^k \subseteq \bar{A}[\alpha]$, and thus $x\circ z^k \ll \bar{A}[\alpha]$ by (p2). Therefore $\bar{A}[\alpha]$ is an $(m,n;k)$-fold $PI(\ll, \ll, \ll)_{BCK}$-ideal of H. □

Now we consider the converse of Theorem 4.8.

Theorem 4.9. *Let $\bar{A}$ be a fuzzy set in H such that $\bar{A}[\alpha]$, $\alpha \in \mathrm{Im}(\bar{A})$, is a closed $(k,m;n)$-fold $PI(\ll, \ll, \ll)_{BCK}$-ideal of H. Then $\bar{A}$ is an $(n,k;m)$-fold fuzzy positive implicative ideal of H.*

Proof. Assume that for $\alpha \in \mathrm{Im}(\bar{A})$, $\bar{A}[\alpha]$ is a closed $(k,m;n)$-fold $PI(\ll, \ll, \ll)_{BCK}$-ideal of H. Then $\bar{A}[\alpha]$ is a hyper BCK-ideal of H (see Theorem 3.12). It follows from Proposition 2.5 that $\bar{A}$ is a fuzzy hyper BCK-ideal of H so that the condition (F1) holds. Let $x, y, z \in H$ and let

$$\beta := \min\Big\{\inf_{b\in(x\circ y)\circ z^k} \bar{A}(b),\ \inf_{c\in y\circ z^m} \bar{A}(c)\Big\}.$$

Then for each $b' \in (x\circ y)\circ z^k$ and $c' \in y\circ z^m$, we have $\bar{A}(b') \geq \inf_{b\in(x\circ y)\circ z^k} \bar{A}(b) \geq \beta$ and $\bar{A}(c') \geq \inf_{c\in y\circ z^m} \bar{A}(c) \geq \beta$. Hence $b', c' \in \bar{A}[\beta]$, and so $(x\circ y)\circ z^k \subseteq \bar{A}[\beta]$ and $y\circ z^m \subseteq \bar{A}[\beta]$. Using (p2), we get $(x\circ y)\circ z^k \ll \bar{A}[\beta]$ and $y\circ z^m \ll \bar{A}[\beta]$, and therefore $x\circ z^n \ll \bar{A}[\beta]$ by (I6). It follows from Lemma 3.6 that $x\circ z^n \subseteq \bar{A}[\beta]$. Thus $\bar{A}(d) \geq \beta$ for all $d \in x\circ z^n$, and so

$$\inf_{a\in x\circ z^n} \bar{A}(a) \geq \beta = \min\Big\{\inf_{b\in(x\circ y)\circ z^k} \bar{A}(b),\ \inf_{c\in y\circ z^m} \bar{A}(c)\Big\}.$$

Consequently, $\bar{A}$ is an $(n,k;m)$-fold fuzzy positive implicative ideal of H □

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