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SOME REMARKS ON SEVEN CONDITIONS APPROXIMATING TO MEASURABLE NORMS

Michie Maeda*, Keiko Harai and Mayumi Shibuya

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Abstract. In this paper, we treat seven conditions approximating to measurable norms.
First we have the equivalency of these conditions except two for rotationally quasi-invariant cylindrical measures. Next it is shown that Yan-Gong's condition is strictly weaker than Gross' condition.

1 Introduction

In 1962 Gross introduced the concept of measurable norms([5]), and in 1971 Dudley et al. introduced another measurability of norms([3]).
Badrikian and Chevet offered the following question([1]).
"Do these concepts of measurable norms coincide with each other for every cylindrical measure?"
The first author solved this problem negatively([8]). Also these two concepts were investigated from different directions and many results were gotten([9, 10, 11]).

On the other hand, several condition, that are similar to these conditions, are given and studied by many mathematicians([2, 4, 14]).

In this paper, we take out seven conditions approximating to measurable norms and research them with respect to some examples.
First we have the equivalency of these conditions except (iv) and (vii) in Theorem 1 for rotationally quasi-invariant cylindrical measures. For four conditions induced in them, it was proved by the first author([10]).
Moreover it is shown that Yan-Gong's condition is strictly weaker than Gross' condition, where Yan-Gong's and Gross' conditions are included in the above seven conditions.

2 Preliminaries

Let X be a locally convex Hausdorff space over the real field $\mathbb{R}$, X' its topological dual, $(\cdot,\cdot)$ the natural pairing between X and X' and $\mathscr{B}(X)$ the Borel $\sigma-$algebra of X. Let $\{\xi_1,\dots,\xi_n\}$ be a finite system of elememts of X'. Then by Ξ we denote the operator from X into $\mathbb{R}^n$ mapping x onto the vector $((x,\xi_1),\dots,(x,\xi_n))$. A set $Z\subset X$ is said to be a cylindrical set if there are $\xi_1,\dots,\xi_n\in X'$ and $B\in\mathscr{B}(\mathbf{R}^n)$ such that $Z=\Xi^{-1}(B)$. Let $\mathscr{C}_X$ denote the collection of all cylindrical sets of X.

A map μ from $\mathscr{C}_X$ into $[0,1]$ is called a cylindrical measure if it satisfies the following conditions:

(1) $\mu(X)=1$;

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*Corresponding author.

(2) Restrict μ to the $\sigma-$algebra of cylindrical sets which are generated by a fixed finite system of functionals. Then each such restriction is countably additive.

By putting $\mu_{\xi_1,\ldots,\xi_n}(B) = \mu(\Xi^{-1}(B))$ each cylindrical measure μ defines a family of Borel probability measures.

Next we interpret two kinds of measurable norms defined on a Hilbert space.
Let H be a real separable Hilbert space with norm $|\cdot| = \sqrt{<\cdot,\cdot>}$. $\mathscr{F}$ will denote the partially ordered set of finite dimensional orthogonal projections of H and $FD(H)$ the family of all finite dimensional subspaces of H. $P > Q$ means $PH \supset QH$ for $P, Q \in \mathscr{F}$. Also a subset E of H of the following form is a cylindrical set, $E = \{x \in H; Px \in F\}$, where $P \in \mathscr{F}$ and F is a Borel subset of PH.

Definition 1 *The canonical Gauss cylindrical measure is the cylindrical measure γ from $\mathscr{C}_H$ into $[0,1]$ defined as follows:*

If $E = \{x \in H; Px \in F\}$, then

$$\gamma(E) = \Big(\frac{1}{\sqrt{2\pi}}\Big)^n \int_F e^{-\frac{|x|^2}{2}}\, dx,$$

where n =dimPH and dx is the Legesgue measure on PH.

Definition 2 *A semi-norm $\|\cdot\|$ in H is called (G)measurable if for every $\varepsilon > 0$, there exists $P_0 \in \mathscr{F}$ such that $\gamma(\{x \in H; \|Px\| > \varepsilon\}) < \varepsilon$ for $\forall P \perp P_0$ and $P \in \mathscr{F}$.*

This concept was introduced by Gross in 1962([5]). It was the starting point of the successive research concerning the abstract Wiener space. In Definition 2 , we can replace γ to μ which is any cylindrical measure defined on H. Such a case we say that $\|\cdot\|$ is $\mu-(G)$measurable.

Also we can redefine the above concept as follows:

Definition 3 *We say that $\|\cdot\|$ is $\mu-(G)$measurable if for every $\varepsilon > 0$, there exists $G \in FD(H)$ such that $\mu(N_\varepsilon \cap F + F^\perp) \geq 1-\varepsilon$ whenever $F \in FD(H)$ and $F \perp G$, where $N_\varepsilon = \{x \in H; \|x\| \leq \varepsilon\}$ and $F^\perp$ is the orthogonal complement of F.*

Definition 4 *A semi-norm $\|\cdot\|$ is said to be $\mu-(D)$measurable if for every $\varepsilon > 0$ there exists $G \in FD(H)$ such that $\mu(P_F(N_\varepsilon) + F^\perp) \geq 1-\varepsilon$ whenever $F \in FD(H)$ and $F \perp G$, where P_F is the orthogonal projection of H onto F.*

This was introduced by Dudley-Feldman-LeCam in 1971([3]).
Let E be the completion of H with respect to the norm $\|\cdot\|$ and i the inclusion map of H into E. If $\|\cdot\|$ is $\gamma-(G)$measurable, then the triple (i, H, E) is called an abstract Wiener space. The norm $\|\cdot\|$ is continuous $\mu-(D)$measurable if and only if $i(\mu)$, where $i(\mu)$ is the image of μ under the map i, is countably additive.
It is easy to see that (G)measurability implies (D)measurability. But the converse is false generally([8], this is the 1984-Example). If μ is a generalized rotationally quasi-invariant cylindrical measure, then the converse is true ([11]). Of course, γ and rotationally invariant cylindrical measures and rotationally quasi-invariant cylindrical measures are genralized rotationally quasi-invariant cylindrical measures.

Definition 5 *Let μ be a cylindrical measure on H. If $\mu(C) = \mu(u(C))$ whenever C is a cylindrical set of H and u is a unitary operator of H, μ is said to be rotationally invariant.*

Definition 6 *If $\mu \sim_c u(\mu)$ whenever u is a unitary operator of H, μ is called rotationally quasi-invariant.*

Remark 1 *"$\sim_c$" means to be cylindrically equivarent, and $u(\mu)$ is the image of μ under u.*

3 Seven conditions

In this section we treat with seven conditions that are similar to measurable norms. The following theorem is all.

Theorem 1 *Let H be a real separable Hilbert space with norm $|\cdot| = \sqrt{<\cdot,\cdot>}$, μ be a cylindrical measure on H, $\|\cdot\|$ be a continuous norm defined on H, B be the completion of H with respect to $\|\cdot\|$ and i be the inclusion map from H into B. Moreover, let Y be the bidual B'' of B with weak*-topology $\sigma(B'', B')$ and j be the inclusion map from H into Y. Then the following seven conditions satisfy the relations:*
$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$, $(i) \Rightarrow (ii) \Rightarrow (v) \Leftrightarrow (vi) \Rightarrow (vii)$
If μ is continuous (this means that the characteristic function of μ is continuous on H), then the following conditions satisfy the relations:
$(iii) \Rightarrow (vi)$ and $(iv) \Rightarrow (vii)$

(i) For any $\varepsilon > 0$ there exists $N \in \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers, such that $n > m \geq N$ implies

$$\mu(\{x \in H; \|P_n x - P_m x\| > \varepsilon\}) < \varepsilon$$

for every sequence $\{P_n\} \subset \mathscr{F}$ such that P_n converges strongly to the identity map I, we write it $P_n \nearrow I$.
(ii) $\|\cdot\|$ is a $\mu-(G)$measurable norm.
(iii) There exists a sequence $\{P_n\} \subset \mathscr{F}$ such that $P_n \nearrow I$, which has the property that for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n > m \geq N$ implies

$$\mu(\{x \in H; \|P_n x - P_m x\| > \varepsilon\}) < \varepsilon.$$

(iv) There exists a sequence $\{P_n\} \subset \mathscr{F}$ such that $P_n \nearrow I$, which has the property that for any $\varepsilon > 0$ there exist $N_\varepsilon \in \mathbb{N}$ and $n_\varepsilon \in \mathbb{N}$ such that $N \geq N_\varepsilon$ and $n \geq n_\varepsilon$ implies

$$\mu(\{x \in H; \sup_{1 \leq k \leq n} \|P_k x\| > N\}) < \varepsilon.$$

(v) $\|\cdot\|$ is a $\mu-(D)$measurable norm.
(vi) $i(\mu)$ (i.e. $\mu \circ i^{-1}$) is extensible to a measure.
(vii) $j(\mu)$ is extensible to a measure.

Proof $(i) \Rightarrow (ii)$ (*see Baxendale* [2])
$(ii) \Rightarrow (iii)$ (*see Baxendale* [2])
$(iii) \Rightarrow (iv)$ (*see Harai* [6])
$(v) \Leftrightarrow (vi)$ (*see Dudley-Feldman-LeCam* [3])
$(iii) \Rightarrow (vi)$ (*see Yan* [14])
$(iv) \Rightarrow (vii)$ (*see Yan* [14] *and Gong* [4])

□

Remark 2 (1) *If B is reflexive, then* (*vi*) *and* (*vii*) *are equivalent.*

(2) *The above* (*i*), (*ii*), (*iii*), (*v*) *and* (*vi*) *conditions are equivalent for γ* ([2]).

We have the following theorem.

Theorem 2 *Let H be a real separable Hilbert space and μ be a rotationally quasi-invariant cylindrical measure. Then conditions* (*i*), (*ii*), (*iii*), (*v*) *and* (*vi*) *in Theorem 1 are equivalent.*

Proof *The first author showed that the conditions (ii), (iii), (v) and (vi) in Theorem 1 are equivalent for rotationally quasi-invariant cylindrical measures ([10]).*
Here we have to show that $(ii) \Rightarrow (i)$. Since μ is a rotationally guasi-invariant cylindrical measure, there exists a rotationally invariant cylindrical measure λ such that $\mu \sim_c \lambda$([10]). If the statement "$(ii) \Rightarrow (i)$" is satisfied for λ, then also for μ. Therefore we have to show that "$(ii) \Rightarrow (i)$" is satisfied for rotationally invariant cylindrical measures.

Here we need one lemma.

Let γ^t be the Gauss cylindrical measure with parameter $t > 0$, i.e. $\gamma^t(C) = \gamma(\frac{C}{t})$ for every $C \in \mathscr{C}_H$ and $\gamma^0 = \delta_0$. Let λ be a rotationally invariant cylindrical measure. Then we have

$$\lambda(\cdot) = \int_{[0,\infty)} \gamma^t(\cdot) d\sigma_\lambda(t)$$

by Yamasaki's theorem ([13]), *where σ_λ is a Borel probability measure on $[0,\infty)$ depending to λ. Let $\|\cdot\|$ be a norm defined on H. Obviously we can see that if $\|\cdot\|$ is $\gamma^{t_0}-(G)$measurable for some $t_0 > 0$, then $\|\cdot\|$ is $\gamma^t-(G)$measurable for all $t > 0$.*

Lemma 1 *Let λ be a rotationally invariant cylindrical measure on H and not δ_0 and $\|\cdot\|$ be a norm defined on H. If $\|\cdot\|$ is $\lambda-(G)$measurable, then $\|\cdot\|$ is $\gamma^t-(G)$measurable for all $t > 0$.*

Proof *For any $\varepsilon > 0$, there exists $P_0 \in \mathscr{F}$ such that $\lambda(\{x \in H; \|Px\| > \varepsilon\}) < \varepsilon$ for every $P \perp P_0$, $P \in \mathscr{F}$.*
Recall that

$$\lambda(\{x \in H; \|Px\| > \varepsilon\}) = \int_{[0,+\infty)} \gamma^t(\{x \in H; \|Px\| > \varepsilon\}) d\sigma_\lambda(t)$$

and

$$= \int_{[0,u)} \gamma^t(\{x \in H; \|Px\| > \varepsilon\}) d\sigma_\lambda(t) + \int_{[u,+\infty)} \gamma^t(\{x \in H; \|Px\| > \varepsilon\}) d\sigma_\lambda(t).$$

It is clear that there exists $u_0 > 0$ such that $\sigma_\lambda([u_0,+\infty)) > 0$, and denote it by α (i.e.$\alpha = \sigma_\lambda([u_0,+\infty)) > 0$).
Since $\int_{[u_0,+\infty)} \gamma^t(\{x \in H; \|Px\| > \varepsilon\}) d\sigma_\lambda(t) < \varepsilon$ and $\alpha = \sigma_\lambda([u_0,+\infty))$, there exists u' such that $u' \geq u_0$ and $\gamma^{u'}(\{x \in H; \|Px\| > \varepsilon\}) < \frac{\varepsilon}{\alpha}$.
u' depends on P, however its existence in $[u_0,+\infty)$ is sure . And so we have

$$\gamma^{u_0}(\{x \in H; \|Px\| > \varepsilon\}) = \gamma^{u'}(\{x \in H; \|Px\| > \frac{u'}{u_0}\varepsilon\}) \leq \gamma^{u'}(\{x \in H; \|Px\| > \varepsilon\}) < \frac{\varepsilon}{\alpha}.$$

Therefore it follows that for any $\varepsilon > 0$ there exists $P_0 \in \mathscr{F}$ such that

$$\gamma^{u_0}(\{x \in H; \|Px\| > \varepsilon\}) < \frac{\varepsilon}{\alpha}$$

for every $P \perp P_0$, $P \in \mathscr{F}$.
This means $\|\cdot\|$ is $\gamma^{u_0} - (G)$measurable. Then $\|\cdot\|$ is $\gamma^t - (G)$measurable for all $t > 0$.

$\square$

Here we return the proof of Theorem 2. Suppose that $\|\cdot\|$ is $\lambda-(G)$measurable, where λ is a rotationally invariant cylindrical measure and not δ_0.

It follows from Lemma 1 that $\|\cdot\|$ is $\gamma^t - (G)$measurable for all $t > 0$.
Given arbitrary $\varepsilon > 0$. Let $\{P_n\}$ be a sequence included in $\mathscr{F}$ which strongly converges to I, and u be a positive real number such that $\sigma_\lambda([u, +\infty)) < \frac{\varepsilon}{2}$, where σ_λ is the induced Borel probability measure with λ on $[0, +\infty)$ (see Yamasaki [13]). Also we check that Remark 2 is satisfied for every γ^t $(t > 0)$.

Since $\|\cdot\|$ is $\gamma^u - (G)$measurable, $\|\cdot\|$ satisfies condition (i). This induces that there exists $n_0 \in \mathbb{N}$ such that $n > m \geq n_0$ implies

$$\gamma^u(\{x \in H; \|P_n x - P_m x\| > \varepsilon\}) < \frac{1}{2}\varepsilon.$$

Therefore for every t such that $0 \leq t \leq u$, we have

$$\begin{aligned}\gamma^t(\{x \in H; \|P_n x - P_m x\| > \varepsilon\}) &= \gamma^u(\{x \in H; \|P_n x - P_m x\| > \tfrac{u}{t}\varepsilon\}) \\ &\leq \gamma^u(\{x \in H; \|P_n x - P_m x\| > \varepsilon\}) \\ &< \tfrac{1}{2}\varepsilon.\end{aligned}$$

Then we have

$$\begin{aligned}\lambda(\{x \in H; \|P_n x - P_m x\| > \varepsilon\}) &= \textstyle\int_{[0,+\infty)} \gamma^t(\{x \in H; \|P_n x - P_m x\| > \varepsilon\}) d\sigma_\lambda(t) \\ &= \textstyle\int_{[0,u)} \gamma^t(\{x \in H; \|P_n x - P_m x\| > \varepsilon\}) d\sigma_\lambda(t) \\ &\quad + \textstyle\int_{[u,+\infty)} \gamma^t(\{x \in H; \|P_n x - P_m x\| > \varepsilon\}) d\sigma_\lambda(t) \\ &< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This completes the proof of Theorem 2. $\square$

4 Some Examples and seven conditions

The 1984-Example([8]) showed that Gross' condition((ii) in Theorem 1) is strictly stronger than D.F.L.'s condition((v) in Theorem 1).

In this section we treat some examples and clarify the relation between (ii) and (iii) (or (iv)).

First we introduce several continuous norms and several cylindrical measures on ℓ^2. Let $e_n = (0, \ldots, 0, 1, 0, \ldots)$, where 1 appears in the $n-$th place. It is clear that $\{e_n\}_{n=1,2,\ldots}$

is a complete orthonormal system (CONS) on ℓ^2.

(i) Construction of $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_3$ and $\|\cdot\|_4$

Let $\{\alpha_n\}_{n=1,2,\dots}$ be the sequence of non-negative real numbers such that $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \dots$ and $\alpha_n \to \infty$ as $n \to \infty$. Denote by Γ_1 the convex hull of the set $\{\pm\alpha_n(e_1+e_2+\dots+e_n); n=1,2,\dots\}$, B_1 the open unit ball of ℓ^2 and $U_1 = \Gamma_1 + B_1$. It is obvious that U_1 is open, convex, absorbing and circled. we denote by $\|x\|_1$ gauge of U_1 at $x \in \ell^2$. $\|\cdot\|_1$ is the continuous norm defined on ℓ^2.

Let $\{\beta_n\}_{n=1,2,\dots}$ be the sequence of non-negative real numbers such that $\beta_{2m} = 0$ for $m = 1,2,\dots$, $\beta_{2m-1} > 0$ for $m = 1,2,\dots$, and the sequence $\{\beta_{2m-1}\}_{m=1,2,\dots}$ is increasing and tends to ∞ as $m \to \infty$. Let Γ_2 be the convex hull of the set $\{\pm\beta_n(e_1+e_2+\dots+e_n); n = 1,2,\dots\}$ and $U_2 = \Gamma_2 + B_1$. U_2 is also open, convex, absorbing and circled. We denote by $\|x\|_2$ the gauge of U_2 at $x \in \ell^2$. $\|\cdot\|_2$ is continuous, too.

Let $\{\eta_n\}_{n=1,2,\dots}$ be the sequence satisfying the same condition as it of $\{\beta_n\}_{n=1,2,\dots}$. Let Γ_3 be the convex hull of the set of $\{\pm\eta_n(e_1+2e_2+\dots+ne_n); n=1,2,\dots\}$, B_2 the open set $\{x=(x_n)\in\ell^2; \sqrt{\sum_{n=1}^{\infty}(\frac{x_n}{n})^2} < 1\}$ and $U_3 = \Gamma_3 + B_2$. The gauge of U_3 at $x \in \ell^2$ defines the continuous norm $\|x\|_3$.

For $x = (x_n) \in \ell^2$, we denote $\|x\|_4 = \sqrt{\sum_{n=1}^{\infty}(\frac{x_n}{n})^2}$.

(ii) Construction of $\mu_{\mathbf{a}}$ and $\mu_{\mathbf{b}}$

Let $(\ell^2)^*$ be the algebraic dual of ℓ^2, equipped with its weak topology $\sigma((\ell^2)^*, \ell^2)$, and $(\cdot,\cdot)$ be the natural pairing $(\ell^2)^* \times \ell^2 \to \mathbb{R}$. Then a cylindrical set in $(\ell^2)^*$ and in ℓ^2 can be described as

$$Z = \{x \in (\ell^2)^*; ((x,\xi_1),\dots,(x,\xi_n)) \in D\}$$

and

$$\tilde{Z} = \{x \in \ell^2; (< x,\xi_1 >,\dots,< x,\xi_n >) \in D\}$$

where $\xi_1,\dots,\xi_n \in \ell^2$ and $D \in \mathscr{B}(\mathbb{R}^n)$, respectively.
We choose an algebraic basis $\mathscr{J}$ of ℓ^2 containing $\{e_n\}_{n=1,2,\dots}$.
Define $\mathbf{a}$ and $\mathbf{b} \in (\ell^2)^*$ as follows:

$$\begin{aligned}&(\mathbf{a}, e_n) = 1 \text{ for } n = 1,2,\dots,\\ &(\mathbf{a}, e_\alpha) = 0 \text{ for } e_\alpha \in \mathscr{J}\backslash\{e_n\}_{n=1,2,\dots}\end{aligned}$$

$$\begin{aligned}&(\mathbf{b}, e_n) = n \text{ for } n = 1,2,\dots\\ &(\mathbf{b}, e_\alpha) = 0 \text{ for } e_\alpha \in \mathscr{J}\backslash\{e_n\}_{n=1,2,\dots}\end{aligned}$$

Let $\delta_{\mathbf{a}}$ and $\delta_{\mathbf{b}}$ denote the Dirac measures at the fixed point $\mathbf{a}$ and $\mathbf{b}$ in $(\ell^2)^*$ respectively.

Then the induced measures $\mu_{\mathbf{a}}$ and $\mu_{\mathbf{b}}$ on ℓ^2 are defined by

$$\mu_{\mathbf{a}}(\tilde{Z}) = \delta_{\mathbf{a}}(Z)\ ,$$

$$\mu_{\mathbf{b}}(\tilde{Z}) = \delta_{\mathbf{b}}(Z)\ , \text{ respectively.}$$

It is known that $\|\cdot\|_2$ is $\mu_{\mathbf{a}}-(D)$measurable but not $\mu_{\mathbf{a}}-(G)$measurable ([8], this is the 1984 Example).

Later, we have that $\|\cdot\|_2$ is not $\gamma-$measurable([12]). Then we construct $\|\cdot\|_3$ and $\mu_{\mathbf{b}}$.

$\|\cdot\|_3$ is $\mu_{\mathbf{b}}-(D)$measurable but not $\mu_{\mathbf{b}}-(G)$measurable ([12]), however it is $\gamma-$measurable.

$\|\cdot\|_4$ is the well known example that is $\gamma-$measurable.

The following theorem is the main result in this section.

Theorem 3 *(i) $\|\cdot\|_2$ satisfies the condition (iii) in Theorem 1 with respect to $\mu_{\mathbf{a}}$.*
(ii) $\|\cdot\|_3$ satisfies (iii) in Theorem 1 with respect to $\mu_{\mathbf{b}}$.

Remark 3 *This theorem implies that Gross' condition ((ii) in Theorem 1) is strictly stronger than Yan-Gong's condition ((iii) and (iv) in Theorem 1).*

Proof *Let P_n be the orthogonal projection from ℓ^2 onto the linear span of $\{e_1, e_2, \dots, e_{2n+1}\}$. Clearly P_n strongly converges to I.*
For $n > m$, we have

$$\mu_{\mathbf{a}} \circ (P_n - P_m)^{-1} = (P_n - P_m)(\mu_{\mathbf{a}}) = \delta_{e_{2m+2}+\ldots+e_{2n+1}}$$

Now we show that for any $\varepsilon > 0$

$$\lim_{n,m\to\infty} \mu_{\mathbf{a}}(\{x \in \ell^2; \|P_n x - P_m x\|_2 \le \varepsilon\}) = 1.$$

It is enough to prove that for any $\varepsilon > 0$ exists $n_0 \in \mathbb{N}$ such that $\|e_{2m+2} + \ldots + e_{2n+1}\|_2 \le \varepsilon$ for every $n > m \ge n_0$.
Put $k = \dfrac{1}{\frac{1}{\beta_{2m+1}} + \frac{1}{\beta_{2n+1}}}$, we have

$$\begin{aligned} & k(e_{2m+2} + \ldots + e_{2n+1}) \\ &= k(e_1 + \ldots + e_{2n+1}) - k(e_1 + \ldots + e_{2m+1}) \\ &= \tfrac{k}{\beta_{2n+1}}\{\beta_{2n+1}(e_1 + \ldots + e_{2n+1})\} + \tfrac{k}{\beta_{2m+1}}\{-\beta_{2m+1}(e_1 + \ldots + e_{2m+1})\}. \end{aligned}$$

It is sure that $\frac{k}{\beta_{2m+1}} + \frac{k}{\beta_{2n+1}} = 1$.

Resure that Γ_2 is the convex hull of $\{\pm\beta_{2j+1}(e_1 + \ldots + e_{2j+1}); j = 1, 2, \ldots\}$ and so $k(e_{2m+2} + \ldots + e_{2n+1}) \in \Gamma_2 \subset U_2$.
This means

$$\|e_{2m+2} + \ldots + e_{2n+1}\|_2 \le \frac{1}{k} = \frac{1}{\beta_{2m+1}} + \frac{1}{\beta_{2n+1}}.$$

For arbitrary $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\frac{2}{\beta_{2n_0+1}} \le \varepsilon$, so we have

$$\|e_{2m+2} + \ldots + e_{2n+1}\|_2 \le \frac{2}{\beta_{2n_0+1}} \le \varepsilon$$

This completes the proof.
The case (ii) is shown by similar methods of the above proof (see [6]). □

The following theorem shows the connection between the cylindrical measure $\mu_{\mathbf{a}}$, and the norms $\|\cdot\|_1$, $\|\cdot\|_3$ and $\|\cdot\|_4$.

Theorem 4 *The norms $\|\cdot\|_1$, $\|\cdot\|_3$ and $\|\cdot\|_4$ satisfies the condition (iii) in Theorem 1 for the cylindrical measure $\mu_{\mathbf{a}}$.*

Proof *First we show the case of $\|\cdot\|_1$. Let P_n be the orthogonal projection from ℓ^2 onto the linear span of the $\{e_1, e_2, \ldots, e_n\}$. P_n converges strongly to I as $n \to \infty$.*
For $n > m$, $(P_n - P_m)$ is the orthogonal projection onto the linear span of $\{e_{m+1}, \ldots, e_n\}$.
Put $k = \dfrac{1}{\frac{1}{\alpha_m} + \frac{1}{\alpha_n}}$.
After is similar to the proof of Theorem 3.
Second we consider the case of $\|\cdot\|_4$.
Let P_n be the same as above.
For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sum_{j=m+1}^{n} \left(\tfrac{1}{j}\right)^2 \leq \varepsilon^2 \text{ for every } n > m \geq N.$$

Since $(P_n - P_m)\mu_{\mathbf{a}} = \delta_{e_{m+1}+\ldots+e_n}$ and $\|e_{m+1} + \ldots + e_n\|_4 = \sqrt{\sum_{j=m+1}^{n} \left(\frac{1}{j}\right)^2} \leq \varepsilon$, we have

$$\mu_{\mathbf{a}}(\{x \in \ell^2; \|P_n x - P_m x\|_4 \leq \varepsilon\}) = \delta_{e_{m+1}+\ldots+e_n}(\{y \in (P_n - P_m)\ell^2; \|y\|_4 \leq \varepsilon\}) = 1$$

This means that $\|\cdot\|_4$ satisfies the condition (iii) for $\mu_{\mathbf{a}}$.
Now check that $\|x\|_3 \leq \|x\|_4$ for all $x \in \ell^2$. Then $\|\cdot\|_3$ satisfies (iii) for $\mu_{\mathbf{a}}$, too. □

We close this section by the next Theorem.

Theorem 5 *$\|\cdot\|_4$ is not $\mu_{\mathbf{b}} - (D)$measurable.*

Proof *It is sufficient to show that there exists a positive number ε_0 satisfying that any $G \in FD(\ell^2)$ there exists $F \in FD(\ell^2)$ such that $F \perp G$ and $\mu_{\mathbf{b}}(P_F(N_{\varepsilon_0}) + F^{\perp}) < 1 - \varepsilon_0$, where $N_{\varepsilon_0} = \{x \in \ell^2; \|x\|_4 \leq \varepsilon_0\}$.*
Let $\varepsilon_0 = \frac{1}{2}$ be given.
Let G be an arbitrary finite dimensional subspace of ℓ^2, and $\{\xi^j\}_{j=1,2,\ldots,n}$ be a CONS of G. Then each $\{\xi^j\}$ is of the form $\xi^j = \sum_{i=1}^{\infty} \alpha_i^j e_i$ where $\alpha_i^j \in \mathbb{R}$ for $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots$.
Then we have the following matrix A :

$$A = \begin{pmatrix} \alpha_1^1 & \ldots & \alpha_n^1 & \ldots & \alpha_{n+m}^1 \\ \vdots & \ldots & \vdots & \ldots & \vdots \\ \alpha_1^n & \ldots & \alpha_n^n & \ldots & \alpha_{n+m}^n \end{pmatrix}$$

where m is chosen such that rank $A = n$. Suppose $N > n + m$. Then the next equation has its solution in $\mathbb{R}^{n+m}$.

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ \vdots \\ x_{n+m} \end{pmatrix} = \begin{pmatrix} -\alpha_N^1 \\ \vdots \\ -\alpha_N^n \end{pmatrix} \quad \ldots (*)$$

By construction we know that $\alpha_i^j \to 0$ as $i \to \infty$ for $j = 1, 2, \ldots, n$. Therefore, for every $\delta > 0$, we may choose a positive integer $N(> n+m)$, N sufficiently large, such that the equation $(*)$ has the solution $x_1 = \eta_1, \ldots, x_{n+m} = \eta_{n+m}$ satisfying

$$\max_{1 \le l \le n+m} |\eta_l| < \delta. \ldots (**)$$

Let $\tau = \eta_1 e_1 + \ldots + \eta_{n+m} e_{n+m} + e_N$ and F be the one dimensional subspace of ℓ^2 generated by τ. Then $F \perp G$.
Upon putting $\phi = \frac{\tau}{|\tau|}$, where $|\cdot|$ is the original norm of ℓ^2, we obtain

$$(b, \phi) = \frac{(b,\tau)}{|\tau|} = (\eta_1 + 2\eta_2 + \ldots + (n+m)\eta_{n+m} + N)/|\tau|$$

We have to show that $(b,\phi)\phi \notin P_F(N_{\varepsilon_0})$. Suppose that $(b,\phi)\phi \in P_F(N_{\varepsilon_0})$.
Let $\delta < \frac{1}{n+m}$.
If $(b,\phi)\phi \in P_F(N_{\varepsilon_0})$, then there exists $x_0 \in N_{\varepsilon_0}$ such that $P_F x_0 = (b,\phi)\phi$.
Now for any $x \in N_{\varepsilon_0}$, we have

$$\begin{aligned}
&|\eta_1|| < x, e_1 > | < \delta\varepsilon_0, \\
&|\eta_2|| < x, e_2 > | < 2\delta\varepsilon_0, \\
&\qquad\vdots \\
&|\eta_{n+m}|| < x, e_{n+m} > | < (n+m)\delta\varepsilon_0 \text{ and} \\
&| < x, e_N > | < N\varepsilon_0.
\end{aligned}$$

Then

$$\begin{aligned}
&| < \tau, x > | \\
&= |\eta_1 < x, e_1 > + \ldots + \eta_{n+m} < x, e_{n+m} > + < x, e_N > | \\
&\le |\eta_1|| < x, e_1 > | + \ldots + |\eta_{n+m}|| < x, e_{n+m} > | + | < x, e_N > | \\
&< \varepsilon_0 + \varepsilon_0 N \\
&= (1+N)\varepsilon_0 \\
&= \tfrac{1}{2}(1+N)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&| < \tau, x > | \\
&= |\eta_1 + 2\eta_2 + \ldots + (n+m)\eta_{n+m} + N| \\
&\ge N - |\eta_1| - 2|\eta_2| - \ldots - (n+m)|\eta_{n+m}| \\
&> N - \{1 + 2 + \ldots + (n+m)\}\delta \\
&> N - \frac{1}{2} \cdot \frac{n+m+1}{n+m} \\
&> N - 1
\end{aligned}$$

This means contradiction. □

Appendix

By the above results and [8] and [12], we have the following table.

The sign ◯ means the norm satisfies the condition with respect to the cylindrical measure, the sign × means it does not satisfy and the blank space means indefiniteness.

norm	**cylindrical measure**	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)
$\|\cdot\|_1$	γ							
	$\mu_{\mathbf{a}}$			$\bigcirc$	$\bigcirc$			
$\|\cdot\|_2$	γ	$\times$	$\times$	$\times$		$\times$	$\times$	
	$\mu_{\mathbf{a}}$	$\times$	$\times$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$
$\|\cdot\|_3$	γ	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$
	$\mu_{\mathbf{a}}$			$\bigcirc$	$\bigcirc$			
	$\mu_{\mathbf{b}}$	$\times$	$\times$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$
$\|\cdot\|_4$	γ	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$
	$\mu_{\mathbf{a}}$			$\bigcirc$	$\bigcirc$			
	$\mu_{\mathbf{b}}$	$\times$	$\times$	$\times$		$\times$	$\times$	

Most blanks except a few ones will be filled by $\bigcirc$ or $\times$ in the near future. It will appear elsewhere.

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Department of Mathematics, Faculty of Science, Ochanomizu University 2-1-1, Otsuka, Bunkyo-ku, Tokyo112-8610, Japan

E-mail address : maedamic@cc.ocha.ac.jp