LORENTZ MULTIPLIERS FOR HANKEL TRANSFORMS

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Abstract. Let ϕ be a function on $(0,\infty)$ continuous except on a null set, and $\phi_\epsilon(\xi) = \phi(\epsilon\xi)$ $(\epsilon > 0)$. Also $\tilde{T}_\epsilon$ be the operator on Jacobi series such that $(\tilde{T}_\epsilon f)^\wedge(n) = \phi_\epsilon(n)\hat{f}(n)$ $(n \in \mathbf{Z})$, where $\hat{f}(n)$ is the coefficient of Jacobi expanstion of f, and $\mathcal{H}_\alpha(Tf)(\xi) = \phi(\xi)\mathcal{H}_\alpha f(\xi)$ $(\xi \in (0,\infty))$, where $\mathcal{H}_\alpha f$ is the modified Hankel transform of f with order α. Then Igari [4] proved that if the operator norm of $\tilde{T}_\epsilon$ is uniformly bounded for all $\epsilon > 0$, T is an operator on Hankel transforms(the details in §1, §2). After that, Connett-Schwartz[2] and Kanjin[5] proved the weak version and the maximal version by using [4], respectively. In this paper, we prove the analogy of Igari[4] in the Lorentz space, in the same way. Also in §3, as an application of this result, we show a result with respect to the partial sum operator of the Jacobi series.

1. Introduction

Let (X,ν) be a measure space, and for any $1 \le p < \infty, 1 \le q \le \infty$, $L^{p,q}(X)$ define the Lorentz space such that

$$L^{p,q}(X) = \{f : f \text{ } is \text{ } measurable, \ \| f \|^*_{p,q} < \infty\},$$

where

$$\| f \|^*_{p,q} = \begin{cases} \{q\int_0^\infty (t\nu(\{| f | > t\})^{1/p})^q \frac{dt}{t}\}^{1/q} & (1 \le q < \infty) \\ \sup_{t>0} t\nu(\{| f | > t\})^{1/p} & (q = \infty). \end{cases}$$

In particular, $L^{p,q}(X) = L^p(X)$ for $p = q$.

Now let $P_n^{(\alpha,\beta)}(x)$ denote the Jacobi polynomial of degree n and order (α,β), $\alpha,\beta > -1$ defined by

$$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}\{(1-x)^{n+\alpha}(1+x)^{n+\beta}\}.$$

The functions $\{P_n^{(\alpha,\beta)}(\cos\theta)\}_{n=0}^\infty$ are orthogonal on $(0,\pi)$ with respect to the measure $d\mu(\theta) = (\sin\frac{\theta}{2})^{2\alpha+1}(\cos\frac{\theta}{2})^{2\beta+1}d\theta$. For a functon $f(\theta)$ integrable on $(0,\pi)$ with respect to $d\mu$, define

$$\hat{f}(n) = \int_0^\pi f(\theta)P_n^{(\alpha,\beta)}(\cos\theta)(\sin\frac{\theta}{2})^{2\alpha+1}(\cos\frac{\theta}{2})^{2\beta+1}d\theta.$$

Put

$$\frac{1}{h_n^{(\alpha,\beta)}} = \int_0^\pi [P_n^{(\alpha,\beta)}(\cos\theta)]^2(\sin\frac{\theta}{2})^{2\alpha+1}(\cos\frac{\theta}{2})^{2\beta+1}d\theta.$$

Then $\{\sqrt{h_n^{(\alpha,\beta)}}P_n^{(\alpha,\beta)}(\cos\theta)\}_{n=0}^\infty$ is a complete orthonormal system in $L^2((0,\pi),\mu)$. For $(X,\nu) = ((0,\pi),\mu)$, we denote the Lorentz norm of $g \in L^{p,q}(0,\pi)$ by $\| g \|^J_{p,q}$. For any

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$\phi \in \ell^\infty (= \ell^\infty(\{0,1,2,.......\}))$, we define a transformation $\tilde{T}_\phi$ by

$$\tilde{T}_\phi g(\theta) = \sum_{n=0}^{\infty} \phi(n)\hat{g}(n)h_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta),$$

and the opetator norm from $L^{p,r}(0,\pi)$ into $L^{p,q}(0,\pi)$ by

$$\| \tilde{T}_\phi \|^J_{M(p,r;p,q)} = \sup\{\| \tilde{T}_\phi g \|^J_{p,q} : \| g \|^J_{p,r} \le 1, g \in C_c^\infty(0,\pi)\},$$

and $M^J(p,r;p,q) = \{\tilde{T}_\phi : \| \tilde{T}_\phi \|^J_{M(p,r;p,q)} < \infty\}$. For $\alpha > -1$, $(X,\nu) = ((0,\infty), d\eta(x) = x^{2\alpha+1}dx)$, and a function f on $(0,\infty)$, we denote the Lorentz norm of $f \in L^{p,q}(0,\infty)$ by $\| f \|^H_{p,q}$. Also the modified Hankel transform of order α is defined by

$$\mathcal{H}_\alpha f(x) = \int_0^\infty f(y)\frac{J_\alpha(xy)}{(xy)^\alpha}d\eta(y),$$

where J_α is the Bessel function of the first kind. Also the multiplier transformation associated with $\phi \in L^\infty(0,\infty)$ is defined formally by

$$T_\phi f(x) = \int_0^\infty \phi(y)\mathcal{H}_\alpha f(y)\frac{J_\alpha(xy)}{(xy)^\alpha}d\eta(y),$$

the operator norm of T_ϕ from $L^{p,r}(0,\infty)$ into $L^{p,q}(0,\infty)$ by

$$\| T_\phi \|^H_{M(p,r;p,q)} = \sup\{\| T_\phi f \|^H_{p,q} : \| f \|^H_{p,r} \le 1, f \in C_c^\infty(0,\infty)\},$$

and $M^H(p,r;p,q) = \{T_\phi : \| T_\phi \|^H_{M(p,r;p,q)} < \infty\}$. For $\epsilon > 0$ and $\phi \in L^\infty(0,\infty)$, let

$$T_\epsilon f(x) = \int_0^\infty \phi(\epsilon y)\mathcal{H}_\alpha f(y)\frac{J_\alpha(xy)}{(xy)^\alpha}d\eta(y),$$

$$T^* f(x) = \sup_{\epsilon>0} | T_\epsilon f(x) |,$$

$$\tilde{T}_\epsilon g(\theta) = \sum_{n=0}^{\infty} \phi(\epsilon n)\hat{g}(n)h_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta),$$

and

$$\tilde{T^*} g = \sup_{\epsilon>0} | \tilde{T}_\epsilon g(x) |,$$

where $f \in C_c^\infty(0,\infty)$ and $g \in C_c^\infty(0,\pi)$. Igari[4] showed the following:

Theorem A Let $1 \le p < \infty$ and $\alpha,\ \beta > -1$. Assume that ϕ is a function on $(0,\infty)$ continuous except on a null set and $\liminf_{\epsilon\to+0}$
$\| \tilde{T}_\epsilon \|^J_{M(p,p;p,p)}$ is finite, then $\| T \|^H_{M(p,p;p,p)} \le \liminf_{\epsilon\to+0} \| \tilde{T}_\epsilon \|^J_{M(p,p;p,p)}$.

After that, Connett-Schwartz[2] showed the analogy of weak type:

Theorem B Let $1 \le p < \infty$ and $\alpha,\ \beta > -1$. Assume that ϕ is a function on $(0,\infty)$ continuous except on a null set and $\liminf_{\epsilon\to+0}$

$\| \tilde{T}_\epsilon \|^J_{M(p,p;p,\infty)}$ is finite, then $\| T \|^H_{M(p,p;p,\infty)} \leq \liminf_{\epsilon \to +0} \| \tilde{T}_\epsilon \|^J_{M(p,p;p,\infty)}$.

Also Kanjin[5] showed the analogy of maximal type:

Theorem C Let $1 < p < \infty$ and $\alpha,\ \beta > -1$. Assume that ϕ is a function on $(0, \infty)$ continuous except on a null set and $\| \tilde{T}^* \|^J_{M(p,p;p,p)}$ is finite, then $\| T^* \|^H_{M(p,p;p,p)} < \infty$.

In §2, we show Theorem 1 that is the analogy of Theorem A and Theorem B on the Lorentz space. Also we prove Theorem 2 that is a generalization of Theorem C by the application of [6]. Also in §3, we show an application of Theorem 1 with respect to the partial sum operator S_N on $L^{p,q}(0,\pi)$:

for $\alpha > -\frac{1}{2},\ 1 < r < \infty$ and the partial sum operators $S_N (N = 1, 2, ...)$,

$$S_N : L^{\frac{4\alpha+4}{2\alpha+3},r}(0,\pi) \longrightarrow L^{\frac{4\alpha+4}{2\alpha+3},\infty}(0,\pi)$$

are unbounded.

Throughout this paper, for $s > 0$, we denote s' the conjugate exponent of s i.e. $1/s + 1/s' = 1$, and the letter C a positive constant that may vary from line to line.

2. Results

First we show the following:

Theorem 1 Let $1 < p < \infty,\ 1 \leq q \leq \infty,\ 1 \leq r < \infty$ and $\alpha,\ \beta > -1$. Assume that ϕ is a function on $(0, \infty)$, continuous except on a null set and $\sup_{\epsilon > 0} \| \tilde{T}_\epsilon \|^J_{M(p,r;p,q)}$ is finite, then $T \in M^H(p, r; p, q)$.

Proof. Let $M > 0,\ f \in C_c^\infty(0, \infty)$ and $f_\epsilon(\theta) = f(\theta/\epsilon)$. Also let ϵ be a positive number such that $\pi/\epsilon > M$ and N a positive integer. We define

$$G(\tau, 1/\epsilon) = \sum_{n=0}^{\infty} \phi(\epsilon n) \hat{f}_\epsilon(n) h_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos \epsilon\tau)\ (= \tilde{T}_\epsilon f_\epsilon),$$

$$G^N(\tau, 1/\epsilon) = \sum_{n=0}^{N[1/\epsilon]} \phi(\epsilon n) \hat{f}_\epsilon(n) h_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos \epsilon\tau),$$

$$H^N(\tau, 1/\epsilon) = G(\tau, 1/\epsilon) - G^N(\tau, 1/\epsilon),$$

and

$$G(\tau) = \int_0^\infty \phi(y) \mathcal{H}_\alpha f(y) \frac{J_\alpha(\tau y)}{(\tau y)^\alpha} d\eta(y)\ (= Tf(\tau)).$$

Also let $K > 0$ and $h \in C_c^\infty(0, K)$ be fixed. Then we obtain

$$\int G^N(\tau, 1/\epsilon) h(\tau) d\eta(\tau) = \int G(\tau, 1/\epsilon) h(\tau) d\eta(\tau) - \int H^N(\tau, 1/\epsilon) h(\tau) d\eta(\tau),$$

and

$$| \int G^N(\tau, 1/\epsilon) h(\tau) d\eta(\tau) |$$

$$\leq C \| \chi_{(0,K)} G(\tau, 1/\epsilon) \|^H_{p,q} \| h \|^H_{p',q'} + \| \chi_{(0,K)} H^N(\tau, 1/\epsilon) \|_{L^2(\eta)} \| f \|_{L^2(\eta)},$$

where $\chi_{(0,K)}$ is the characteristic function on $(0,K)$. Here, we estimate $\| \chi_{(0,K)}G(\tau,1/\epsilon) \|^H_{p,q}$. Let $0<\delta<1$ be fixed. Then for there exists $\epsilon_0>0$ with $\pi/\epsilon_0>K$ such that for any $0<\epsilon<\epsilon_0$ we have

$$\| \chi_{(0,K)}G(\tau,1/\epsilon) \|^H_{p,q} \leq \epsilon^{-(2\alpha+2)/p}2^{(2\alpha+1)/p}(1+\delta)^{1/p} \| G(\theta/\epsilon,1/\epsilon) \|^J_{p,q}$$

by the change of variables and the definition of the Lorentz space, and we obtain that for $0<\epsilon<\epsilon_0$

$$\| \chi_{(0,K)}G(\tau,1/\epsilon) \|^H_{p,q}$$
$$\leq \epsilon^{-(2\alpha+2)/p}2^{(2\alpha+1)/p}(1+\delta)^{1/p}(\sup_{\epsilon>0} \| \tilde{T}_\epsilon \|^J_{M(p,r;p,q)}) \| f_\epsilon \|^J_{p,r}$$

by the assumption and $G(\theta/\epsilon,1/\epsilon)=\tilde{T}_\epsilon f_\epsilon(\theta)$. By the change of variables, we can show that in the case of $r<\infty$

$$\epsilon^{-(2\alpha+2)/p}2^{(2\alpha+1)/p} \| f_\epsilon \|^J_{p,r}$$
$$=\epsilon^{-(2\alpha+2)/p}2^{(2\alpha+1)/p}\times$$
$$(r\int_0^\infty (t\mu(\{\theta\leq\pi : | f(\theta/\epsilon) |>t\})^{1/p})^r\frac{dt}{t})^{1/r}$$
$$=(r\int_0^\infty (t(\int_{\{0\leq\tau\leq M:|g(\tau)|>t\}}(\frac{\sin(\epsilon\tau/2)}{\epsilon\tau/2})^{2\alpha+1}\cos(\epsilon\tau/2)^{2\beta+1}d\eta(\tau))^{1/p})^r\frac{dt}{t})^{1/r},$$

and there exists $\epsilon_1>0$ with $\epsilon_1<\epsilon_0$ such that

$$\epsilon^{-(2\alpha+2)/p}2^{(2\alpha+1)/p} \| f_\epsilon \|^J_{p,r} \leq C \| f \|^J_{p,r} \ @(0<\epsilon<\epsilon_1)$$

for some $C>0$ by the dominated convergence theorem with $\| f \|_\infty<\infty$. In the case of $r=\infty$, we can show , similarly. Therefore, for any $\epsilon_1>\epsilon>0$, we have

$$| \int_0^K G^N(\tau,1/\epsilon)h(\tau)d\eta(\tau) |$$
$$\leq C(1+\delta)^{2/p} \| f \|^H_{p,r} (\sup_{\epsilon>0} \| \tilde{T}_\epsilon \|^J_{M(p,r;p,q)}) \| h \|^H_{p',q'} + \| \chi_{(0,K)}H^N(\tau,1/\epsilon) \|_{L^2(\eta)} \| h \|_{L^2(\eta)} .$$

Here, we remark $| G^N(\tau,1/\epsilon) | \leq C@(\epsilon>0)$ by the estimates of $G^N(\tau,1/\epsilon)$ (cf.[4;p.205]). Then, $G^N(\tau,1/\epsilon)\longrightarrow G^N(\tau)$ $(\epsilon\to 0)$ weakly and pointwisely for some $G^N(\tau)$ by [4]. After all, we get

$$| \int_0^K G^N(\tau)h(\tau)d\eta(\tau) |$$
$$\leq C(1+\delta)^{2/p} \| f \|^H_{p,r} (\sup_{\epsilon>0} \| \tilde{T}_\epsilon \|^J_{M(p,r;p,q)}) \| h \|^H_{p',q'} + \frac{B}{N^2} \| h \|_{L^2(\eta)},$$

where B is a constant independent on ϵ and N. Since it is shown that
$\| G^N - G \|_{L^2((0,K),\eta)}\longrightarrow 0$ as $N\to\infty$ by [4], we obtain that

$$| \int G(\tau)h(\tau)d\eta(\tau) |$$
$$\leq C(1+\delta)^{1/p} \| f \|^H_{p,r} (\sup_{\epsilon>0} \| \tilde{T}_\epsilon \|^J_{M(p,r;p,q)}) \| h \|^H_{p',q'} .$$

Therefore, we have that

$$\| Tf \|_{p,q}^{H} \leq C(\sup_{\epsilon>0} \| \tilde{T}_{\epsilon} \|_{M(p,r;p,q)}^{J}) \| f \|_{p,r}^{H}$$

and

$$\| T \|_{M(p,r;p,q)}^{H} \leq C \sup_{\epsilon>0} \| \tilde{T}_{\epsilon} \|_{M(p,r;p,q)}^{J} .$$

q.e.d.

Next we show a generalization of Theorem C.

Theorem 2 Let $1 < p < \infty$, $1 < q \leq \infty$, $1 \leq r < \infty$ and α, $\beta > -1$. Assume that ϕ is a bounded continuous function on $(0,\infty)$and $\| \tilde{T}^{*} \|_{M(p,r;p,q)}^{J}$ is finite, then

$$\| T^{*} \|_{M(p,r;p,q)}^{H} < \infty.$$

To prove this statement, we show the following Lemma:

Lemma(cf.[6])

(1) We have $\| \tilde{T}^{*} \|_{M(p,r;p,q)}^{J} < \infty$, if and only if, there exists a constant C such that for any positive integer N,

$$\| \sum_{j=1}^{N} \tilde{T}_{\epsilon_j} g_j \|_{p',r'}^{J} \leq C \| \sum_{j=1}^{N} | g_j | \|_{p',q'}^{J}$$

for all $\epsilon_j > 0$ and $g_j \in C_c^{\infty}(0,\pi)$ $(j = 1, 2, ..., N)$.

(2) We have $\| T^{*} \|_{M(p,r;p,q)}^{H} < \infty$, if and only if, there exists a constant C such that for any positive integer N,

$$\| \sum_{j=1}^{N} T_{\epsilon_j} f_j \|_{p',r'}^{H} \leq C \| \sum_{j=1}^{N} | f_j | \|_{p',q'}^{H}$$

for all $\epsilon_j > 0$ and $f_j \in C_c^{\infty}(0,\infty)$@$(j = 1, 2, ..., N)$.

Proof. (1)By Hunt[3], for any $g_j \in C_c^{\infty}(0,\pi)$ $(j = 1, ..., N)$, we may assume that

$$\| \sum_{j=1}^{N} \tilde{T}_{\epsilon_j} g_j \|_{p',r'}^{J}$$

$$= \sup\{ | \int \sum_{j=1}^{N} \tilde{T}_{\epsilon_j} g_j h d\mu | : \| h \|_{p,r}^{J} \leq 1,\ h \in C_c^{\infty}(0,\pi)\}.$$

Then we have that for any $h \in C_c^{\infty}(0,\pi)$

$$\int \sum_{j} \tilde{T}_{\epsilon_j} g_j h d\mu = \int \sum_{j} \tilde{T}_{\epsilon_j} h g_j d\mu,$$

and

$$| \int \sum_j \tilde{T}_{\epsilon_j} g_j h d\mu | \leq \int \sum_j (\tilde{T}^* h)(\sum_j | g_j |) d\mu.$$

By the assumption, we obtain

$$\| \sum_{j=1}^{N} \tilde{T}_{\epsilon_j} g_j \|_{p',r'}^{J} \leq C \| \tilde{T}^* \|_{M(p,r;p,q)}^{J} \| \sum_j | g_j | \|_{p',r'}^{J} .$$

Next we show the inverse. For any $g \in C_c^\infty(0,\pi)$, we can show

$$\tilde{T}^* g = \sup_{\epsilon>0} | \tilde{T}_\epsilon g | = \sup_{\epsilon_j>0} | \tilde{T}_{\epsilon_j} g |$$

for some $\{\epsilon_j\}_{j=1}^\infty$. In fact, by the definition of $\tilde{T}_\epsilon f$, the estimates of $h_n^{(\alpha,\beta)}$ and $\| P_n^{(\alpha,\beta)} \|_\infty$(cf.[7]), and the assumption of ϕ, $F(\epsilon,\theta) = \tilde{T}_\epsilon g(\theta)$ is continuous on $(0,\infty)\times(0,\pi)$. On the other hand, for any $\epsilon_0 > 0$, by the duality[3], we may assume

$$\| \max_{1\leq j\leq N} | \tilde{T}_{\epsilon_j} g | \|_{p,q}^{J} - \epsilon_0 \leq \int \max_{1\leq j\leq N} | \tilde{T}_{\epsilon_j} g | h d\mu$$

for some $h \geq 0$ with $\| h \|_{p',r'}^J \leq 1$. Also let $0 < \epsilon < \epsilon_0$ be fixed and $1 \leq j \leq N$. We define $E_j(\epsilon) = \{\max_{1\leq k\leq N} | \tilde{T}_{\epsilon_k} g | - \epsilon < | \tilde{T}_{\epsilon_j} g |\}$, $F_j(\epsilon) = E_j(\epsilon) - \cup_{k=1}^{j-1} E_k(\epsilon)$, $E_0 = \phi$, and $h_j = \chi_{F_j(\epsilon)} h \ sgn(\tilde{T}_{\epsilon_j} g)$. Then we obtain

$$\begin{aligned}
\sum_j \tilde{T}_{\epsilon_j} g h_j &= \sum_j | \tilde{T}_{\epsilon_j} g | \chi_{F_j(\epsilon)} h \\
&\geq \sum_j (\max_{1\leq j\leq N} | \tilde{T}_{\epsilon_j} g | - \epsilon) h \chi_{F_j(\epsilon)} \\
&= (\max_{1\leq j\leq N} | \tilde{T}_{\epsilon_j} g | - \epsilon) h \sum_j \chi_{F_j(\epsilon)},
\end{aligned}$$

and

$$| \int \sum_j \tilde{T}_{\epsilon_j} g h_j d\mu | \geq \int (\max_{1\leq j\leq N} | \tilde{T}_{\epsilon_j} h | - \epsilon_0) h d\mu.$$

Then we may assume $h_j \in C_c^\infty(0,\pi)$ $(j = 1, ..., N)$, since $C_c^\infty(0,\pi)$ is dense in $L^{p',q'}((0,\pi), d\mu)$. Therefore, we get that

$$\begin{aligned}
\int \max_{1\leq j\leq N} | \tilde{T}_{\epsilon_j} g | h d\mu &\leq C \| g \|_{p,r}^{J} \| h \|_{p',q'}^{J} \\
&\leq C \| \sum_j \tilde{T}_{\epsilon_j} h_j \|_{p',r'}^{J} \| g \|_{p,r}^{J},
\end{aligned}$$

and

$$\int \max_{1\leq j\leq N} | \tilde{T}_{\epsilon_j} g | h d\mu \leq C \| g \|_{p,r}^{J} \| \sum | h_j | \|_{p',q'}^{J} + \epsilon_0 | \int h d\mu | .$$

Hence, it is shown that

$$\int \max_{1\leq j\leq N} | \tilde{T}_{\epsilon_j} g | h d\mu \leq C \| g \|_{p,q}^{J} \| h \|_{p',q'}^{J}$$

by $\sum_j | h_j | \leq | h |$, and

$$\| \max_{1\leq j\leq N} | \tilde{T}_{\epsilon_j} g | \|^J_{p,q} \leq C \| g \|^J_{p,r} .$$

So by the usual method, we get

$$\| \tilde{T}^* g \|^J_{p,q} \leq C \| g \|^J_{p,r} .$$

(2): By the duality, we may assume

$$\| \sum_{j=1}^N T_{\epsilon_j} f_j \|^H_{p',q'}$$

$$= \sup\{| \int \sum_{j=1}^N (T_{\epsilon_j} f_j) h d\eta |: \| h \|^H_{p,r} \leq 1,\ h \in C_c^\infty(0,\infty)\}.$$

Here, we have

$$\int \sum_{j=1}^N (T_{\epsilon_j} f_j) h d\eta = \int \sum_{j=1}^N (T_{\epsilon_j} h) f_j d\eta$$

by $\mathcal{H}_\alpha f_j,\ \mathcal{H}_\alpha h \in L^1(\eta)$ and the definition of T_{ϵ_j}. Then we get that by the assumption

$$| \int \sum_j (T_{\epsilon_j} f_j) d\eta | \leq \int (\sup | T_{\epsilon_j} h |) \sum | f_j | d\eta$$

$$\leq C \| T^* h \|^H_{p,q} \| \sum | f_j | \|^H_{p',q'},$$

and

$$\| \sum_{j=1}^N T_{\epsilon_j} f_j \|^H_{p',q'} \leq C \| T^* h \|^H_{p,q} \| \sum | f_j | \|^H_{p',q'} .$$

In the inverse case, as we remember $\phi \in C(0,\infty) \cap L^\infty(0,\infty)$ and $f \in C_c^\infty(0,\infty)$, we can show the result as same as the proof of (1). We omit the details.
q.e.d.

The proof of Theorem 2.
Let L be any positive integer, $\{f_j\}_{j=1}^L \subset C_c^\infty(0,\infty)$ with *supp* $f_j \subset (0,M)$ for some $M>0$, and $f_{j,\epsilon}(\theta) = f_j(\theta/\epsilon)$ for $\epsilon > 0$. Also let ϵ_0 be a positive number such that $\pi/\epsilon_0 > M$. For $0<\epsilon<\epsilon_0$ and a positive integer N, we define

$$G_j(\tau, 1/\epsilon) = \sum_{n=0}^\infty \phi(\epsilon_j \epsilon n) \hat{f}_{j,\epsilon}(n) h_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos \epsilon\tau),$$

$$G_j^N(\tau, 1/\epsilon) \sum_{n=0}^{N[1/\epsilon]} \phi(\epsilon_j \epsilon n) \hat{f}_{j,\epsilon}(n) h_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos \epsilon\tau),$$

and

$$H_j^N(\tau, 1/\epsilon) = G_j(\tau, 1/\epsilon) - G_j^N(\tau, 1/\epsilon)\ (j = 1, ..., L),$$

where $\{\epsilon_j\}_{j=1}^{\infty}$ is dense in $(0,\infty)$. By the application of Lemma, we shall show $\| T^* \|_{M(p,r;p,q)}^{H} < \infty$ in the same manner of the proof of Theorem 1. Let $0 < K < \pi/\epsilon_0$ be fixed, and $h \in C_c^{\infty}(0,\infty)$ with *supp* $h \subset (0,K)$. By the definition of $G_j^N(\tau, 1/\epsilon)$, we have

$$\int \sum G_j^N(\tau,1/\epsilon)h(\tau)d\eta(\tau)$$

$$= \sum \int G_j(\tau,1/\epsilon)h(\tau)d\eta(\tau) - \sum \int H_j^N(\tau,1/\epsilon)h(\tau)d\eta(\tau),$$

and

$$| \sum \int_0^K G_j^N(\tau,1/\epsilon)h(\tau)d\eta(\tau) |$$

$$\leq C \| \chi_{(0,K)} \sum_j G_j(\tau,1/\epsilon) \|_{p',r'}^{H} \| h \|_{p,r}^{H} + \| \sum_j H_j^N(\tau,1/\epsilon) \|_{L^2((0,K),\eta)} \| h \|_{L^2(\eta)} .$$

Then in the similar way to Theorem 1, we get that for any $0 < \delta < 1$ and sufficiently small $\epsilon > 0$

$$\| \chi_{(0,K)} \sum_j G_j(\tau,1/\epsilon) \|_{p',r'}^{H} \leq (1+\delta)^{1/p'} \| \sum \tilde{T}_{\epsilon_j \epsilon} f_{j,\epsilon} \|_{p',r'}^{J},$$

and by Lemma (1) and the definition of $\tilde{T}^*$

$$\| \chi_{(0.K)} \sum_j G_j(\tau,1/\epsilon) \|_{p',q'}^{H}$$

$$\leq C(1+\delta)^{1/p'} 2^{(2\alpha+1)/p'} \epsilon^{-(2\alpha+2)/p'} \| \tilde{T}^* \|_{M(p,r;p,q)}^{J} \| \sum f_{j,\epsilon} \|_{p',q'}^{J}$$

for sufficiently small $\epsilon > 0$. After all, we have in the same way of the proof of Theorem 1 that for sufficiently small $\epsilon > 0$,

$$| \int_0^K \sum_j G_j(\tau,1/\epsilon)h(\tau)d\eta(\tau) |$$

$$\leq C(1+\delta)^{1/p'} \| \sum | f_j | \|_{p',q'}^{H} \| \tilde{T}^* \|_{M(p,r;p,q)}^{J} \| h \|_{p,r}^{H}$$
$$+ \| \sum_j H_j^N(\tau,1/\epsilon) \|_{L^2((0,K),\eta)} \| h \|_{L^2(\eta)},$$

and

$$| \int \sum_{j=1}^{N} G_j(\tau)h(\tau)d\eta(\tau) | \leq C(1+\delta)^{2/p'} \| \tilde{T}^* \|_{M(p,r;p,q)}^{J} \| \sum_{j=1}^{L} | f_j | \|_{p',q'}^{H}$$

by Igari[4], and we get

$$\| \sum_{j=1}^{L} T_{\epsilon_j \epsilon} f_j \|_{p',r'}^{H} \leq C \| \tilde{T}^* \|_{M(p,r;p,q)}^{J} \| \sum_{j=1}^{L} | f_j | \|_{p',q'}^{H} .$$

Hence, by Lemma (2), we obtain the desired result:

$$\| T^* \|_{M(p,r;p,q)}^{H} \leq C \| \tilde{T}^* \|_{M(p,r;p,q)}^{J} .$$

q.e.d.

3. An application.

Colzani[1] showed the following:

Theorem D Let $\alpha > -1/2$ and $1 < r \leq \infty$. The partial sum operators $\{S_R\}$ are not bounded from $L^{(4\alpha+4)/(2\alpha+3),r}(\eta)$ into $L^{(4\alpha+4)/(2\alpha+3),\infty}(\eta)$, where

$$S_R f(x) = \int_0^R \frac{J_\alpha(xy)}{(xy)^\alpha} \mathcal{H}_\alpha f(y) d\eta(y) \ (f \in C_c^\infty(0,\infty)).$$

By the application of Theorem D, we can show the following result:

Theorem 3 Let $\alpha, \beta > -1/2$ and $1 < r \leq \infty$. The partial sum operators $\{S_N\}$ are not bounded from
$L^{(4\alpha+4)/(2\alpha+3),r}((0,\pi),\mu)$ into $L^{(4\alpha+4)/(2\alpha+3),\infty}((0,\pi),\mu)$, where

$$S_N g(\theta) = \sum_{n=0}^{N} \hat{g}(n) h_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta) \ (g \in C_c^\infty(0,\pi)).$$

Proof. We may assume $r < \infty$ by the property of the Lorentz norm. Also we assume that $\{S_N\}$ are bounded from $L^{(4\alpha+4)/(2\alpha+3),r}((0,\pi),\mu)$ into $L^{(4\alpha+4)/(2\alpha+3),\infty}((0,\pi),\mu)$. Then we define that for $\epsilon, R > 0$

$$\phi_R(\xi) = \chi_{(0,R)}(\xi),$$

and

$$(\tilde{T}_\epsilon g)^\wedge(n) = \phi_R(\epsilon n)\hat{g}(n) \ (g \in C_c^\infty(0,\pi)).$$

Here, by the assumption of $\{S_N\}$ and $\phi_R(\epsilon n) = \chi_{(0,R/\epsilon)}(n)$, we obtain

$$\sup_{\epsilon>0} \| \tilde{T}_\epsilon \|^J_{M(\frac{4\alpha+4}{2\alpha+3},r;\frac{4\alpha+4}{2\alpha+3},\infty)} < \infty.$$

On the other hand, by Theorem 1, for $\alpha > -\frac{1}{2}$ there exists a positive constant $C > 0$ such that

$$\| T_{\phi_R} \|^H_{M(\frac{4\alpha+4}{2\alpha+3},r;\frac{4\alpha+4}{2\alpha+3},\infty)} \leq C \sup_{\epsilon>0} \| \tilde{T}_\epsilon \|^J_{M(\frac{4\alpha+4}{2\alpha+3},r;\frac{4\alpha+4}{2\alpha+3},\infty)} .$$

Therefore, we get that $\{S_R\}$ are bounded from $L^{(4\alpha+4)/(2\alpha+3),r}((0,\infty),\eta)$ into $L^{(4\alpha+4)/(2\alpha+3),\infty}((0,\pi),\eta)$. This is a contradiction to Theorem D. Hence, we get the desired result.
q.e.d.

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