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NORMAL BCI-ALGEBRAS

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Abstract. In this paper we generalize the following five notions from BCK-algebras into BCI-algebras: stabilizer, left and right stabilizers, normal BCK-algebra and normal ideal, and investigate some basic properties of them.

§0. Introduction and preliminaries

In [6], by using stabilizers and left and right stabilizers in BCK-algebras, we introduced and investigated normal BCK-algebras. In [5] we considered normal ideals in BCK-algebras (early in 1991, Hoo in [4] had actually got involved in the consideration of them in BCI-algebras). In this paper we will generalize each of these notions from BCK-algebras into BCI-algebras, and investigate a number of basic properties of it.

Throughout this paper, for the symbols and terminologies concerned, we refer the reader to [2], [7], [8] and [9], and we will use some familiar properties without explanation.

Recall that given a BCI-algebra X, the BCI-ordering $\leq$ on X is defined by which $x \leq y$ if and only if $x * y = 0$ for any $x, y \in X$. A *positive element* x of X means $x \geq 0$ (i.e., $0 * x = 0$), and the set of all positive elements of X is just the BCK-part B of X; a *minimal element* x of X means that $y \leq x$ (i.e., $y * x = 0$) implies $x = y$ for any $y \in X$, and the set of all minimal elements of X is just the p-semisimple part P of X. It is known that for any $x, y \in X$, if $x \leq y$, then $y * x$ is a positive element of X, and that for any $x \in X$ there is one and only one minimal element a of X, satisfying $a \leq x$ (refer to [9, §1.3]). An *ideal* A of X is a subset of X such that (i) $0 \in A$ and (ii) $x, y * x \in A$ imply $y \in A$ for any $x, y \in X$. A *subalgebra* Y of X is a nonempty subset of X such that Y is closed under the BCI-operation $*$ on X. If A is both an ideal and a subalgebra of X, we call it a *closed ideal* of X. An ideal A of X is closed if and only if $0 * x \in A$ for any $x \in A$. The BCK-part B of X is a closed ideal of X and the p-semisimple part P of X is a subalgebra of X. The generated ideal $\langle S \rangle$ of X by a subset S of X can be expressed as

$$\langle S \rangle = \{0\} \bigcup \left\{ x \in X \;\middle|\; \begin{array}{l} (\cdots((x * a_1) * a_2) * \cdots) * a_n = 0 \\ \text{for some } a_1, a_2, \ldots, a_n \in S \end{array} \right\}.$$

If $S = \{a\}$, we denote $\langle a \rangle$ for $\langle \{a\} \rangle$ in brevity. In the following let's write down several results: for any $x, y, z \in X$,

(0.1) $(x * y) * (x * z) \leq z * y$;
(0.2) $(x * y) * (z * y) \leq x * z$;
(0.3) $0 * (x * y) = (0 * x) * (0 * y)$;
(0.4) $x * y = x * (x * (x * y))$;
(0.5) $0 * x$ is a minimal element of X;
(0.6) $0 * (0 * x) = x$ whenever x is a minimal element of X;
(0.7) $x * y \leq x$, i.e., $(x * y) * x = 0$, whenever y is a positive element of X.

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Every ideal A of X determines a congruence $\equiv$ on X in the sense that $x \equiv y \,(\mathrm{mod}\ A)$ if and only if $x * y \in A$ and $y * x \in A$ for any x, $y \in X$. The symbol X/A will be used instead of the quotient algebra $X/\equiv$, which is still a BCI-algebra.

If A and I are ideals of X such that $X = \langle A \cup I \rangle$ and $A \cap I = \{0\}$, then X is called the *subdirect sum* of A and I, denoted by $X = A \overline{\oplus} I$. It is known that if A, I are closed ideals of X and if $X = A \overline{\oplus} I$, then for any $x \in X$, there are uniquely $a \in A$ and $b \in I$ such that $x \equiv a \,(\mathrm{mod}\ I)$ and $x \equiv b \,(\mathrm{mod}\ A)$ (see [2, Theorem 2.1]). The element a is said the *component* of x in A, and b of x in I.

Proposition 0.1. *Let A, I be two closed ideals of a BCI-algebra X such that $X = A \overline{\oplus} I$ and let $a \in A$ and $b \in I$. Then a is the component of x in A and b of x in I if and only if $x * a = b$ and $x * b = a$.*

Proof. The necessity is a special case of [2, Proposition 2.6], and we only need to show the sufficiency. In fact, since I is closed, our supposition of sufficiency means that $x * a = b \in I$ and $a * x = (x * b) * x = 0 * b \in I$, then $x \equiv a \,(\mathrm{mod}\ I)$, and so a is the component of x in A. Similarly, b is the component of x in I.

Proposition 0.2. *Let A, I be two ideals of a BCI-algebra X such that $X = A \overline{\oplus} I$ and let x, x' be any elements in X.*

(1) *If a and a' are respectively the components of x and x' in A, then $a * a'$ is the component of $x * x'$ in A.*
(2) *If x and x' have the same components in both A and I, then $x = x'$.*

Proof. (1) It is got by the substitution property of congruences.

(2) Since $X = A \overline{\oplus} I$, we have $A \cap I = \{0\}$. If x and x' have the same components in A, by (1), 0 is the component of $x * x'$ in A, then $x * x' \in A$. Similarly, $x * x' \in I$. Hence $x * x' \in A \cap I = \{0\}$ and $x * x' = 0$. Likewise, $x' * x = 0$. Therefore $x = x'$.

Assume that $X = A \overline{\oplus} I$. If for any $a \in A$ and $b \in I$, there exists $x \in X$ such that a is the component of x in A and b of x in I, we say X is the *direct sum* of A and I, denoted by $X = A \oplus I$.

Proposition 0.3. *If the p-semisimple part P of a BCI-algebra X is an ideal of X, then $X = B \oplus P$ where B is the BCK-part of X.*

Proof. For any $x \in X$, letting a be a minimal element of X, satisfying $a \leq x$, we have $a \in P$ and $x * a \in B$, then $x \in \langle B \cup P \rangle$, and so $X = \langle B \cup P \rangle$. It is obvious that $B \cap P = \{0\}$. Thus $X = B \overline{\oplus} P$. Also, for any $b \in B$ and $p \in P$, putting $x = b * (0 * p)$, by (0.5), we have

$$x * b = (b * (0 * p)) * b = 0 * (0 * p) \in P,$$
$$b * x = b * (b * (0 * p)) \leq 0 * p \in P.$$

Then $x \equiv b \,(\mathrm{mod}\ P)$. On the other hand, by (0.3) and (0.4), we obtain

$$0 * x = 0 * (b * (0 * p)) = (0 * b) * (0 * (0 * p)) = 0 * (0 * (0 * p)) = 0 * p.$$

Then (0.2) and (0.6) together give

$$x * p = (b * (0 * p)) * p = (b * p) * (0 * p) \leq b \in B,$$
$$p * x = (0 * (0 * p)) * x = (0 * x) * (0 * p) = (0 * p) * (0 * p) = 0 \in B.$$

So, $x \equiv p \,(\mathrm{mod}\ B)$. Hence $X = B \oplus P$.

A BCK-algebra X is called *normal* if for any $a \in X$, the *right stabilizer* $\{a\}^*_R$, i.e., the set $\{x \in X \mid x * a = x\}$, is an ideal of X (see [6]).

Proposition 0.4. *A BCK-algebra X is normal if and only if $x * y = x$ implies $y * x = y$ for any x, $y \in X$* (see [6, Theorem 2]).

An ideal A of a BCK-algebra X is called *normal* if $x * (x * y) \in A$ implies $y * (y * x) \in A$ for any x, $y \in X$ (see [5]).

Proposition 0.5. *A BCK-algebra is normal if and only if the zero ideal $\{0\}$ of it is normal* (see [5, Theorem 2.3]).

§1. Stabilizers

Definition 1.1. Given a nonempty subset S of a BCI-algebra X, the sets

$$S^*_L = \{x \in X \mid a * x = a \text{ for any } a \in S\},$$
$$S^*_R = \{x \in X \mid x * a = x \text{ for any } a \in S\}$$

are called the *left* and *right stabilizers* of S, respectively. And the set

$$S^* = \{x \in X \mid a * x = a \text{ and } x * a = x \text{ for any } a \in S\},$$

i.e., $S^* = S^*_L \cap S^*_R$, is called the *stabilizer* of S.

These are the natural generalization of the corresponding notions in BCK-algebras, thus there are many similar properties, but their proofs need to be made suitable change. It is obvious that if S, T are nonempty subsets of X, then

(1.1) $S^*_L = \bigcap_{a\in S}\{a\}^*_L$, $S^*_R = \bigcap_{a\in S}\{a\}^*_R$ and $S^* = \bigcap_{a\in S}\{a\}^*$;
(1.2) $(S \cup T)^*_L = S^*_L \cap T^*_L$, $(S \cup T)^*_R = S^*_R \cap T^*_R$ and $(S \cup T)^* = S^* \cap T^*$;
(1.3) if $S \subseteq T$, then $T^*_L \subseteq S^*_L$, $T^*_R \subseteq S^*_R$ and $T^* \subseteq S^*$.

For convenience we call S a *positive subset* of a BCI-algebra X if S is a nonempty subset of X and every element in X is positive. Similarly, we have the notions of *positive ideals* and *positive subalgebras* of X.

Proposition 1.1. *The left stabilizer S^*_L of any nonempty subset S of a BCI-algebra X is a positive ideal of X, thus it is a closed ideal of X.*

Proof. Clearly, $0 \in S^*_L$, then $S^*_L \neq \emptyset$. For any $x \in S^*_L$ and any $a \in S$, since

$$0 * x = (a * x) * a = a * a = 0,$$

S^*_L is a positive subset of X. Also, if x, $y * x \in S^*_L$, then

$$a = a * (y * x) = (a * x) * (y * x).$$

So, (0.2) implies

$$a * (a * y) = ((a * x) * (y * x)) * (a * y) \leq (a * y) * (a * y) = 0.$$

On the other hand, note that S^*_L is a positive subset of X, by (0.3), the following holds:

$$(a * y) * a = 0 * y = (0 * y) * 0 = (0 * y) * (0 * x) = 0 * (y * x) = 0.$$

Hence $a * y = a$ and $y \in S^*_L$. Therefore S^*_L is a positive ideal of X. Finally, it is obvious from S^*_L being positive that S^*_L is a closed ideal of X.

It is a pity that a right stabilizer or a stabilizer may be empty. But we have the following results.

Proposition 1.2. *Let S be a nonempty subset of a BCI-algebra X. Then*

(1) *S^*_R (or S^*) is not empty if and only if S is a positive subset of X;*
(2) *if S is a positive subset of X, then S^*_R is a subalgebra of X, containing the whole minimal elements of X;*
(3) *if S is a positive subset of X, then S^* is a positive subalgebra of X.*

Proof. (1) If S^*_R or S^* is not empty, putting $x \in S^*_R$ (or $x \in S^*$), for any $a \in S$, we have

$$0 * a = (x * a) * x = x * x = 0,$$

that is, a is a positive element of X. Hence S is a positive subset of X.

Conversely, if S is positive, then $0 * a = 0$ for any $a \in S$. So, $0 \in S^*_R$ and $S^*_R \neq \emptyset$. Also, clearly $0 \in S^*_L$, then $0 \in S^*_L \cap S^*_R$, that is, $0 \in S^*$, and so $S^* \neq \emptyset$.

(2) If S is positive, by (1), $S^*_R \neq \emptyset$. Putting $x,\ y \in S^*_R$, for any $a \in S$, we have

$$(x * y) * a = (x * a) * y = x * y,$$

then $x * y \in S^*_R$. Hence S^*_R is a subalgebra of X. Also, since S is positive, by (0.7), $x * a \leq x$ for all $x \in X$. Now, if x is minimal, then $x * a = x$, and so $x \in S^*_R$. Hence S^*_R contains the whole minimal elements of X.

(3) Since $S^* = S^*_L \cap S^*_R$, Proposition 1.1 together with (2) gives that S^* is a positive subalgebra of X.

However, even if S is a positive subset of X, S^*_R and S^* are generally not ideals of X.

Example 1.1. Let $X = \{0, 1, 2, 3, a\}$ and define a binary operation $*$ on X by

$*$	0	1	2	3	a
0	0	0	0	0	a
1	1	0	0	0	a
2	2	2	0	2	a
3	3	3	3	0	a
a	a	a	a	a	0

Then $(X; *, 0)$ is a BCI-algebra (refer to [9, Theorem 5.1.1]). Obviously, $\{2\}^*_R = \{0, 3, a\}$ and $\{2\}^* = \{0, 3\}$. Since $3 \in \{2\}^*_R$ and $1 * 3 = 0 \in \{2\}^*_R$, but $1 \notin \{2\}^*_R$, $\{2\}^*_R$ is not an ideal of X. Similarly, $\{2\}^*$ is not either.

Proposition 1.3. *Let S be a nonempty subset of a BCI-algebra X.*

(1) *If $0 \in S$, then $S \cap S^*_L = \{0\}$, otherwise, $S \cap S_L = \emptyset$.*
(2) *$S \subseteq (S^*_L)^*_R$.*
(3) *$S^*_L = ((S^*_L)^*_R)^*_L$.*

Proof. (1) If $0 \in S$, since S^*_L is an ideal of X, we have $S \cap S^*_L \neq \emptyset$. For any $x \in S \cap S^*_L$, by $x = x * x = 0$, we obtain $S \cap S^*_L = \{0\}$.

Next, if it is false, then $S \cap S^*_L \neq \emptyset$. By the proof we just now give, $S \cap S^*_L = \{0\}$, then $0 \in S$, a contradiction with $0 \notin S$. Hence $S \cap S^*_L = \emptyset$.

(2) By virtue of Proposition 1.1, $S^*_L \neq \emptyset$. If $a \in S$, then $a * x = a$ for any $x \in S^*_L$, and so $a \in (S^*_L)^*_R$, and hence $S \subseteq (S^*_L)^*_R$.

(3) By (2), $(S_L^*)_R^*$ is non-vacuous, then $((S_L^*)_R^*)_L^*$ is well-defined. Using (2) and (1.3), we obtain $((S_L^*)_R^*)_L^* \subseteq S_L^*$. On the other hand, if $a \in S_L^*$, then $a * x = a$ for any $x \in (S_L^*)_R^*$. Hence $a \in ((S_L^*)_R^*)_L^*$ and $S_L^* \subseteq ((S_L^*)_R^*)_L^*$. Therefore $S_L^* = ((S_L^*)_R^*)_L^*$.

Proposition 1.4. *Let S be a positive subset of a BCI-algebra X.*

(1) *If $0 \in S$, then $S \cap S_R^* = S \cap S^* = \{0\}$, otherwise, $S \cap S_R^* = S \cap S^* = \emptyset$.*
(2) *$S \subseteq (S_R^*)_L^*$ and $S \subseteq S^{**}$ where $S^{**} = (S^*)^*$.*
(3) *$S_R^* = ((S_R^*)_L^*)_R^*$ and $S^* = S^{***}$.*

The proof is similar to Proposition 1.3 and omitted.

Proposition 1.5. *Let S be a positive subset of a BCI-algebra X. Then $S_R^* = \langle S\rangle_R^*$ and $S_R^* \cap \langle S\rangle = \{0\}$ where $\langle S\rangle$ is the generated ideal of X by S.*

Proof. By (1.3), $\langle S\rangle_R^* \subseteq S_R^*$. Letting $x \in S_R^*$, we have

$$x * a = x \quad \text{for any } a \in S. \tag{1.4}$$

For all $b \in \langle S\rangle$, if $b = 0$, of course, $x * b = x$; if $b \neq 0$, there are $a_1, a_2, \ldots, a_n \in S$ such that

$$(\cdots((b * a_1) * a_2) * \cdots) * a_n = 0. \tag{1.5}$$

Repeatedly applying (1.4), the following holds:

$$x = (\cdots((x * a_1) * a_2) * \cdots) * a_n. \tag{1.6}$$

Putting (1.5) and (1.6) together, we obtain

$$x = (\cdots((x * a_1) * a_2) * \cdots) * a_n) * (\cdots((b * a_1) * a_2) * \cdots) * a_n).$$

Now, using (0.2) step by step, it follows

$$x \leq (\cdots((x * a_1) * a_2) * \cdots) * a_{n-1}) * (\cdots((b * a_1) * a_2) * \cdots) * a_{n-1}) \leq \cdots \leq x * b,$$

that is, $x \leq x * b$. On the other hand, it is easily seen from (1.5) and (0.3) that b is a positive element of X, then $x * b \leq x$ by (0.7). So, $x * b = x$. Thus $x \in \langle S\rangle_R^*$. Hence $S_R^* \subseteq \langle S\rangle_R^*$. Therefore $S_R^* = \langle S\rangle_R^*$. Also, by Proposition 1.4(1), $S_R^* \cap \langle S\rangle = \langle S\rangle_R^* \cap \langle S\rangle = \{0\}$.

It is interesting that if S is a positive ideal of X, we have some unusual properties, including that S^* must be an ideal of X.

Theorem 1.6. *Let A be a positive ideal of a BCI-algebra X. Then $A^* = A_L^* \subseteq A_R^*$, thus A^* is a positive ideal of X. Moreover, if A_R^* is an ideal of X, then $A^* = A_L^* = A_R^* \cap B$ where B is the BCK-part of X.*

Proof. For the first half part, if $A_L^* \subseteq A_R^*$, then $A^* = A_L^* \cap A_R^* = A_L^*$ and A^* is a positive ideal of X by A_L^* being a positive ideal of X. It remains to show $A_L^* \subseteq A_R^*$. Put $x \in A_L^*$. For any $a \in A$, by $x * (x * a) \leq a$, we obtain $x * (x * a) \in A$, then

$$(x * (x * a)) * x = x * (x * a).$$

As A and A_L^* are positive ideals of X, a and x are positive elements of X, then

$$(x * (x * a)) * x = 0.$$

Comparison gives $x * (x * a) = 0$. Also, by (0.7), the equality $(x * a) * x = 0$ holds. Hence $x * a = x$ and $x \in A_R^*$. Therefore $A_L^* \subseteq A_R^*$.

For the second half part, because A_L^* is a positive ideal of X (i.e., $A_L^* \subseteq B$) and because $A^* = A_L^* \subseteq A_R^*$, it suffices to show $A_R^* \cap B \subseteq A_L^*$. Put $x \in A_R^* \cap B$. For any $a \in A$, since x and a are positive elements of X, we obtain $a * (a * x) \leq a$, then $a * (a * x) \in A$. As A_R^* is an ideal of X and $a * (a * x) \leq x$, we derive $a * (a * x) \in A_R^*$. So, $a * (a * x) \in A \cap A_R^*$. Note that $A \cap A_R^* = \{0\}$. It follows $a * (a * x) = 0$. Obviously, $(a * x) * a = 0$. Thus $a * x = a$ and $x \in A_L^*$. Hence $A_R^* \cap B \subseteq A_L^*$.

Theorem 1.7. *Let X be a BCI-algebra, A a positive ideal of X, and I a closed ideal of X. If $A \cap I = \{0\}$, then $I \subseteq A_R^*$. Further, if $X = A \overline{\oplus} I$, then $I = A_R^*$.*

Proof. Assume that $A \cap I = \{0\}$. Putting $x \in I$, for any $a \in A$, since A is an ideal of X, by $x * (x * a) \leq a$, we obtain $x * (x * a) \in A$. Because A is positive, we have

$$(x * (x * a)) * x = 0 * (x * a) = (0 * x) * (0 * a) = 0 * x.$$

Then the fact that I is a closed ideal of X implies $x * (x * a) \in I$. Hence

$$x * (x * a) \in A \cap I = \{0\}.$$

Thus $x * (x * a) = 0$. Also, by (0.7), $(x * a) * x = 0$. Therefore $x * a = x$ and $x \in A_R^*$. We have shown that $I \subseteq A_R^*$.

Further, if $X = A \overline{\oplus} I$, then $A \cap I = \{0\}$. By the above proof, $I \subseteq A_R^*$. Also, for any $x \in A_R^*$, letting a be the component of x in A, we have $a \in A$ and $x * a \in I$. Note that $x * a = x$. It follows $x = x * a \in I$. Hence $A_R^* \subseteq I$. Therefore $I = A_R^*$.

§2. Normal BCI-algebras

Definition 2.1. A BCI-algebra X is called *normal* if for any positive element a of X, the right stabilizer $\{a\}_R^*$ is an ideal of X.

It is evident that any normal BCK-algebra is a normal BCI-algebra.

Example 2.1. Let $X = \{0, 1, a, b\}$ and define two binary operations $*$ and $*'$ on X by

$*$	0	1	a	b
0	0	0	a	a
1	1	0	b	a
a	a	a	0	0
b	b	a	1	0

$*'$	0	1	a	b
0	0	0	a	a
1	1	0	a	a
a	a	a	0	0
b	b	a	1	0

Then $(X; *, 0)$ and $(X; *', 0)$ are BCI-algebras (see [9, page 276]). It is easy to verify that the former is normal, but the latter is not.

Using (1.1) we directly have the next result.

Proposition 2.1. *A BCI-algebra X is normal if and only if the right stabilizer S_R^* of any positive subset S of X is an ideal of X.*

Note that $\langle S_R^* \rangle$ is the least ideal of X, containing S_R^*, the following holds.

Corollary 2.2. *A BCI-algebra X is normal if and only if $S_R^* = \langle S_R^* \rangle$ for all positive subset S of X.*

If X is a p-semisimple BCI-algebra, then 0 is the only positive element of X. It is evident that $\{0\}_R^* = X$ and X is an ideal of itself. We have then had the next assertion.

Proposition 2.3. *Every p-semisimple BCI-algebra is normal.*

A nonzero BCI-algebra X is called *J-semisimple* if X contains at least a maximal ideal and the intersection of the whole maximal ideals of X is equal to $\{0\}$. If $X = \{0\}$, we provide that it is J-semisimple (see [10]). It has been known that if X is J-semisimple, the right stabilizer of any positive element of X is a closed ideal of X (see [10, Theorem 13]). Thus this assertion can be rewritten as follows.

Theorem 2.4. *Every J-semisimple BCI-algebra is normal.*

Proposition 2.5. *A BCI-algebra X is normal if and only if every subalgebra Y of X is normal.*

Proof. Since X is a subalgebra of itself, the sufficiency is naturally true, and we only need to show the necessity. For any positive element a in the subalgebra Y of X, we denote $(\{a\}_R^*)_Y$ for the right stabilizer in Y and $(\{a\}_R^*)_X$ for that in X, namely,

$$(\{a\}_R^*)_Y = \{x \in Y \mid x * a = x\} \quad \text{and} \quad (\{a\}_R^*)_X = \{x \in X \mid x * a = x\}.$$

Obviously, a is also a positive element of X. By the normality of X, $(\{a\}_R^*)_X$ is an ideal of X. Then $(\{a\}_R^*)_X \cap Y$ is an ideal of Y. It is obvious that $(\{a\}_R^*)_Y = (\{a\}_R^*)_X \cap Y$. Hence $(\{a\}_R^*)_Y$ is an ideal of Y, proving Y is normal.

Theorem 2.6. *If X is a normal BCI-algebra, then the p-semisimple part P of X is an ideal of X and $X = B \oplus P$ where B is the BCK-part of X.*

Proof. Put $x, y \in X$ with $x, y * x \in P$ and let $a \in P$ be the minimal element satisfying $a \leq y$. Since P is a subalgebra of X, we have $(y * x) * a \in P$, i.e., $(y * a) * x \in P$. Also, since $y * a$ is a positive element of X, Proposition 1.2(2) gives $P \subseteq \{y * a\}_R^*$. Then $x \in \{y * a\}_R^*$ and $(y * a) * x \in \{y * a\}_R^*$. Moreover, by the normality of X, $\{y * a\}_R^*$ is an ideal of X. Hence $y * a \in \{y * a\}_R^*$. ¿From this we have

$$y * a = (y * a) * (y * a) = 0.$$

Thus $y = a \in P$ by a being a minimal element of X. Therefore P is an ideal of X. Finally, by Proposition 0.3, $X = B \oplus P$.

Corollary 2.7. *If X is a J-semisimple BCI-algebra, then the p-semisimple part P of X is an ideal of X and $X = B \oplus P$ in which B is the BCK-part of X.*

Theorem 2.8. *Suppose that A_1 and A_2 are two closed ideals of a BCI-algebra X such that $X = A_1 \overline{\oplus} A_2$. Then X is normal if and only if A_1 and A_2 are normal subalgebras of X.*

Proof. As any closed ideal of X is a subalgebra of X, by Proposition 2.5, the necessity holds.

Conversely, assume that A_1 and A_2 are two normal subalgebras of X. For any positive element $a \in X$, let a_1 be the component of a in A_1 and a_2 of a in A_2. By Proposition 0.2(1), $0 * a_1$ is the component of $0 * a$ in A_1. Since $0 * a = 0$ and the component of 0 in A_1 is 0 itself, by the uniqueness of components, we have $0 * a_1 = 0$. Thus a_1 is a positive element of X and of A_1. Similarly, a_2 is a positive element of X and of A_2. Denote

$$I_1 = \{x \in A_1 \mid x * a_1 = x\} \quad \text{and} \quad I_2 = \{x \in A_2 \mid x * a_2 = x\}.$$

Obviously, I_1 is the right stabilizer of $\{a_1\}$ in A_1, and I_2 of $\{a_2\}$ in A_2. Since A_1 is normal, I_1 is an ideal of A_1. By the transitivity of ideals, it is also an ideal of X. Likewise, I_2 is an ideal of A_2 and of X. Denote I for the generated ideal $\langle I_1 \cup I_2 \rangle$ of X. It is easy to verify from Proposition 1.2(2) that I is a closed ideal of X, thus it is a subalgebra of X. Note that

$$I_1 \cap I_2 \subseteq A_1 \cap A_2 = \{0\}.$$

We have the representation:

$$I = I_1 \overline{\oplus} I_2. \tag{2.1}$$

Now, in order to show the right stabilizer $\{a\}_R^*$ of $\{a\}$ in X is an ideal of X, we turn to prove $\{a\}_R^* = I$. For any $x \in \{a\}_R^*$, if x_1 is the component of x in A_1 and x_2 of x in A_2, by Proposition 0.1, we have $x * x_1 = x_2$. Also, by Proposition 0.2(1), the component of $x * a$ in A_1 is $x_1 * a_1$. Since $x * a = x$, the uniqueness of components implies $x_1 * a_1 = x_1$, then $x_1 \in I_1$. Similary, $x_2 \in I_2$. Note that $x * x_1 = x_2$. It yields $x \in \langle I_1 \cup I_2 \rangle$, that is, $x \in I$, in other words, $\{a\}_R^* \subseteq I$. On the other hand, for any $x \in I$, by (2.1), we are able to assume that x_1 is the component of x in I_1 and x_2 of x in I_2, then $x * x_1 = x_2$ and $x * x_2 = x_1$ by Proposition 0.1. It is obvious that $x_1 \in A_1$ and $x_2 \in A_2$. Applying Proposition 0.1 to the representation $X = A_1 \overline{\oplus} A_2$, we see that x_1 is just the component of x in A_1 and x_2 of x in A_2. Hence the component of $x * a$ in A_1 is $x_1 * a_1$. Since $x_1 \in I_1$, we have $x_1 * a_1 = x_1$. Therefore $x * a$ and x have the same components in A_1. Likewise, their components in A_2 are the same. By Proposition 0.2(2), $x * a = x$, that is, $x \in \{a\}_R^*$, in other words, $I \subseteq \{a\}_R^*$. We have then shown that $\{a\}_R^* = I$. Now, it is evident that $\{a\}_R^*$ is an ideal of X. Therefore X is normal.

Putting Proposition 2.3, Theorems 2.6 and 2.8 together, we obtain the next corollary.

Corollary 2.9. *A BCI-algebra X is normal if and only if the BCK-part B of X is a normal BCK-algebra and the p-semisimple part P of X is an ideal of X.*

Using Proposition 0.4, the last corollary can be rewritten as follows.

Corollary 2.10. *A BCI-algebra X is normal if and only if it satisfies the following:*

(1) *$x * y = x$ implies $y * x = y$ for any positive elements x and y of X;*
(2) *the p-semisimple part P of X is an ideal of X.*

Before concluding this section let's consider the normality of weakly implicative BCI-algebras. A BCI-algebra X is called *weakly implicative* if

$$(x * (y * x)) * (0 * (y * x)) = x$$

for all x, $y \in X$ (see [1]).

Theorem 2.11. *Every weakly implicative BCI-algebra X is normal.*

Proof. Assume that B and P are the BCK-part and p-semisimple part of X. By the weakly implicativity of X, we have $x * (y * x) = x$ for any x, $y \in B$, then B is an implicative BCK-algebra, thus it is a commutative BCK-algebra. Applying the commutativity of B, we obtain that $x * y = x$ implies $y * x = y$ for any x, $y \in B$.

Next, it is clear that $0 \in P$. If x, $y * x \in P$, letting $a \in P$ such that $a \leq y$, since P is a subalgebra of X, we have $(y * x) * a \in P$, i.e., $(y * a) * x \in P$. Obviously, $0 \leq y * a$. Denote $u = y * a$, then $0 \leq u$ and $u * x \in P$. By $0 \leq u$, we obtain $0 * x \leq u * x$ and $x * u \leq x$, then $0 * x = u * x$ and $x * u = x$ by $u * x$, $x \in P$ (i.e., $u * x$ and x are minimal elements of X). Now, by the weakly implicativity of X, we derive

$$y * a = u = (u * (x * u)) * (0 * (x * u)) = (u * x) * (0 * x) = (0 * x) * (0 * x) = 0.$$

Hence $y = a$ by a being a minimal element of X. Therefore $y \in P$, proving P is an ideal of X. Now, by Corollary 2.10, X is normal.

Corollary 2.12. *If X is a weakly implicative BCI-algebra, then the p-semisimple part P of X is an ideal of X and X can be expressed as the direct sum $X = B \oplus P$ in which B is the BCK-part of X.*

We remark that Corollary 2.12 was actually obtained by S.M. Wei and J. Meng who considered it from the way of KL-products (for detail, see [9, §4.2]).

Corollary 2.13. *A normal BCI-algebra is weakly implicative if and only if the BCK-part of it is an implicative BCK-algebra.*

§3. Normal ideals

Normal ideals were considered by C.S. Hoo in [4] who calls them *commutative ideals*.

Definition 3.1. An ideal A of a BCI-algebra X is called *normal* if $x * (x * y) \in A$ implies $y * (y * x) \in A$ for any x, $y \in X$.

Every BCI-algebra X contains at least a normal ideal, e.g., X itself is just one.

Example 3.1. Let X be the first algebra in Example 2.1, then there are altogether four ideals of it, which are X, $\{0, a\}$, $\{0, 1\}$ and $\{0\}$. Routine verification gives that the first two ideals are normal, but the others are not.

It is worth attending that if X is a proper BCI-algebra, a positive ideal of X is never normal.

Proposition 3.1. *Let A be a normal ideal of a BCI-algebra X. Then A is positive if and only if X is a BCK-algebra.*

Proof. The sufficiency is evident and we only need to show the necessity. Assume that A is a positive ideal of X, then 0 is the only minimal element of X, contained in A. For any $x \in X$, since $x * (x * 0) = 0 \in A$, by the normality of A, we have $0 * (0 * x) \in A$. Note that $0 * (0 * x)$ is a minimal element of X, it follows $0 * (0 * x) = 0$. Hence (0.4) implies

$$0 * x = 0 * (0 * (0 * x)) = 0 * 0 = 0.$$

Therefore X is a BCK-algebra.

For convenience we denote $x * y^n = (\cdots((x * y) * y) * \cdots) * y$ in which y occurs n times.

Lemma 3.2. *If A is an ideal of a BCI-algebra X, then $x*(x*y) \in A$ implies $x*(x*y^n) \in A$ for any x, $y \in X$ and any natural number n* (refer to [5, Lemma 2.1]).

Proposition 3.3. *Suppose that M is a maximal ideal of a BCI-algebra X. If M contains the whole minimal elements of X, then M is a normal ideal of X.*

Proof. Assume that $x * (x * y) \in M$. If $y \in M$, since $0 * (y * x)$ is a minimal element of X and $(y * (y * x)) * y = 0 * (y * x)$, our hypotheses imply $y * (y * x) \in M$. If $y \notin M$, by the maximality of M, there is a natural number n such that $x * y^n \in M$. Also, by Lemma 3.2, $x * (x * y^n) \in M$. Hence $x \in M$. Note that $y * (y * x) \leq x$, it yields $y * (y * x) \in M$. Therefore M is normal.

Proposition 3.4. *Let A be an ideal of a BCI-algebra X. Then A is normal if and only if $x * (x * y) \in A$ implies $y * (y * x^n) \in A$ for any x, $y \in X$ and any natural number n.*

Proof. Assume that A is normal and $x, y \in X$. If $x * (x * y) \in A$, then $y * (y * x) \in A$. By Lemma 3.2, $y * (y * x^n) \in A$ for any natural number n.

Conversely, putting $n = 1$, our assumption of sufficiency gives that $x * (x * y) \in A$ implies $y * (y * x) \in A$ for any $x, y \in X$. Hence A is normal.

Proposition 3.5. *Let A be a normal ideal of a BCI-algebra X. Then A is closed, containing the entire minimal elements of X* (see [4, Proposition 2.16]).

Corollary 3.6. *If X is a p-semisimple BCI-algebra, then the normal ideal of X can only be X itself.*

Theorem 3.7. *An ideal A of a BCI-algebra X is normal if and only if the quotient algebra X/A is a normal BCK-algebra* (see [4, Theorem 2.17]).

Theorem 3.8. *A BCI-algebra X is normal if and only if the p-semisimple part P of X is a normal ideal of X.*

Proof. Assume that X is normal. By Theorem 2.6, P is an ideal of X and $X = B \oplus P$ in which B is the BCK-part of X. Then B is a normal BCK-algebra by Proposition 2.5. For any $x, x' \in X$, letting b, b' be respectively the components of x and x' in B, by Proposition 0.2(1), $b * (b * b')$ is the component of $x * (x * x')$ in B. Now, if $x * (x * x') \in P$, it is easily seen from Proposition 0.1 that 0 is the component of $x * (x * x')$ in B. By the uniqueness of components, we obtain $b * (b * b') = 0$. So, Proposition 0.5 gives $b' * (b' * b) = 0$. Thus the component of $x' * (x' * x)$ in B is 0. Hence $x' * (x' * x) \in P$. Therefore P is normal.

Conversely, since P is an ideal of X, by Proposition 0.3, $X = B \oplus P$. Using the uniqueness of components, we can define the mapping f from X to B sending x to the component of x in B. Obviously, f is a surjection. By the substitution property of congruences, f is a homomorphism. It is easy to verify that the kernel of f is P. So, the first isomorphic theorem (see [3, Theorem 3.2]) gives that X/P is isomorphic to B. Also, since P is a normal ideal of X, by Theorem 3.7, X/P is a normal BCK-algebra. Thus B is a normal BCK-algebra too. Now, by Corollary 2.9, X is normal.

Because the p-semisimple part of a BCK-algebra is $\{0\}$, Proposition 0.5 becomes a direct corollary of Theorem 3.8.

Corollary 3.9. *A BCK-algebra X is normal if and only if the zero ideal $\{0\}$ of X is normal, or if and only if $x * (x * y) = 0$ implies $y * (y * x) = 0$ for any $x, y \in X$.*

Note that every normal ideal of X contains all minimal elements of X, we obtain

Corollary 3.10. *A BCI-algebra X is normal if and only if the intersection of all normal ideals of X is exactly the p-semisimple part P of X.*

Combining Corollaries 2.9 with 3.9, we also obtain

Corollary 3.11. *A BCI-algebra X is normal if and only if the zero ideal $\{0\}$ of X is a normal ideal of the BCK-part B of X and the p-semisimple part P of X is an ideal of X.*

Finally, we remark that the following assertion is not true: if X is a BCI-algebra which is not p-semisimple and if A is a nonzero normal ideal of X, then $A \cap B \neq \{0\}$ where B is the BCK-part of X (refer to [4, Proposition 2.28]). For instance, the algebra X in Example 3.1 is not p-semisimple and $\{0, a\}$ is a nonzero normal ideal of it, but $\{0, a\} \cap B = \{0\}$ where $B = \{0, 1\}$. Following the ideas of this assertion, we give the next assertion.

Proposition 3.12. *A BCI-algebra X is not normal if and only if $A \cap B \neq \{0\}$ where A is* *an arbitrary normal ideal of X and B is the BCK-part of X.*

Proof. Assume that X is not a normal BCI-algebra and A is a normal ideal of X. By Theorem 3.8, $A \neq P$ where P is the p-semisimple part of X, then P is properly contained in A by Proposition 3.5. Putting $x \in A - P$ and letting a be a minimal element of X such that $a \leq x$, we have $x * a \neq 0$ and $x * a \in B$. Also, since A is a closed ideal of X and $a \in P \subset A$, we obtain $x * a \in A$. Hence $0 \neq x * a \in A \cap B$ and $A \cap B \neq \{0\}$.

Conversely, if it is false, then X is normal. By Theorem 3.8, P is a normal ideal of X. However, $P \cap B = \{0\}$, a contradiction with our assumption of sufficiency.

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