WEAKLY ALMOST PERIODICITY AND CHAOS *

LIDONG WANG , ZHENYAN CHU, GONGFU LIAO AND XIAODONG DUAN

Received June 6, 2002; revised June 24, 2003

ABSTRACT. In this paper , we consider a continuous map $f : X \to X$, where X is a compact metric space , and discuss the existence of chaotic set of f , specially (as X=[0,1]) . We prove that f has a positively topological entropy if and only if it has an uncountably chaotic set in which each point is weakly almost periodic and is not almost periodic .

1 . INTRODUCTION

Interference or false appearance often appears in the discussion of dynamical systems . As we all know , importantly dynamical properties of a system focus on its non-wandering set , a wandering set can be regarded as a kind of interference . This point of view can be interpreted reasonably from ergodic theory , since a wandering set is a set of absolute measure zero[4] and the phenomenon which happens on it is unimportant or false , but the wandering set is not all the interference of the system . To obtain a subsystem which not only can get rid of all interference but also can make the importantly dynamical properties of original system invariant , Zhou[4] introduced measure center . To decide the concept of measure center , he defined weakly almost periodic point too , showed that the closure of a set of weakly almost periodic points equals to its measure center and the set of weakly almost periodic points is a set of absolutely ergodic measure 1[4] . These show that it is more significant to discuss problems on a set of weakly almost periodic points .

Throughout this paper , X will denote a compact metric space with metric d , I is the closed interval [0,1] .

For a continuous map $f : X \to X$, we denote the set of periodic points , almost periodic points , weakly almost periodic points , recurrent points , non-wandering points and chain recurrent points of f by $P(f)$, $A(f)$, $W(f)$, $R(f)$, $\Omega(f)$ and $CR(f)$ respectively , denote the topological entropy of f by $ent(f)$, whose definitions are as usual ; f^n will denote the n-fold iterate of f .

In this paper , we first derive a sufficient condition for a map having an uncountably chaotic set in which each point is weakly almost periodic and is not almost periodic from Theorem A . As an application , we prove Theorem B .

The main result are stated as follows .

THEOREM A. *Let $f : X \to X$ be continuous . If f has an almost shift invariant set , it has an uncountably chaotic set in which each point is weakly almost periodic and is not almost periodic .*

THEOREM B. *Let $f : I \to I$ be continuous , then $ent(f) > 0 \Leftrightarrow$ there exists an uncountably chaotic set of f in which each point is weakly almost periodic and is not almost periodic .*

*Project supported by the National Natural Science Foundation of China.
2000 *Mathematics Subject Classification.* 37B10, 37B40, 74H65.
Key words and phrases. weakly almost periodic point, chaos, shift.

2 . Basic definitions and preparations

Let $S=\{0,1\}$, $\Sigma=\{x=x_0x_1\cdots \mid x_i\in S, i=0,1,2,\cdots\}$ and define $\rho:\Sigma\times\Sigma\to R$ as follows : for any $x,y\in\Sigma$, if $x=x_0x_1\cdots$ and $y=y_0y_1\cdots$ then

$$\rho(x,y)=\begin{cases}0 & if\;\; x=y\\ \frac{1}{2^k} & if\;\; x\neq y,\;\; and\;\; k=\min\{n\mid x_n\neq y_n\}\end{cases}$$

It is not difficult to check that ρ is a metric on Σ .The space (Σ,ρ) is a compact metric space and called the one-sided symbolic space with two symbols .

Define $\sigma:\Sigma\to\Sigma$ by $\sigma(x_0x_1\cdots)=x_1x_2\cdots$ for any $x=x_0x_1\cdots\in\Sigma$, then σ is a continuous map on Σ and called the shift on the one-sided symbolic space Σ . Hence (Σ,σ) is a compact system .

Definition 2.1. *$D\subset X$ is said to be a chaotic set of f , if for any different points x , $y\in D$,*

$$\liminf_{n\to\infty} d\,(f^n(x),f^n(y)\,)=0 \quad \limsup_{n\to\infty} d\,(f^n(x),f^n(y)\,)>0\ .$$

f is said to be chaotic if it has a chaotic set which is uncountable .

Definition 2.2. *$x\in X$ is called a weakly almost periodic point of f , if for any $\varepsilon>0$ there exists an integer $N_\varepsilon>0$ such that for any $n>0$,*

$$\#\,(\{\,r\mid f^r(x)\in V(x,\varepsilon)\,,\,0\le r<nN_\varepsilon\,\})\ge n\ ,$$

where $\#(\cdot)$ denotes the cardinal number of a set and $V(x,\varepsilon)$ the spherical neighborhood . Denote the set of weakly almost periodic points of f by $W(f)$. It is easy to see that

$$P(f)\subset A(f)\subset W(f)\subset R(f)\subset\overline{P(f)}\subset\Omega(f)\subset CR(f)\ .$$

Definition 2.3. *A compact set $\Lambda\subset X$ is said to be almost shift invariant if :*

(1) $f(\Lambda)\subset\Lambda$,

(2) There exists a continuous surjection $h:\Lambda\to\Sigma$ satisfying the following conditions :

a) Set$\{\,y\in\Sigma\mid h^{-1}(y)$ contains at least two points $\}$ is countable .

b) $h\circ f|_\Lambda=\sigma\circ h$.

Definition 2.4. *Let $x\in\Sigma$ with $x=x_0x_1\cdots$. It is called a repeating sequence with recurring period of length m if $x_{m+t}=x_t$ for any $t\in\{0,1,2,\cdots\}$, we then write $x=(\dot{x}_0\dot{x}_1\cdots\dot{x}_{m-1})$.*

Definition 2.5. *$Y\subset X$ is called a minimal set of f if for any $x\in Y$, $\omega(x,f)=Y$.*

Lemma 2.1. *Let (Σ,ρ) be the one-sided symbolic space , and let σ be the shift on Σ. Then*

(1) For any $s\in\Sigma$ and any $m>0$, $\sigma^m(s)=s$ if and only if $s=(\dot{s}_0\dot{s}_1\cdots\dot{s}_{m-1})$ (i.e. s is a repeating sequence with recurring period of length m) .

(2)There are exactly 2^n elements s in Σ such that $\sigma^n(s)=s$.

For a proof see [8] .

Lemma 2.2. *Let $f:X\to X$, $g:Y\to Y$ be continuous , where X and Y are both compact metric spaces . If there exists a continuous surjection $h:X\to Y$ such that $g\circ h=h\circ f$, then*

(1) $h(W(f))=W(g)$,

(2) $h(A(f))=A(g)$.

For a proof see [5] .

Lemma 2.3. *$W(f^m) = W(f)$ and $A(f^m) = A(f)$ for any $m > 0$.* For a proof see [2] and [4] .

Lemma 2.4. *For any $x \in X$ and any $N > 0$, the following are equivalent :*

(1) $x \in A(f)$.

(2) $x \in \omega(x, f)$ and $\omega(x, f)$ is a minimal set of f . For a proof see [10] and [11] .

Lemma 2.5. *One-sided shift is Li-Yorke chaotic and there is a chaotic set $\mathcal{T}$ satisfying $\mathcal{T} \subset W(\sigma) - A(\sigma)$.*

Proof: Let p_1 be an arrangement of the recurring periods of two repeating sequences of length 1 such that $\sigma(s) = s$, *e.g.* $p_1 = 01$.

Let p_2 be an arrangement of the recurring periods of the 2^2 repeating sequences of length 2 such that $\sigma^2(s) = s,$ *e.g.* $p_2 = 00\,01\,10\,11$.

p_n is an arrangement of the recurring periods of the 2^n repeating sequences of length n such that $\sigma^n(s) = s,$ *e.g.* $p_n = 000 \cdots 0 \cdots 11 \cdots 1$.

Let $a = p_1 p_2 \cdots p_n \cdots = 0100011011000 \cdots 001 \cdots = a_0 a_1 \cdots a_n \cdots$. It is essy to see that $\omega(a, \sigma) = \Sigma$. In fact , for any $x = x_0 x_1 x_2 \cdots \in \Sigma$, Let T_n be a periodic point of σ with period n and with recurring period $(\dot{x}_0 \dot{x}_1 \cdots \dot{x}_{n-1})$. Then $T_n \to x\,(n \to \infty)$. By the construction of a , for any $\varepsilon > 0$, there exists $N_i(\varepsilon)$ such that

$$|\sigma^{N_i}(a) - x| < \sum_{n=N_i}^{\infty} \frac{1}{2^n} < \varepsilon$$

So $\sigma^{N_i}(a) \to x\,(i \to \infty)$. This shows $x \in \omega(a, \sigma)$, *i.e.* $\Sigma \subset \omega(a, \sigma)$. On the other hand , $\omega(a, \sigma) \subset \Sigma$, hence $\Sigma = \omega(a, \sigma)$.

The following we will construct a set B_a .

We use $(x)_n$ to denote the first $n+1$ symbols of x for any $x \in \Sigma$, *i.e.* $(x)_n = (x_0 x_1 \cdots x_n)$ for any $n \geq 0$. Let

$$B_a = \{ b_a = (a)_0 (b)_0 (a)_1 (b)_1 \cdots (a)_n (b)_n \cdots \mid \forall\, b = b_0 b_1 \cdots \in \Sigma ,\, a = a_0 a_1 \cdots \} ,$$

then B_a is an uncountable set .

(1) We will construct an uncountable set $\mathcal{T}$.

For any $e = e_0 e_1 \cdots e_n \cdots \in B_a$, let $Q_0 = (0)$, $Q_1 = (Q_0 Q_0 e_0)$, $Q_2 = (Q_1 Q_1 e_1), \cdots$, $Q_n = (Q_{n-1} Q_{n-1} e_{n-1}) \cdots$. We take $x(e) \in \Sigma$ satisfying

$$(x(e))_{2^{n+1}-2} = Q_n = (Q_{n-1} Q_{n-1} e_{n-1}) \quad for\ n = 1, 2, \cdots .$$

Let $\mathcal{T} = \{\, x(e) \,|\, \forall\, e \in B_a \,\}$. Obviously $\mathcal{T}$ is an uncountable set .

(2) We will prove that $\sigma|_{\mathcal{T}}$ is chaotic .

Firstly , we put $n_i = 2^{(i+1)^2+1} - 1 - (i+1)^2 + i(i+1)$ $(i = 1, 2, \cdots)$, then for any $x \in \mathcal{T}$, $x = x(e)$ and $e \in B_a$,

$$\sigma^{n_i}(x(e)) = \sigma^{i(i+1)}((e)_{(i+1)^2-1} \cdots) = ((a)_i \cdots) .$$

Moreover ,

$$\rho(\sigma^{n_i}(x(e)), a) \leq \sum_{n=i}^{+\infty} \frac{1}{2^n} = \frac{1}{2^{i-1}} .$$

Hence ,

$$\lim_{i\to+\infty}\rho(\sigma^{n_i}(x(e)),a)=0 \quad (i.e.\ \lim_{i\to+\infty}\sigma^{n_i}(x(e))=a\ ) .$$

Since $\omega(a,\sigma)=\Sigma$, by the arbitrariness of x , for any $x,y\in\mathcal{T}$, we obtain

$$\liminf_{n\to\infty}\rho(\sigma^n(x),\sigma^n(y))=0 .$$

Secondly , we put $q_i=2^{(2i+1)^2+1}-1-(2i+1)^2+(i+1)^2$ $(i=1,2,\cdots)$, then for any $x\in\mathcal{T}$, $x=x(e)$ and $e\in B_a$,

$$\sigma^{q_i}(x)=\sigma^{q_i}(x(e))=\sigma^{(i+1)^2}((e)_{(2i+1)^2-1}\cdots)=((b)_i\cdots) .$$

For any $x,y\in\mathcal{T}$, $x\neq y$, where $x=x(\overline{\beta})$, $y=y(\overline{\gamma})$, $\overline{\beta}\in B_a$, $\overline{\gamma}\in B_a$, $\overline{\beta}=(a)_0(\beta)_0(a)_1(\beta)_1\cdots$, $\overline{\gamma}=(a)_0(\gamma)_0(a)_1(\gamma)_1\cdots$. By the construction of $\mathcal{T}$, we know that

$$\beta=\beta_0\beta_1\beta_2\cdots\neq\gamma=\gamma_0\gamma_1\gamma_2\cdots ,$$

so

$$\lim_{i\to+\infty}\rho(\sigma^{q_i}(x),\sigma^{q_i}(y))=\rho(\beta,\gamma)>0 ,$$

moreover ,

$$\limsup_{n\to\infty}\rho(\sigma^n(x),\sigma^n(y))>0 .$$

Summing up , $\mathcal{T}$ is a chaotic set , *i.e.* $\sigma|_{\mathcal{T}}$ is chaotic .

(3) We will prove that $\mathcal{T}\subset W(\sigma)-A(\sigma)$.

For any $x(e)\in\mathcal{T}$ where $e\in B_a$, $e=e_0e_1\cdots e_n\cdots\in\Sigma$, and for any given $\varepsilon>0$, there exists $m\in Z^+$ such that $0<\dfrac{1}{2^m}<\varepsilon$, let $N_\varepsilon=2^{m+2}$, since for any $n>0$ with $n\in Z^+$, there exists a nonnegative p such that $2^p\leq n<2^{p+1}$, we have

$$2^pN_\varepsilon\leq nN_\varepsilon<2^{p+1}N_\varepsilon \quad (i.e.\ 2^{m+p+2}\leq nN_\varepsilon<2^{m+p+3}\ ) .$$

Since there are at least two Q_m in $Q_{m+1}=(Q_mQ_me_m)$, generally , we have that there are at least 2^n Q_m in Q_{m+n} for $n=1,2,\cdots$, and since $2^{m+p+2}\leq nN_\varepsilon$, we obtain that the first $nN_\varepsilon+1$ symbols of $(x(e))$, $(x(e))_{nN_\varepsilon+1}=(x_0x_1\cdots x_{nN_\varepsilon})$, contain Q_{m+p+1} , so

$$\#(\{\,r\,|\,\sigma^r(x(e))\in V(x(e),\varepsilon)\,,\,0\leq r<nN_\varepsilon\,\})>2^{p+1}>n .$$

Hence , by the definition of $W(\sigma)$, we know that $x(e)\in W(\sigma)$, and by (2) we know that

$$\lim_{i\to+\infty}\sigma^{n_i}(x(e))=a \ \ and\ \ \omega(a,\sigma)=\Sigma ,$$

then we have $\omega(x(e),\sigma)=\Sigma$. Since Σ is not a minimal set of σ , by Lemma 2.4 $x(e)\notin A(\sigma)$, furthermore $x(e)\in W(\sigma)-A(\sigma)$. By the arbitrariness of x , we can obtain $\mathcal{T}\subset W(\sigma)-A(\sigma)$.

3 . Proofs of the main theorem

Proof of theorem A : By the hypothesis of the theorem , f has an almost shift invariant set Λ , thus there is a continuous surjection $h:\Lambda\to\Sigma$ such that for any $x\in\Lambda$,

$$\sigma\circ h(x)=h\circ f|_\Lambda(x) .$$

By Lemma 2.5 , there is an uncountably chaotic set $\mathcal{T} \subset W(\sigma) - A(\sigma)$, for simplicity, let $g = f|_\Lambda$. By Lemma 2.2 , for each $y \in \mathcal{T}$, we take an $x \in W(g) - A(g)$ such that $h(x) = y$. All of these points form an uncountable subset of Λ , which we will denote it by D . To complete the proof of the theorem , it suffices to show that D is a chaotic set of f .

For any $x_1, x_2 \in D$, there exists $y_1, y_2 \in \mathcal{T}$ such that $h(x_i) = y_i$ for $i = 1, 2$.

Firstly , it is easily seen that

$$\limsup_{n\to\infty} \rho(\sigma^n(y_1), \sigma^n(y_2)) > 0 \quad (\textit{i.e.} \limsup_{n\to\infty} \rho(hg^n(x_1), hg^n(x_2)) > 0 \ ) \ ,$$

implies

$$\limsup_{n\to\infty} d(g^n(x_1), g^n(x_2)) > 0 \ .$$

Secondly , since Λ is an almost shift invariant set of g and $\mathcal{T}$ is uncountable , it follows that there exists $y_0 \in \Sigma$ such that $h^{-1}(y_0)$ contains only one point x_0 . By the chaotic property and construction of $\mathcal{T}$, there exists $m_i \to +\infty$ such that

$$\lim_{i\to+\infty} \sigma^{m_i}(y_1) = \lim_{i\to+\infty} \sigma^{m_i}(y_2) = y_0 \ ,$$

which implies

$$\lim_{i\to+\infty} g^{m_i}(x_1) = \lim_{i\to+\infty} g^{m_i}(x_2) = x_0 \ .$$

Thus ,

$$\liminf_{n\to\infty} d(g^n(x_1), g^n(x_2)) = 0 \ .$$

Since $g = f|_\Lambda$, we can see that D is a chaotic set of f .

Proof of theorem B : Since $ent(f) > 0$, by [1] , form some $N > 0$, f^N has an almost shift invariant set (cf. the proof prop. 15 of chap.II in [1]) . It follows from Theorem A that f^N has an uncountably chaotic set , say D , in which each point is weakly almost periodic and is not almost periodic under f^N . Obviously D is also a chaotic set of f . And by Lemma 2.3 , $D \subset W(f) - A(f)$. Hence the result follows .

By [9] , $ent(f) > 0 \Leftrightarrow R(f)$ contains an uncountably chaotic set of f , and $W(f) \subset R(f)$, sufficient condition is proved .

References

[**1**] L.S.Block and W.A.Coppel , *Dynamics in One Dimension* , Lecture Notes in Math . **1513** , Springer , Berlin , 1992.

[**2**] G.F.Liao and L.Y.Wang , Almost Periodicity , Chain Recurrence and Chaos , *Israel. J. Math.* **93** (1996) , 145-156.

[**3**] H.J.Jiang , A Note on Weakly Almost Periodic Point , *J.of Math. (PRC)* , **19** (1999) , 56-60(in Chinese).

[**4**] Z.L.Zhou , Weakly Almost Points and Measure Center , *Science of China (A)* , **22(6)** (1992) , 572-581(in Chinese).

[**5**] Z.L.Zhou and W.H.He , Administrative Levels of Orbit Structure and Topological Semi-conjugate , *Science of China (A)* , **25(5)** (1995) , 457-464(in Chinese).

[**6**] Z.L.Zhou , Chaos and Topological Entropy , *Acta Math. Sinica* , **31** (1998) , 83-87(in Chinese).

[**7**] Z.L.Zhou , G.F.Liao and L.Y.Wang , The Positively Topological Entropy is not Equivalent to Chaos-a Class of Subshift , *Science of China (A)* , **37** (1994) , 653-660(in Chinese).

[**8**] R.L.Devaney , *An Introduction to Chaotic Dynamical Systems* , 2nd ed. , Addison-Wesley , Redwood City , CA , 1989.

[**9**] B. S. Du , Every Chaotic Interval Map has a Scrambled Set in the Recurrent Set, *Bull. Austral. Math. Soc.* **39** (1989) , 259-264.

[**10**] P.Erdős and A.H.Stone , Some Remarks on Almost Periodic Transformations, *Bull. Amer. Math. Soc.* **51** (1945) , 126-130.

[**11**] W.H.Gottschalk , Orbit-closure Decompositions and Almost Periodic Properties, *Bull. Amer. Math. Soc.* **50** (1944) , 915-919.

Institute of Nolinear information technology
Dalian Nationalities University
Dalian 116600, P.R.China
E-mail:wldwyylfx@hotmail.com
Institute of Nolinear information technology
Dalian Nationalities University
Dalian 116600, P.R.China
E-mail:chuzhenyan8@sohu.com
Institute of Mathematics
Jilin University
Changchun 130023, P.R. China
E-mail:liaogf@email.jlu.edu.cn
and
Institute of Nolinear information technology
Dalian Nationalities University
Dalian 116600, P.R.China