ON POSITIVE IMPLICATIVE HYPER BCK-IDEALS

R. A. BORZOOEI AND M. BAKHSHI

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ABSTRACT. In this note first we define the notions of *positive implicative hyper BCK-ideals of types 1,2,...,8*. Then, giving some examples, we show that these notions are different. After that we state and prove some theorems which determine the relationship between these notions and (strong, weak) hyper BCK-ideals. Finally will be presented, a classification of hyper BCK-algebras of order 3.

1. Introduction

The study of BCK-algebras was initiated by Y. Imai and K. Iséki[5] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculi. The hyperstructure theory was introduced in 1934 by F. Marty [11]. In [8], Y.B. Jun, M.M. Zahedi, R.A. Borzooei et al. applied the hyperstructures to BCK-algebras, and introduced the notion of a hyper BCK-algebra which is a generalization of BCK-algebra, and investigated some related properties. Now we follow [1,2,10] and obtain some results, as mentioned in the abstract.

2. Preliminaries

Definition 2.1 (8)**.** By a *hyper BCK-algebra* we mean a nonempty set H endowed with a hyperoperation "$\circ$" and a constant 0 satisfies the following axioms:
(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x \circ H \ll \{x\}$,
(HK4) $x \ll y$ and $y \ll x$ imply $x = y$.
for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$. In such a case, we call "$\ll$" the *hyperorder* in H.

Example 2.2 (8)**.** (i) Let $H = \{0, 1, 2\}$. Consider the following table:

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0\}$	$\{0\}$
3	$\{3\}$	$\{3\}$	$\{2\}$	$\{0, 2\}$

Then $(H, \circ, 0)$ is a hyper BCK-algebra.

Proposition 2.3 (8,9)**.** *In any hyper BCK-algebra H, the following hold:*
(i) $0 \circ 0 = \{0\}$,

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(ii) $0 \ll x$,
(iii) $x \ll x$,
(iv) $A \ll A$,
(v) $A \subseteq B$ *implies* $A \ll B$,
(vi) $0 \circ x = \{0\}$,
(vii) $0 \circ A = \{0\}$,
(viii) $A \ll \{0\}$ *implies* $A = \{0\}$,
(ix) $x \circ y \ll x$,
(x) $x \circ 0 = \{x\}$,
(xi) $A \circ \{0\} = \{0\}$ *implies* $A = \{0\}$,
(xii) $y \ll z$ *implies* $x \circ z \ll x \circ y$,
(xiii) $x \circ y = \{0\}$ *implies* $(x \circ z) \circ (y \circ z) = \{0\}$ *and* $x \circ z \ll y \circ z$.

for all $x, y, z \in H$ *and for all nonempty subsets* A *and* B *of* H.

Definition 2.4 (8)**.** Let H be a hyper BCK-algebra and let S be a subset of H containing 0. If S is a hyper BCK-algebra with respect to the hyperoperation "$\circ$" on H, we say that S is a *hypersubalgebra* of H.

Theorem 2.5 (8)**.** *Let S be a nonempty subset of a hyper BCK-algebra H. Then S is a hypersubalgebra of H if and only if $x \circ y \subseteq S$ for all $x, y \in S$.*

Definition 2.6 (7,8)**.** Let I be a nonempty subset of a hyper BCK-algebra H and $0 \in I$. Then I is said to be a *hyper BCK-ideal* of H if for all $x, y \in H$, $x \circ y \ll I$ and $y \in I$ imply $x \in I$, *weak hyper BCK-ideal* of H if for all $x, y \in H$, $x \circ y \subseteq I$ and $y \in I$ imply $x \in I$,*strong hyper BCK-ideal* of H if for all $x, y \in H$, $(x \circ y) \cap I \neq \emptyset$ and $y \in I$ imply $x \in I$.

Theorem 2.7 (7,8)**.** *Let I be a nonempty subset of a hyper BCK-algebra H. Then the following statements are hold.*
(i) Any strong hyper BCK-ideal of H is a hyper BCK-ideal.
(ii) Any hyper BCK-ideal of H is a weak hyper BCK-ideal.

Definition 2.8 (9)**.** Let H be a hyper BCK-algebra. An element $a \in H$ is said to be *left* (resp. *right*) *scalar* if $|a \circ x| = 1$ (resp. $|x \circ a| = 1$) for all $x \in H$. If $a \in H$ is both left and right scalar, we say that a is a *scalar* element.

Theorem 2.9. *There are 19 non-isomorphic hyper BCK-algebras of order 3.*

Definition 2.10 (3)**.** Let $H = \{0, 1, 2\}$ be a hyper BCK-algebra of order 3. Then we say that H satisfies the normal condition, if one of the conditions $1 \ll 2$ or $2 \ll 1$ holds. If none of them hold, then we say that H satisfies the simple condition.

Lemma 2.11 (3)**.** *Let $H = \{0, 1, 2\}$ be a hyper BCK-algebra of order 3. Then,*

(a) *If H satisfies the simple condition, then*
(i) $1 \circ 1 = \{0\}$ *or* $\{0, 1\}$ *and* $1 \circ 2 = \{1\}$,
(ii) $2 \circ 1 = \{2\}$ *and* $2 \circ 2 = \{0\}$ *or* $\{0, 2\}$.
(b) *If H satisfies the normal condition, then*
(iii) $1 \circ 1 = \{0\}$ *or* $\{0, 1\}$,
(iv) $1 \circ = \{0\}$ *or* $\{0, 1\}$,
(v) $2 \circ 1 = \{1\}$, $\{2\}$, $\{1, 2\}$,
(vi) $2 \circ 2 = \{0\}$, $\{0, 1\}$, $\{0, 2\}$ *or* $\{0, 1, 2\}$.

Theorem 2.12 (3)**.** *Let H be a hyper BCK-algebra of order 3 which satisfies the normal condition. Then H has at most one proper hyper BCK-ideal which is $I = \{0, 1\}$.*

3. **Positive implicative hyper BCK-ideals**

Note. From now on in this paper we let H denote a hyper BCK-algebra.

Definition 3.1. Let I be a nonempty subset of H and $0 \in I$. Then I is said to be a *positive implicative hyper BCK-ideal* of

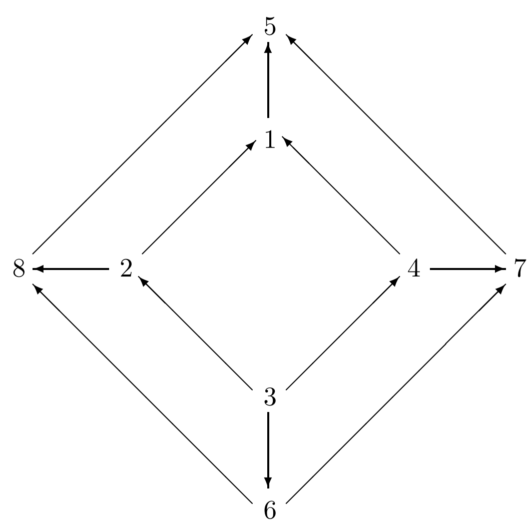
(i) *type 1*, if $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$ implies that $x \circ z \subseteq I$ for all $x, y, z \in H$,
(ii) *type 2*, if $(x \circ y) \circ z \ll I$ and $y \circ z \subseteq I$ implies that $x \circ z \subseteq I$ for all $x, y, z \in H$,
(iii) *type 3*, if $(x \circ y) \circ z \ll I$ and $y \circ z \ll I$ implies that $x \circ z \subseteq I$ for all $x, y, z \in H$,
(iv) *type 4*, if $(x \circ y) \circ z \subseteq I$ and $y \circ z \ll I$ implies that $x \circ z \subseteq I$ for all $x, y, z \in H$,
(v) *type 5*, if $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$ implies that $x \circ z \ll I$ for all $x, y, z \in H$,
(vi) *type 6*, if $(x \circ y) \circ z \ll I$ and $y \circ z \ll I$ implies that $x \circ z \ll I$ for all $x, y, z \in H$,
(vii) *type 7*, if $(x \circ y) \circ z \subseteq I$ and $y \circ z \ll I$ implies that $x \circ z \ll I$ for all $x, y, z \in H$,
(viii) *type 8*, if $(x \circ y) \circ z \ll I$ and $y \circ z \subseteq I$ implies that $x \circ z \ll I$ for all $x, y, z \in H$.

Theorem 3.2. *Let I be a nonempty subset of H. Then,*

(i) *If I is a positive implicative hyper BCK-ideal of type 3, then I is a positive implicative hyper BCK-ideal of types 2,4 and 6,*
(ii) *If I is a positive implicative hyper BCK-ideal of type 2, then I is a positive implicative hyper BCK-ideal of types 1 and 8,*
(iii) *If I is a positive implicative hyper BCK-ideal of type 4, then I is a positive implicative hyper BCK-ideal of types 1 and 7,*
(iv) *If I is a positive implicative hyper BCK-ideal of type 6, then I is a positive implicative hyper BCK-ideal of types 7 and 8,*
(v) *If I is a positive implicative hyper BCK-ideal of type 1 (type 7,8), then I is a positive implicative hyper BCK-ideal of type 5.*

Proof. (i) Let I be a positive implicative hyper BCK-ideal of type 3 and $(x \circ y) \circ z \ll I$ and $y \circ z \subseteq I$, for $x, y, z \in H$. Then by Proposition 2.3(v), $y \circ z \ll I$ and so by hypothesis $x \circ z \subseteq I$. Therefore I is a positive implicative hyper BCK-ideal of type 2. The proofs of types 4 and 6 are similar. The proofs of other cases are similar to the above by suitable modifications. □

In the following diagram, we can see the summary of the Theorem 3.2.



Example 3.3. (i) Let H be the hyper BCK-algebra which is defined as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

Then $I=\{0,1\}$ is a positive implicative hyper BCK ideal of type 1,4,6 and 8, but it is not of type 2 and 3.

(ii) Let $H=\{0,1,2,3\}$ .Consider the following tables,

$\circ_1$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{1\}$	$\{0\}$	$\{0\}$
3	$\{3\}$	$\{1\}$	$\{1\}$	$\{0,1\}$

$\circ_2$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{1\}$	$\{0\}$	$\{0\}$
3	$\{3\}$	$\{2\}$	$\{1\}$	$\{0,1\}$

Then $(H,\circ_1)$ and $(H,\circ_2)$ are hyper BCK-algebras and $I=\{0,2\}$ is a positive implicative hyper BCK-ideal of type 7 of $(H,\circ_1)$, not of type 4 and 6, but it is a positive implicative hyper BCK-ideal of type 5 of $(H,\circ_2)$, not of other types.

Note. Nonempty subset I of H is proper, if $\{0\}\neq I\neq H$.

Theorem 3.4. *Let $H=\{0,1,2\}$ be a hyper BCK-algebra of order 3 and I be a proper subset of H. Then,*

(i) I is a positive implicative hyper BCK-ideal of type 1 if and only if I is a positive implicative hyper BCK-ideal of type 4.

(ii) I is a positive implicative hyper BCK-ideal of type 5 if and only if I is a positive implicative hyper BCK-ideal of type 6,7 or 8.

(iii) H has at least one proper positive implicative hyper BCK-ideal of type 5.

Proof. (i) By Theorem 3.2(iii), every positive implicative hyper BCK-ideal of type 4 is a positive implicative hyper BCK-ideal of type 1.

Conversely, let $I_1=\{0,1\}$ be a positive implicative hyper BCK-ideal of type 1. Let $(x\circ y)\circ z\subseteq I_1$ and $y\circ z\ll I_1$ but $x\circ z\not\subseteq I_1$. Then $2\in x\circ z$ and so $x\neq 0$. Since if $x=0$, then $2\in x\circ z=0\circ z=\{0\}$, which is impossible. Now we consider the following two cases.

Case 1. H satisfies the simple condition.

Then by Lemma 2.11(a), $x\neq 1$. Thus $x=2$ and so

$$(2\circ y)\circ z\subseteq I_1 \text{ and } y\circ z\ll I_1$$

Since $2\in x\circ z=2\circ z$, then by Lemma 2.11(a), $z=1$ or 2.

If $z=1$, then

$$(2\circ y)\circ 1\subseteq I_1 \text{ and } y\circ 1\ll I_1$$

Now, if $y=0$ then by Lemma 2.11(ii), $2\in 2\circ 1=(2\circ 0)\circ 1\subseteq I_1$, which is impossible. If $y=1$, then by Lemma 2.11(ii), $2\in 2\circ 1\subseteq(2\circ 1)\circ 1\subseteq I_1$, which is a contradiction. If $y=2$, then $2\in 2\circ 1\ll I_1=\{0,1\}$. Hence $2\ll 1$, which is impossible.

If $z=2$, then $(2\circ y)\circ 2=(2\circ y)\circ z\subseteq I_1$. Since $2\in x\circ z=2\circ 2$, then by Lemma 2.11(ii), $2\circ 2=\{0,2\}$. If $y=0$, then $2\in 2\circ 2\subseteq(2\circ 0)\circ 2\subseteq I_1$, which is impossible. If $y=1$, then $2\in 2\circ 2\subseteq(2\circ 1)\circ 2\subseteq I_1$, which is a contradiction. If $y=2$ then $2\in 2\circ 2\subseteq(2\circ 2)\circ 2\subseteq I_1$, which is impossible.

Case 2. H satisfies the normal condition.

Since $2\in x\circ z$, then by Lemma 2.11(b), $x=2$ and so by (HK2)

$$2\circ y\subseteq(x\circ z)\circ y=(x\circ y)\circ z\subseteq I_1$$

If $y = 0$, then $2 \in 2 \circ 0 = 2 \circ y \subseteq I_1$, which is impossible. If $y = 1$, then $(2 \circ 1) \circ 0 = 2 \circ 1 = 2 \circ y \subseteq I_1$. Since I_1 is a positive implicative hyper BCK-ideal of type 1 and $1 \circ 0 = \{1\} \subseteq I_1$, then $\{2\} = 2 \circ 0 \subseteq I_1$, which is a contradiction. If $y = 2$, then $2 \circ 2 = 2 \circ y \subseteq I_1$ and so by Lemma 2.11(vi), $2 \circ 2 = \{0\}$ or $\{0, 1\}$. Since $2 \in x \circ z = 2 \circ z$, then $z \neq 2$. Thus $z = 0$ or 1. Moreover, $2 \circ z = y \circ z \ll I_1$. If $z = 0$, then $\{2\} = 2 \circ 0 \ll I_1 = \{0, 1\}$. Thus $2 \ll 1$ and so $0 \in 2 \circ 1$, which is a contradiction. If $z = 1$, then $2 \circ 1 = 2 \circ z \ll I_1 = \{0, 1\}$ and so by Lemma 2.11(v), $2 \circ 1 = \{1\}$. Thus $(2 \circ 1) \circ 0 = 2 \circ 1 = \{1\} \subseteq I_1$. Since I_1 is a positive implicative hyper BCK-ideal of type 1 and $1 \circ 0 = \{1\} \subseteq I_1$, then $\{2\} = 2 \circ 0 \subseteq I_1$ which is impossible. Therefore, we prove that $I_1 = \{0, 1\}$ is a positive implicative hyper BCK-ideal of type 4.

Now, let $I_2 = \{0, 2\}$ be a positive implicative hyper BCK-ideal of type 1, $(x \circ y) \circ z \subseteq I_2$ and $y \circ z \ll I_2$ but $x \circ z \not\subseteq I_2$. Then $1 \in x \circ z$ and so $x \neq 0$. Since if $x = 0$, then $1 \in x \circ z = 0 \circ z = \{0\}$ which is impossible. Now we considering two following cases.

Case 1. H satisfies the simple condition.

Then by Lemma 2.11(a), $x = 1$ and so $1 \in x \circ z = 1 \circ z$. Now we consider the following cases for z.

Case 1-1. If $z = 0$, then

$$1 \circ y = (1 \circ y) \circ 0 = (x \circ y) \circ z \subseteq I_2 \text{ and } \{y\} = y \circ 0 \ll I_2$$

Thus by Lemma 2.11(a), $y = 0$ or 2. If $y = 0$, then $1 \in 1 \circ 0 = 1 \circ y \subseteq I_2$, which is impossible. If $y = 2$, then by Lemma 2.11(a), $\{1\} = 1 \circ 2 = 1 \circ y \subseteq I_2$, which is impossible.

Case 1-2. If $z = 1$, then by (HK2) we get that

$$1 \circ y \subseteq (1 \circ 1) \circ y = (1 \circ y) \circ 1 = (x \circ y) \circ z \subseteq I_2$$

If $y = 0$, then $1 \in 1 \circ 0 = 1 \circ y \subseteq I_2$, which is impossible. If $y = 1$, then $1 \in x \circ z = 1 \circ 1 = 1 \circ y \subseteq I_2$, which is a contradiction. If $y = 2$, then by Lemma 2.11(a), $\{1\} = 1 \circ 2 = 1 \circ y \subseteq I_2$, which is impossible.

Case 1-3. If $z = 2$, then $y \circ 2 \ll I_2$ and so by Lemma 2.11(a), $y = 0$ or 2. Hence $y \circ 2 = \{0\}$ or $\{0, 2\}$ and this implies that $y \circ 2 \subseteq I_2$. Moreover, $(1 \circ y) \circ 2 = (x \circ y) \circ z \subseteq I_2$. Since I_2 is a positive implicative hyper BCK-ideal of type 1, then by Lemma 2.11(a) we get that $\{1\} = 1 \circ 2 \subseteq I_2$, which is a contradiction.

Case 2. Let H satisfy the normal condition.

Since $1 \in x \circ z$, then by (HK2) we get that

$$1 \circ y \subseteq (x \circ z) \circ y = (x \circ y) \circ z \subseteq I_2$$

If $y = 0$, then $1 \in 1 \circ 0 \subseteq I_2$, which is impossible. If $y = 2$, then $(1 \circ 2) \circ 0 = 1 \circ 2 \subseteq I_2$. Since I_2 is a positive implicative hyper BCK-ideal of type 1 and $2 \circ 0 = \{2\} \subseteq I_2$, then $\{1\} = 1 \circ 0 \subseteq I_2$, which is a contradiction. If $y = 1$, then $1 \circ 1 = 1 \circ y \subseteq I_2$ and so by Lemma 2.11(iii), $1 \circ 1 = \{0\}$. Moreover, $1 \circ 2 = \{0\}$ or $\{0, 1\}$. If $1 \circ 2 = \{0, 1\}$, then by (HK1) we get that

$$\{0, 1\} = (1 \circ 2) \circ (1 \circ 2) \ll 1 \circ 1 = \{0\}$$

which is impossible. If $1 \circ 2 = \{0\}$, then $(1 \circ 2) \circ 0 = \{0\} \subseteq I_2$ and $2 \circ 0 = \{2\} \subseteq I_2$. Since I_2 is a positive implicative hyper BCK-ideal of type 1, then $\{1\} = 1 \circ 0 \subseteq I_2$, which is a contradiction. Therefore, $I_2 = \{0, 2\}$ is a positive implicative hyper BCK-ideal of type 4.

(ii) Let I be a positive implicative hyper BCK-ideal of type 5 and

$$(x \circ y) \circ z \ll I \text{ and } y \circ z \ll I$$

but $x \circ z \not\ll I$. Then there exists $a \in x \circ z$ such that for all $s \in I$, $a \not\ll s$. If $I = \{0, 1\}$ then $a = 2$ and so by hypothesis, $2 \notin (x \circ y) \circ z$ and $2 \notin y \circ z$. But this implies that $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$. Since I is a positive implicative hyper BCK-ideal of type 5, then $x \circ z \ll I$ which is a contradiction. The proof of the case $I = \{0, 2\}$ is similar. Therefore, I is a

positive implicative hyper BCK-ideal of type 6 and so by Theorem 3.2(iv), I is a positive implicative hyper BCK-ideal of type 7 and 8.
(iii) We show that $I_2 = \{0, 2\}$ is a positive implicative hyper BCK-ideal of type 5, in any hyper BCK-algebra of order 3. Let $(x \circ y) \circ z \subseteq I_2$ and $y \circ z \subseteq I_2$, but $x \circ z \not\ll I_2$, for $x, y, z \in H$. Thus $1 \in x \circ z$ and $1 \not\ll 2$. Now we consider two following cases.
<u>Case 1</u>. Let H satisfy the simple condition.
Since $1 \in x \circ z$, then by (HK2) we get that

$$1 \circ y \subseteq (x \circ z) \circ y = (x \circ y) \circ z \subseteq I_2$$

If $y = 0$, then $\{1\} = 1 \circ 0 = 1 \circ y \subseteq I_2$, which is impossible. If $y = 2$, then by Lemma 2.11(i), $\{1\} = 1 \circ 2 = 1 \circ y \subseteq I_2$, which is a contradiction. If $y = 1$, then $1 \circ 1 = 1 \circ y \subseteq I_2$ and so by Lemma 2.11(i), $1 \circ 1 = \{0\}$. Moreover, by hypothesis we know that

$$(x \circ 1) \circ z \subseteq I_2 \text{ and } 1 \circ z \subseteq I_2$$

Since $1 \in x \circ z$, then by Lemma 2.11(a), $x = 1$ and $z = 0$ or $x = z = 1$ or $x = 1$ and $z = 2$. If $x = 1$ and $z = 0$, then $\{1\} = 1 \circ 0 = 1 \circ z \subseteq I_2$, which is a contradiction. If $x = z = 1$, then $1 \in x \circ z = 1 \circ 1 = \{0\}$, which is impossible.
If $x = 1$ and $z = 2$, then by Lemma 2.11(i), $\{1\} = 1 \circ 2 = 1 \circ z \subseteq I_2$, which is a contradiction.
<u>Case 2</u>. H satisfies the normal condition.
Since $1 \not\ll 2$, then $2 \ll 1$ and so $0 \in 2 \circ 1$. But by Lemma 2.11(v), $2 \circ 1 = \{1\}, \{2\}$ or $\{1, 2\}$, which is impossible. Therefore, $I_2 = \{0, 2\}$ is a positive implicative hyper BCK-ideal of type 5 in any hyper BCK-algebra of order 3. □

Definition 3.5. We say that subset I of H satisfies the *closed* condition, if $x \ll y$ and $y \in I$ implies $x \in I$, for all $x, y \in H$.

Lemma 3.6. *Let I and A be nonempty subsets of H and I satisfy the closed condition. If $A \ll I$, then $A \subseteq I$.*

Proof. The proof is easy. □

Theorem 3.7. *Let I be a nonempty subset of H and satisfies the closed condition. If I is a positive implicative hyper BCK-ideal of type i, then I is a positive implicative hyper BCK-ideal of type j, for all $1 \leq i, j \leq 8$.*

Proof. By considering the Lemma 3.6 the proofs are similar to the proof of Theorem 3.2, by some modification. □

Lemma 3.8. *Let I be a hyper BCK-ideal and A be a nonempty subset of H. Then,*
(i) If $A \ll I$, then $A \subseteq I$.
(ii) I satisfies the closed condition.

Proof. (i) Lemma 3.6[9]
(ii) The proof follows by (i). □

Theorem 3.9. *Let I be a nonempty subset of H. Then the following statements are held.*
(i) If I is a positive implicative hyper BCK-ideal of type 2(3), then I is a hyper BCK-ideal of H.
(ii) Let I be a hyper BCK-ideal of H. If I is a positive implicative hyper BCK-ideal of type i, then I is a positive implicative hyper BCK-ideal of type j, for $1 \leq i, j \leq 8$.
(iii) I is a positive implicative hyper BCK-ideal of type 3 if and only if I is a positive implicative hyper BCK-ideal of type 2.
(iv) If I is a positive implicative hyper BCK-ideal of type 1(4), then I is a weak hyper BCK-ideal of H.

Proof. (i) Let I be a positive implicative hyper BCK-ideal of type 2, $x \circ y \ll I$ and $y \in I$, for $x, y \in H$. Then $(x \circ y) \circ 0 = x \circ y \ll I$ and $y \circ 0 = \{y\} \subseteq I$. Hence by hypothesis $\{x\} = x \circ 0 \subseteq I$. Therefore, I is a hyper BCK-ideal of H. The proof of type 3 is similar.
(ii) By considering the Lemma 3.8(ii), the proof follows by Theorem 3.7.
(iii) ($\Rightarrow$) The proof follows by Theorem 3.2(i).
($\Leftarrow$) Let I be a positive implicative hyper BCK-ideal of type 2. Then by (i), I is a hyper BCK-ideal of H and so by (ii) I is a positive implicative hyper BCK-ideal of type 3.
(iv) The proof is similar to the proof of (i), by some modifications. □

Example 3.10. In example 2.2(ii), $I = \{0, 1\}$ is a hyper BCK-ideal (and so is a weak hyper BCK-ideal) of H but it is not a positive implicative hyper BCK-ideal of type $1, 3, \ldots, 8$.

Example 3.11. Let $H = \{0, 1, 2, 3\}$ be a hyper BCK-algebra which is defined as follows:

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{1\}$	$\{0, 1\}$	$\{0, 1\}$
3	$\{3\}$	$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\{0, 2, 3\}$

Then $I = \{0, 2\}$ is a positive implicative hyper BCK-ideal of type 5,6,7,8 but it is not a weak hyper BCK-ideal(and so is not a hyper BCK-ideal) of H, since $1 \circ 2 = \{0\} \subseteq I$ and $2 \in I$ but $1 \notin I$.

Lemma 3.12. *Let A,B and I are nonempty subsets of H. Then,*
(i) If I is a weak hyper BCK-ideal of H, then $A \circ B \subseteq I$ and $B \subseteq I$ imply $A \subseteq I$.
(ii) If I is a hyper BCK-ideal of H, then $A \circ B \ll I$ and $B \subseteq I$ imply $A \subseteq I$.

Proof. (i) The proof is easy.
(ii) The proof follows by (i) and Lemma 3.8(i). □

Theorem 3.13. *Let $H = \{0, 1, 2\}$ be a hyper BCK-algebra of order 3 and I be a nonempty subset of H. Then,*
(i) I is a positive implicative hyper BCK-ideal of type 3 if and only if I is a hyper BCK-ideal.
(ii) I is a positive implicative hyper BCK-ideal of type 1 if and only if I is a weak hyper BCK-ideal of H.

Proof. (i) By Theorem 3.9(i), any positive implicative hyper BCK-ideal of type 3 is a hyper BCK-ideal of H.

Conversely, let I be a hyper BCK-ideal of H. We consider the following two cases.
Case 1. H satisfies the normal condition.
By Theorem 2.12, H has at most one proper hyper BCK-ideal which is $I = \{0, 1\}$.
Now, let $I = \{0, 1\}$ be a hyper BCK-ideal of H. Then $2 \circ 1 \not\ll I$. Since $1 \in I$, if $2 \circ 1 \ll I$, then $2 \in I$, which is impossible. Hence $2 \in 2 \circ 1$ and so by Lemma 2.11(v), $2 \circ 1 = \{2\}$ or $\{1, 2\}$.
Now, let $(x \circ y) \circ z \ll I$ and $y \circ z \ll I$, but $x \circ z \not\subseteq I$. Then $2 \in x \circ z$. By Lemma 2.11(iii) and (iv), $x \neq 1$. Moreover, $x \neq 0$. Since if $x = 0$, then $2 \in x \circ z = 0 \circ z = \{0\}$, which is impossible. Thus $x = 2$. Since I is a hyper BCK-ideal of H, then by Lemma 3.8(i),

$$(x \circ y) \circ z \subseteq I \text{ and } y \circ z \subseteq I$$

Now, we consider the following cases.
Case 1-1. If $z = 0$, since $\{y\} = y \circ 0 = y \circ z \subseteq I$, then $y = 0$ or 1. If $y = 0$, then $\{2\} = (2 \circ 0) \circ 0 = (x \circ y) \circ z \subseteq I$, which is a contradiction. If $y = 1$, then $2 \in 2 \circ 1 =$

$(2 \circ 1) \circ 0 = (x \circ y) \circ z \subseteq I$, which is impossible.
Case 1-2. If $z = 1$, then $y \circ 1 = y \circ z \subseteq I$. Since I is a hyper BCK-ideal of H and $1 \in I$, then $y \in I$ and so $y = 0$ or 1. If $y = 0$, then by (HK2)

$$2 \in 2 \circ 1 = (2 \circ 1) \circ 0 = (2 \circ 0) \circ 1 = (x \circ y) \circ z \subseteq I$$

which is a contradiction. If $y = 1$, then

$$2 \in 2 \circ 1 \subseteq (2 \circ 1) \circ 1 = (x \circ y) \circ z \subseteq I$$

which is impossible.
Case 1-3. If $z = 2$, since $2 \in x \circ z$ and $x = z = 2$, then $2 \in 2 \circ 2$. Hence, by Lemma 2.11(vi), $2 \circ 2 = \{0, 2\}$ or $\{0, 1, 2\}$. If $y = 0$, then

$$2 \in 2 \circ 2 = (2 \circ 0) \circ 2 = (x \circ y) \circ z \subseteq I$$

which is a contradiction. If $y = 1$, then by (HK2)

$$2 \in 2 \circ 1 \subseteq (2 \circ 2) \circ 1 = (2 \circ 1) \circ 2 = (x \circ y) \circ z \subseteq I$$

which is impossible. If $y = 2$, then

$$2 \in 2 \circ 2 \subseteq (2 \circ 2) \circ 2 = (x \circ y) \circ z \subseteq I$$

which is impossible. Therefore, $x \circ z \subseteq I$ and so I is a positive implicative hyper BCK-ideal of type 3.
Case 2. H satisfies the simple condition.
By Theorem 3.1[3], there are only three following hyper BCK-algebras of order 3 which satisfy the simple condition.

$\circ_1$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0\}$

$\circ_2$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0, 2\}$

$\circ_3$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0, 1\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0, 2\}$

Clearly, we can show that the $I_1 = \{0, 1\}$ and $I_2 = \{0, 2\}$ are hyper BCK-ideals and positive implicative hyper BCK-ideal of type 3 in the above hyper BCK-algebras.
(ii) By Theorem 3.9(iv), any positive implicative hyper BCK-ideal of type 1 is a weak hyper BCK-ideal of H.

Conversely, let $I_1 = \{0, 1\}$ be a weak hyper BCK-ideal of H.
Let $(x \circ y) \circ z \subseteq I_1$ and $y \circ z \subseteq I_1$ but $x \circ z \not\subseteq I_1$ for $x, y, z \in H$. Then $2 \in x \circ z$. Now we consider the following cases.
Case 1. H satisfies the normal condition.
By similar way in the proof of (i), we get that $x = 2$. Now, we consider the following two cases.
Case 1-1. If $z = 0$, since $\{y\} = y \circ 0 \subseteq I_1$, then $y = 0$ or 1. If $y = 0$, then $\{2\} = (2 \circ 0) \circ 0 \subseteq I_1$ which is impossible. If $y = 1$, then $2 \circ 1 = (2 \circ 1) \circ 0 \subseteq I_1$. Since I_1 is a weak hyper BCK-ideal and $1 \in I_1$, then $2 \in I_1$ which is a contradiction.
Case 1-2. If $z = 1$, then $(2 \circ y) \circ 1 \subseteq I_1$ and $y \circ 1 \subseteq I_1$. Since I_1 is a weak hyper BCK-ideal and $1 \in I_1$, then $2 \circ y \subseteq I_1$ and $y \in I_1$. Moreover, since I_1 is a weak hyper BCK-ideal , then $2 \in I_1$, which is impossible.
Case 1-3. If $z = 2$ since $2 \in x \circ z = 2 \circ 2$, then

$$2 \circ y \subseteq (2 \circ 2) \circ y = (2 \circ y) \circ 2 \subseteq I_1$$

If $y = 0$ then $2 \in 2 \circ 0 \subseteq I_1$ which is a contradiction. If $y = 1$, then $2 \circ 1 \subseteq I_1$. Since I_1 is a weak hyper BCK-ideal and $1 \in I_1$, then $2 \in I_1$ which is impossible. If $y = 2$, then $2 \in 2 \circ 2 = y \circ z \subseteq I_1$ which is impossible.

<u>Case 2</u>. H satisfies the simple condition.
Since $2 \in x \circ z$, then by Lemma 2.11(a), $x = 2$. Therefore,

$$(2 \circ y) \circ z \subseteq I_1 \text{ and } y \circ z \subseteq I_1$$

Now, we consider the following cases.
<u>Case 2-1</u>. If $z = 0$, then $\{y\} = y \circ 0 \subseteq I_1$ and $2 \circ y = (2 \circ y) \circ 0 \subseteq I_1$. Since I_1 is a weak hyper BCK-ideal and $y \in I_1$, then $2 \in I_1$ which is impossible.
<u>Case 2-2</u>. If $z = 1$, then $(2 \circ y) \circ 1 \subseteq I_1$ and $y \circ 1 \subseteq I_1$. Since I_1 is a weak hyper BCK-ideal and $1 \in I_1$, then $2 \circ y \subseteq I_1$ and $y \in I_1$ and thus $2 \in I_1$, which is a contradiction.
<u>Case 2-3</u>. If $z = 2$, then $2 \in x \circ z = 2 \circ 2$. Moreover, by Lemma 2.11(a), $2 \circ 1 = \{2\}$. Thus

$$2 \circ y \subseteq (2 \circ 2) \circ y = (2 \circ y) \circ 2 \subseteq I_1$$

If $y = 0$, then $2 \in 2 \circ 0 \subseteq I_1$, which is impossible. If $y = 1$, then $\{2\} = 2 \circ 1 \subseteq I_1$, which is a contradiction. If $y = 2$, then $2 \in 2 \circ 2 = 2 \circ y \subseteq I_1$, which is impossible. Therefore, I_1 is a positive implicative hyper BCK-ideal of type 1 of H.
Now, let $I_2 = \{0, 2\}$ be a weak hyper BCK-ideal of H and $(x \circ y) \circ z \subseteq I_2$ and $y \circ z \subseteq I_2$ but $x \circ z \not\subseteq I_2$. Then $1 \in x \circ z$ and so $x \neq 0$. Since if $x = 0$, then $1 \in x \circ z = 0 \circ z = \{0\}$ which is impossible. Now, we considering the following two cases.
<u>Case 1</u>. H satisfies the simple condition.
By Lemma 2.11(a), $x = 1$ and so $(1 \circ y) \circ z \subseteq I_2$ and $y \circ z \subseteq I_2$. Now we consider the following cases for z.
<u>Case 1-1</u>. If $z = 0$, then $1 \circ y = (1 \circ y) \circ 0 \subseteq I_2$ and $\{y\} = y \circ 0 \subseteq I_2$. Since I_2 is a weak hyper BCK-ideal of H and $y \in I_2$, then $1 \in I_2$ which is impossible.
<u>Case 1-2</u>. If $z = 1$, since $1 \in x \circ z = 1 \circ 1$, then by Lemma 2.11(a), $1 \circ 1 = \{0, 1\}$ and $1 \circ 2 = \{1\}$. If $y = 0$, then $1 \in 1 \circ 1 = (1 \circ 0) \circ 1 = (x \circ y) \circ z \subseteq I_2$, which is impossible. If $y = 1$, then $1 \in 1 \circ 1 \subseteq (1 \circ 1) \circ 1 = (x \circ y) \circ z \subseteq I_2$, which is a contradiction. If $y = 2$, then $1 \in 1 \circ 2 \subseteq (1 \circ 1) \circ 2 = (1 \circ 2) \circ 1 = (x \circ y) \circ z \subseteq I_2$, which is impossible.
<u>Case 1-3</u>. If $z = 2$, then

$$(1 \circ y) \circ 2 \subseteq I_2 \text{ and } y \circ 2 \subseteq I_2$$

Since I_2 is a weak hyper BCK-ideal of H and $2 \in I_2$, then $1 \circ y \subseteq I_2$ and $y \in I_2$. Hence $1 \in I_2$, which is a contradiction.
<u>Case 2</u>. H satisfies the normal condition.
Since $1 \in x \circ z$, then by (HK2),

$$1 \circ y \subseteq (x \circ z) \circ y = (x \circ y) \circ z \subseteq I_2$$

If $y = 0$, then $1 \in 1 \circ 0 \subseteq I_2$, which is impossible. If $y = 2$, then $1 \circ 2 \subseteq I_2$. Since I_2 is a weak hyper BCK-ideal of H and $2 \in I_2$, then $1 \in I_2$ which is a contradiction. If $y = 1$, then $1 \circ 1 \subseteq I_2$ and so by Lemma 2.11(b), $1 \circ 1 = \{0\}$ and $1 \circ 2 = \{0\}$ or $\{0, 1\}$. If $1 \circ 2 = \{0, 1\}$ then by (HK1),

$$\{0, 1\} = (1 \circ 2) \circ (1 \circ 2) \ll 1 \circ 1 = \{0\}$$

which is impossible. If $1 \circ 2 = \{0\}$, then $1 \circ 2 \subseteq I_2$. Since I_2 is a weak hyper BCK-ideal of H and $2 \in I_2$, then $1 \in I_2$ which is impossible. Therefore, I_2 is a positive implicative hyper BCK-ideal of type 1. □

Theorem 3.14 (3)**.** *There are 16 non-isomorphic hyper BCK-algebras of order 3 such that each of them has at least one proper hyper BCK-ideal.*

Theorem 3.15. *There are 16 non-isomorphic hyper BCK-algebras of order 3 such that each of them has at least one proper positive implicative hyper BCK-ideal of type 3.*

Proof. The proof follows by Theorems 3.13(i) and 3.14. □

Definition 3.16 (6)**.** A nonempty subset I of H is said to be *reflexive* if $x \circ x \subseteq I$, for all $x \in H$.

Lemma 3.17 (7)**.** *If I is a reflexive hyper BCK-ideal of H, then*

$$(x \circ y) \cap I \neq \emptyset \text{ implies } x \circ y \subseteq I$$

for all $x, y \in H$.

Theorem 3.18. *Let I be a nonempty subset of H and for any $a \in H$, I_a is defined as follows,*

$$I_a = \{x \in H : x \circ a \subseteq I\}$$

Then the following statements are held:

(i) If I is a positive implicative hyper BCK-ideal of type 1 (3,4), then I_a is a weak hyper BCK-ideal of H, for all $a \in H$,

(ii) Let I be a reflexive positive implicative hyper BCK-ideal of type 3, then I_a is a hyper BCK-ideal of H, for all $a \in H$,

(iii) If for all $a \in H$, I_a is a weak hyper BCK-ideal of H, then I is a positive implicative hyper BCK-ideal of type 1 and 5,

(iv) Let I be reflexive and satisfies the closed condition. Then I is a positive implicative hyper BCK-ideal of type 1(3,...,8) if and only if, I_a is a hyper BCK-ideal of H, for all $a \in H$.

Proof. (i) Let I be a positive implicative hyper BCK-ideal of type 1, $x \circ y \subseteq I_a$ and $y \in I_a$, for $x, y, a \in H$. Then for all $t \in x \circ y$, $t \circ a \subseteq I$ and $y \circ a \subseteq I$. Thus, $(x \circ y) \circ a = \bigcup_{t \in x \circ y} t \circ a \subseteq I$ and $y \circ a \subseteq I$. Since I is a positive implicative hyper BCK-ideal of type 1, then $x \circ a \subseteq I$ and this implies that $x \in I_a$. Therefore I_a is a weak hyper BCK-ideal of H. The proof of types 3 and 4 is similar.
(ii) Let I be a reflexive positive implicative hyper BCK-ideal of type 3. Then by Theorem 3.9(i), I is a hyper BCK-ideal of H. Let $x \circ y \ll I_a$ and $y \in I_a$, for $x, y, a \in H$. Then for all $t \in x \circ y$ there is $s \in I_a$ such that $t \ll s$ i.e $0 \in t \circ s$ and so $(t \circ s) \bigcap I \neq \emptyset$. Since I is a reflexive hyper BCK-ideal of H, then by (HK1) and Lemma 3.17, $(t \circ a) \circ (s \circ a) \ll t \circ s \subseteq I$ and so $(t \circ a) \circ (s \circ a) \ll I$. Since $s \circ a \subseteq I$ and I is a hyper BCK-ideal of H, then by Lemma 3.12(ii), $t \circ a \subseteq I$. Hence $t \in I_a$ and so $x \circ y \subseteq I_a$. Now, since $y \in I_a$ and by (i), I_a is a weak hyper BCK-ideal of H, then $x \in I_a$. Therefore, I_a is a hyper BCK-ideal of H.
(iii) Let $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$ for $x, y, z \in H$. Then $x \circ y \subseteq I_z$ and $y \in I_z$. Since I_z is a hyper BCK-ideal of H, then $x \in I_z$ and this implies that $x \circ z \subseteq I$. Therefore I is a positive implicative hyper BCK-ideal of type 1 and so by Theorem 3.2(v) is of type 5.
(iv) The proof of this case follows by (ii), (iii) and Theorem 3.7. □

Theorem 3.19. *Let I be a nonempty subset of H and for all $a \in I$, $I_a^{\ll}$ is defined as follows:*

$$I_a^{\ll} = \{x \in H : x \circ a \ll I\}$$

Then,

(i) If I and $I_a^{\ll}$ are hyper BCK-ideals of H, for all $a \in H$, then I is a positive implicative hyper BCK-ideal of type 3,

(ii) If I is a reflexive positive implicative hyper BCK-ideal of type 3, then $I_a^{\ll}$ is a hyper BCK-ideal of H, for all $a \in H$.

Proof. (i) Let for all $a \in H$, $I_a^{\ll}$ be a hyper BCK-ideal of H, $(x \circ y) \circ z \ll I$ and $y \circ z \ll I$ for $x, y, z \in H$. Then for all $t \in x \circ y$, $t \circ z \ll I$ and so $t \in I_z^{\ll}$. Thus $x \circ y \subseteq I_z^{\ll}$ and this implies that $x \circ y \ll I_z^{\ll}$. Since $y \in I_z^{\ll}$ and $I_z^{\ll}$ is a hyper BCK-ideal of H, then $x \in I_z^{\ll}$

and so $x \circ z \ll I$. Hence by Lemma 3.8(i), $x \circ z \subseteq I$. Therefore I is a positive implicative hyper BCK-ideal of type 3.
(ii) The proof is similar to the proof of Theorem 3.18(ii) by considering Lemma 3.8(i). $\square$

Example 3.20. Let $H = \{0, 1, 2, 3\}$. Consider the following table:

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0\}$	$\{2\}$
3	$\{3\}$	$\{3\}$	$\{0, 3\}$	$\{0, 3\}$

Then $(H, \circ, 0)$ is a hyper BCK-algebra. $I = \{0, 1\}$ is a positive implicative hyper BCK-ideal of type 3(which is not reflexive). But $I_2 = I_2^{\ll} = \{x \in H : x \circ 2 \subseteq I\} = \{0, 1, 2\}$ are not hyper BCK-ideals of H, because $3 \circ 1 = \{3\} \ll \{0, 1, 2\} = I_2$ and $1 \in I_2$, but $3 \notin I_2$. Therefore, the reflexivity condition in the Theorems 3.18(ii) and 3.19(ii) is necessary.

Theorem 3.21. *Let I be a nonempty subset of H and $a \in H$. Then,*

(i) If I is a hyper BCK-ideal of H and $a \in I$, then $I_a = I = I_a^{\ll}$.

(ii) If $H = \{0, 1, 2\}$ is a hyper BCK-algebra of order 3 and I is a positive implicative hyper BCK-ideal of type 3 of H, then I_a and $I_a^{\ll}$ are hyper BCK-ideals of H for all $a \in H$.

Proof. (i) Let I be a hyper BCK-ideal of H and $a \in I$. Let $x \in I_a$. Thus $x \circ a \subseteq I$ and so $x \circ a \ll I$. Since $a \in I$, then $x \in I$. Therefore, $I_a \subseteq I$.

Now, let $x \in I$. By (HK3), for all $a \in H$ we get that $x \circ a \ll x \in I$ and so $x \circ a \ll I$. Since I is a hyper BCK-ideal of H, then by Lemma 3.8(i), $x \circ a \subseteq I$ which implies $x \in I_a$. Hence $I \subseteq I_a$. Therefore $I = I_a$. The proof of the case $I_a^{\ll}$ is similar.
(ii) Let $I = \{0, 1\}$ be a positive implicative hyper BCK-ideal of type 3. Then by Theorem 3.13(i), I is a hyper BCK-ideal of H. If $a = 0$ or 1 then by (i), $I_a = I$. Thus I_a is a hyper BCK-ideal of H. Now, let $a = 2$. We consider the following cases.
<u>Case 1</u>. H satisfies the simple condition.
Let $x \circ y \ll I_2$ and $y \in I_2$ but $x \notin I_2$ where $I_2 = \{x \in H : x \circ 2 \subseteq I\}$. Thus $x \circ 2 \nsubseteq I$ and so $2 \in x \circ 2$. Hence by Lemma 2.11(a), $x = 2$ and $2 \notin I_2$. Thus $2 \circ y \ll I_2$ and $y \in I_2$.
If $y = 0$, then $\{2\} = 2 \circ 0 \ll I_2$. Then there is $a \in I_2$ such that $2 \ll a$. It is clear that $a \neq 0$. Moreover, since H satisfies the simple condition, then $a \neq 1$. Hence $a = 2$ and so $2 = a \in I_2$, which is a contradiction.
If $y = 1$, then by Lemma 2.11(ii), $\{2\} = 2 \circ 1 \ll I_2$. Similar to the proof of case $y = 0$, we get a contradiction.
If $y = 2$, since $y \in I_2$, then $2 \circ 2 = y \circ 2 \subseteq I$ and this implies that $2 \in I_2$, which is impossible.
<u>Case 2</u>. H satisfies the normal condition.
Let $x \circ y \ll I_2$ and $y \in I_2$ but $x \notin I_2$. By Lemma 2.11(b) and by similar way in the proof of Case 1, we get that $x = 2$ and $2 \notin I_2$. Thus $2 \circ y \ll I_2$ and $y \in I_2$.
If $y = 0$, then $\{2\} = 2 \circ 0 \ll I_2$. Then there is $a \in I_2$ such that $2 \ll a$. Clear that $a \neq 0$ and by Lemma 2.11(v) $a \neq 1$. Thus $a = 2$ and so $2 \in I_2$, which is a contradiction.
If $y = 1$, then by Lemma 2.11(b), $x \circ y = 2 \circ 1 = \{1\}$, $\{1, 2\}$ or $\{2\}$. If $2 \circ 1 = \{2\}$ or $\{1, 2\}$, then $2 \in 2 \circ 1 \ll I_2$, which is impossible. If $2 \circ 1 = \{1\}$, since $1 = y \in I_2$ then $1 \circ 2 \subseteq I$ and $2 \circ 1 = \{1\} \subseteq I_2$. Thus $(2 \circ 1) \circ 2 \subseteq I$ and since I is a positive implicative hyper BCK-ideal of type 3 and $1 \circ 2 \subseteq I$, then $2 \circ 2 \subseteq I$. This implies that $2 \in I_2$, which is a contradiction.
If $y = 2$, since $y \in I_2$, then $2 \circ 2 = y \circ 2 \subseteq I$ and this implies that $2 \in I_2$, which is impossible.
The proof of the case $I_a^{\ll}$ is similar.

Now let $I = \{0, 2\}$ be a positive implicative hyper BCK-ideal of type 3. Then by considering the Lemma 2.11, the proof is similar to the proof of case $I = \{0, 1\}$ by some modifications. $\square$

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R. A. Borzooei, M. Bakhshi
Department of Mathematics, Sistan and Baluchestan University
Zahedan, Iran.

e-mail : borzooei@hamoon.usb.ac.ir