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ON THE NUMBER OF NON-EQUIVALENT ODD 1-FACTORS OF A COMPLETE GRAPH

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Abstract. For even $n > 0$, let K_n be the complete graph with vertices $v_0, v_1, \cdots, v_{n-1}$. An edge $v_i v_j$ is called odd or even accordingly as $|i-j|$ is odd or even. An odd(even) 1-factor of K_n is a 1-factor of K_n whose edges are all odd(even). The Dihedral group D_n acts on K_n naturally, and this action induces an action of D_n on the family of all 1-factors of K_n. In this paper, by applying Burnside's lemma, we calculate the number of the equivalence classes of odd(even) 1-factors under the action of D_n.

1 Introduction

For even $n > 0$, let K_n be the complete graph with vertices $v_0, v_1, \cdots, v_{n-1}$. An edge $v_i v_j$ is called odd or even accordingly as $|i-j|$ is odd or even. An odd(even) 1-factor of K_n is a 1-factor of K_n whose edges are all odd (even). Let X_n be the set of the odd 1-factors of K_n and let Y_n be the set of the even 1-factors of K_n.

The action of the Dihedral group $D_n = \{\rho_0, \rho_1, \cdots, \rho_{n-1}, \sigma_0, \sigma_1, \cdots, \sigma_{n-1}\}$ on K_n is defined by

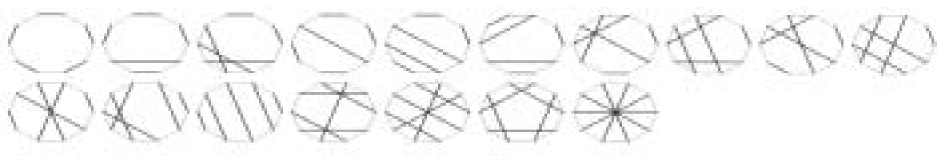
$$\rho_i(v_k) = v_{(k+i) \pmod n} \quad for \quad 0 \le i \le n-1,\ 0 \le k \le n-1$$

$$\sigma_i(v_k) = v_{(n+i-k) \pmod n} \quad for \quad 0 \le i \le n-1,\ 0 \le k \le n-1$$

Then this action induces the action of D_n on X_n and Y_n.

The equivalence classes of X_8 are given with the next figure.

The equivalence classes of X_{10} are given with the next figure.



By applying Burnside's lemma, we calculate the number of the equivalence classes of X_n and Y_n under this group action. This problem was presented by Dr. Shun-ichiro Koh who is a physicist of Kochi University. This problem is related to Feynman diagram in quantum mechanical many body problem.

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Notation 1. *For each integer i such that $0 \le i \le n-1$, let $d = (n, i)$ and R_i^n be defined by the following formula:*

$$R_i^n = \begin{cases} 0 & \text{if } d \text{ is odd and } n/2 \equiv 0 \pmod 2 \\ \displaystyle\sum_{\substack{d=2s+t \\ s\ge 0, t\ge 0}} \frac{d!}{2^s s! t!} \left(\frac{n}{2d}\right)^s & \text{if } d \text{ is odd and } n/2 \equiv 1 \pmod 2 \\ \left(\dfrac{d}{2}\right)! \times \left(\dfrac{n}{d}\right)^{d/2} & \text{if } d \text{ is even} \end{cases}$$

Remark 1. *It is easily checked that R_0^n is equal to $(n/2)!$.*

Notation 2. *Let S_0^n and S_1^n be defined by the following formula:*

$$S_0^n = \begin{cases} 0 & \text{if } n/2 \equiv 0 \pmod 2 \\ \left(\dfrac{n-2}{4}\right)! \times 2^{\frac{n-2}{4}} & \text{if } n/2 \equiv 1 \pmod 2 \end{cases}$$

$$S_1^n = \sum_{\substack{\frac{n}{2}=2s+t \\ s\ge 0, t\ge 0}} \frac{(\frac{n}{2})!}{2^s s! t!}$$

Theorem 1. *For even $n > 0$, the number of non-equivalent odd 1-factors of K_n under the action of the Dihedral group D_n is*

$$\frac{1}{2n}\{\sum_{i=0}^{n-1} R_i^n + \frac{n}{2}(S_0^n + S_1^n)\}$$

Remark 2. *We calculated the non-equivalent odd 1-factors of K_n, $n \le 16$, under the action of the Dihedral group D_n by a computer. The numbers agreed with the numbers that are given by Theorem 1. The results is as follows:*

n=2	1
n=4	1
n=6	3
n=8	5
n=10	17
n=12	53
n=14	260
n=16	1466

Notation 3. *Let $n \equiv 0 \pmod 4$. For each integer i such that $0 \le i \le n-1$, let $d = (n, i)$ and P_i^n be defined by the following formula:*

$$P_i^n = \begin{cases} \displaystyle\sum_{\substack{d/2=2s_1+t_1 \\ d/2=2s_2+t_2 \\ s_1,t_1,s_2,t_2\ge 0}} \frac{(d/2)! \times (d/2)!}{2^{s_1+s_2} s_1! s_2! t_1! t_2!} \left(\frac{n}{d}\right)^{s_1+s_2} & \text{if } d \text{ is even and } n/d \equiv 0 \pmod 2 \\ \left(\dfrac{d-2}{2}\right)!! \times \left(\dfrac{d-2}{2}\right)!! \times \left(\dfrac{n}{d}\right)^{\frac{d}{2}} & \text{if } d \text{ is even and } n/d \equiv 1 \pmod 2 \\ \displaystyle\sum_{\substack{d=2s+t \\ s\ge 0, t\ge 0}} \frac{d!}{2^s s! t!} \left(\frac{n}{2d}\right)^s & \text{if } d \text{ is odd} \end{cases}$$

Remark 3. *It is easily checked that if* $n \equiv 0 \pmod 4$ *then* P_0^n *is equal to* $(n/2-1)!! \times (n/2-1)!!$.

Notation 4. *Let* $n \equiv 0 \pmod 4$*. Let* Q_0^n *and* Q_1^n *be defined by the following formula:*

$$\begin{aligned} Q_0^n &= \sum_{\substack{\frac{n}{4}=2s_1+t_1 \\ \frac{n}{4}=2s_2+t_2 \\ s_1,s_2,t_1,t_2\geq 0}} \frac{(\frac{n}{4})!(\frac{n-4}{4})!}{2^{s_1+s_2}s_1!s_2!t_1!t_2!} \times 2^{s_1+s_2} \\ Q_1^n &= \left(\frac{n-2}{2}\right)!!. \end{aligned}$$

Theorem 2. *If* $n \equiv 2 \pmod 4$ *then* K_n *has no even 1-factors. If* $n \equiv 0 \pmod 4$ *then the number of the non-equivalent even 1-factors of* K_n *under the action of the Dihedral group* D_n *is*

$$\frac{1}{2n}\{\sum_{i=0}^{n-1} P_i^n + \frac{n}{2}(Q_0^n + Q_1^n)\}.$$

Remark 4. *We calculated the non-equivarent even 1-factors of* K_n*,* $n \leq 16$*, under the action of the Dihedral group* D_n *by a computer. The numbers agreed with the numbers that are given by Theorem 2. The results is as follows:*

n=4	1
n=8	3
n=12	22
n=16	436

These computations can be done by applying Burnside's lemma.

Theorem 3. *(Burnside's lemma) Let* G *be a group of permutations acting on a set* S*. Then the number of orbits induced on* S *is given by*

$$\frac{1}{|G|}\sum_{\pi\in G}|fix(\pi)|$$

where $fix(\pi) = \{x \in S | \pi(x) = x\}$.

2 Odd 1-factors

We prove Theorem 1. We must determine the numbers of the fixed points of each permutation ρ_i and σ_i to prove the Theorem by applying Burnside's Lemma.

Lemma 1. *The number of the odd 1-factors of* K_n *is* $(n/2)!$*. This is the number of the fixed points of* ρ_0*.*

Proof. v_0 is able to join any vertex of $\{v_1, v_3, \cdots, v_{n-1}\}$. Then the number of ways of joining is $n/2$. Since the suffix of one vertex of the first edge is even and the suffix of the other is odd, the number of ways of choosing of the second edge is $(n-2)/2$ even if we choose any vertex as one end vertex of the edge. When we choose j edges, we use j vertices which have even suffix and j vertices which have odd suffix. Then the number of ways of choosing of the $j+1$th edge is $(n-2j)/2$ even if we choose any vertex as one end vertex of the edge. Therefore, the number of the odd 1-factors of K_n is (n/2)!. □

Lemma 2. *If (n,i)=1 then the number of the fixed points of* ρ_i *is one if* $n \equiv 2 \pmod 4$ *and is zero if* $n \equiv 0 \pmod 4$.

Proof. Let M_n be the 1-factor $\{v_\alpha v_{n/2+\alpha} | 0 \le \alpha \le n/2-1\}$ of K_n. If $n \equiv 2 \pmod 4$ then $n/2$ is odd and M_n is the odd 1-factor of K_n and $\rho_i(M_n) = M_n$. Conversely, let H be a odd 1-factor of K_n which is fixed by ρ_i and let $v_0 v_m$ be an edge of H. Since (n,i)=1, there is an integer α such that $\alpha i \equiv m \pmod n$. Then $\rho_i^\alpha(v_0) = v_m$ and $\rho_i^\alpha(v_m) = v_{(m+i\alpha) \pmod n}$. Since $\rho_i(H) = H$, we have $v_0 v_m = v_m v_{(m+i\alpha) \pmod n}$. Then we have $m + i\alpha \equiv 0 \pmod n$ and $2m \equiv 0 \pmod n$ and therefore $m = n/2$ and $v_0 v_{n/2} \in H$. If $n \equiv 0 \pmod 4$ then $n/2$ is even. This is contradiction. Then if $n \equiv 0 \pmod 4$ then the number of fixed points of ρ_i is zero. We assume that $n \equiv 2 \pmod 4$. Since $\{\rho_i^\alpha(0) | 0 \le \alpha \le n-1\} = \{0, 1, 2, \cdots, n-1\}$, H is uniquely determined by $v_0 v_{n/2}$ and $H = \{v_\alpha v_{n/2+\alpha} | 0 \le \alpha \le n/2-1\}$. Then the number of the fixed points of ρ_i is one. □

Lemma 3. *Let* $(n, i) = d$ *be greater than one. The number of the fixed points of* ρ_i *is given by the following formula:*

1. *In the case that d is odd:*

   (a) *if* $n/2 \equiv 0 \pmod 2$ *then 0.*

   (b) *if* $n/2 \equiv 1 \pmod 2$ *then*

$$\sum_{\substack{d=2s+t \\ s\ge 0, t \ge 0}} \frac{d!}{2^s s! t!} \left(\frac{n}{2d}\right)^s .$$

2. *If d is even then*

$$\left(\frac{d}{2}\right)! \times \left(\frac{n}{d}\right)^{\frac{d}{2}} .$$

Proof. Let $V_0 = \{v_0, v_d, v_{2d}, \cdots, v_{n-d}\}, V_1 = \{v_1, v_{d+1}, v_{2d+1}, \cdots, v_{n-d+1}\}$,
$V_2 = \{v_2, v_{d+2}, v_{2d+2}, \cdots, v_{n-d+2}\}$, $\cdots$, $V_{d-1} = \{v_{d-1}, v_{2d-1}, v_{3d-1}, \cdots, v_{n-1}\}$.

Since $(n, i) = d$, the equation $xi \equiv m \pmod n$ has a solution if and only if d divides m. Then we have $\rho_i(V_k) = V_k$ *for* $0 \le k \le d-1$.

Let H be an odd 1-factor of K_n which is fixed by ρ_i and let $v_\alpha v_\beta$ be an edge of H. If $v_\alpha \in V_k$ and $v_\beta \in V_k$ then the induced subgraph $H|V_k$ is an odd 1-factor of $K_{n/d}$ which is fixed by $\rho_{i/d}$ and it is unique odd 1-factor $M_{n/d}$ by Lemma 2 . If $v_\alpha \in V_{k_1}$ and $v_\beta \in V_{k_2}$ then the induced subgraph $H|V_{k_1} \cup V_{k_2}$ is an odd 1-factor of $K_{2n/d}$ which is fixed by $\rho_{i/d}$.

We first consider the case that d is odd. Since n/d is even, the number of the vertices in V_k whose suffixes are odd is equal to the number of the vertices in V_k whose suffixes are even for any k.

If $n/2$ is even then V_k is not able to make the odd 1-factor which is fixed by ρ_i alone, because $\frac{n}{2d}$ is even. Furthermore, we can not partition $\{V_0, V_1, V_2, \cdots, V_{d-1}\}$ into $d/2$ pairs, since d is odd. Therefore, if $n/2$ is even then the number of the fixed points of ρ_i is zero.

On the other hand, if $n/2$ is odd then $v_0 v_{d(\frac{n}{2d})}$ is an odd edge. Then V_k is able to make the unique odd 1-factor which is fixed by ρ_i alone. Furthermore, any V_{k_1} and V_{k_2} are able to make $\frac{n}{2d}$ odd 1-factors which are fixed by ρ_i. The number of ways to partition $\{V_0, V_1, V_2, \cdots, V_{d-1}\}$ into s pairs and t singletons is equal to

$$\frac{d!}{2^s s! t!}.$$

Then, the number of the odd 1-factors fixed by ρ_i is

$$\sum_{\substack{d=2s+t\\ s\geq 0, t\geq 0}} \frac{d!}{2^s s! t!}\left(\frac{n}{2d}\right)^s .$$

Next we consider the case that d is even.

$V_0, V_2, V_4, \cdots, V_{d-2}$ are the set whose elements have even suffix and $V_1, V_3, V_5, \cdots, V_{d-1}$ are the set whose elements have odd suffix. Then V_k is not able to make odd 1-factor which is fixed by ρ_i alone. But V_k, for k is even, and V_l, for l is odd, are able to make $\frac{n}{d}$ odd 1-factors fixed by ρ_i. The number of ways to partition $\{V_0, V_1, V_2, \cdots, V_{d-1}\}$ into $d/2$ pairs of the type $\{V_{even}, V_{odd}\}$ is equal to $(\frac{d}{2})!$. Then, the number of the odd 1-factors fixed by ρ_i is

$$\left(\frac{d}{2}\right)! \times \left(\frac{n}{d}\right)^{\frac{d}{2}} .$$

We have the results. □

Lemma 4. *The number of the fixed points of σ_0 is equal to the number of the fixed points of σ_{2d} for all $1 \leq d \leq n/2 - 1$.*

Proof. Let H be an odd 1-factor of K_n fixed by σ_0. Then it is easily verified that $\rho_d(H)$ is an odd 1-factor of K_n fixed by σ_{2d}. Conversely, if H is an odd 1-factor of K_n fixed by σ_{2d} then $\rho_d^{-1}(H)$ is an odd 1-factor of K_n fixed by σ_0. Then we have the results. □

Similarly, we have the next lemma.

Lemma 5. *The number of the fixed points of σ_1 is equal to the number of the fixed points of σ_{2d+1} for all $1 \leq d \leq n/2 - 1$.*

Lemma 6. *The number of the fixed points of σ_0 is given by the following formula:*

1. *if $n/2 \equiv 0 \pmod 2$ then 0.*
2. *if $n/2 \equiv 1 \pmod 2$ then $\left(\frac{n-2}{4}\right)! \times 2^{\frac{n-2}{4}}$.*

Proof. Since the axis of σ_0 passes through v_0 and $v_{n/2}$, the odd 1-factor of K_n fixed by σ_0 must contain the edge $v_0 v_{n/2}$. If $n/2 \equiv 0 \pmod 2$ then the edge $v_0 v_{n/2}$ is even and the number of the fixed points of σ_0 must be zero. Therefore we assume that $n/2 \equiv 1 \pmod 2$. Let $V_1 = \{v_1, v_{n-1}\}, V_2 = \{v_2, v_{n-2}\}, V_3 = \{v_3, v_{n-3}\}, \cdots, V_{n/2-1} = \{v_{n/2-1}, v_{n/2+1}\}$. Then we have $\sigma_0(V_k) = V_k$ for $1 \leq k \leq n/2 - 1$. $V_1, V_3, V_5, \cdots, V_{n/2-2}$ are the sets whose elements have odd suffixes and $V_2, V_4, V_6, \cdots, V_{n/2-1}$ are the sets whose elements have even suffixes.

Let H be an odd 1-factor of K_n fixed by σ_0 and let $v_\alpha v_\beta$ be an edge of H. If α and β are not equal to zero then $v_\alpha \in V_{2k}$ and $v_\beta \in V_{2l+1}$ or $v_\alpha \in V_{2k+1}$ and $v_\beta \in V_{2l}$ for some k and l. The number of ways to partition $\{V_1, V_2, \cdots, V_{n/2-1}\}$ into $(n-2)/4$ pairs of the type $\{V_{even}, V_{odd}\}$ is equal to $(\frac{n-2}{4})!$. The number of ways to choosing the edge between V_{2k} and V_{2l+1} is two. Then the number of the fixed points of σ_0 is

$$\left(\frac{n-2}{4}\right)! \times 2^{\frac{n-2}{4}} .$$

We have the results. □

Lemma 7. *The number of the fixed points of σ_1 is*

$$\sum_{\substack{n/2=2s+t\\ s\geq 0, t\geq 0}} \frac{(\frac{n}{2})!}{2^s s! t!}.$$

Proof. Let $V_1 = \{v_1, v_0\}, V_2 = \{v_2, v_{n-1}\}, V_3 = \{v_3, v_{n-2}\}, \cdots, V_{n/2} = \{v_{n/2}, v_{n/2+1}\}$. Then we have $\sigma_1(V_k) = V_k$ for $1 \leq k \leq n/2$. Each V_k contains one vertex whose suffix is even and one vertex whose suffix is odd. Let H be an odd 1-factor of K_n fixed by σ_1 and let $v_\alpha v_\beta$ be an edge of H. Then v_α and v_β are contained in same V_k or $v_\alpha \in V_{k_1}$ and $v_\beta \in V_{k_2}$ for some k_1 and k_2. The number of ways to partition $\{V_1, V_2, V_3, \cdots, V_{n/2}\}$ into s pairs and t singletons is equal to

$$\frac{(\frac{n}{2})!}{2^s s! t!}.$$

Then the number of the fixed points of σ_1 is

$$\sum_{\substack{n/2=2s+t\\ s\geq 0, t\geq 0}} \frac{(\frac{n}{2})!}{2^s s! t!}.$$

We have the results. □

Then we completely proved Theorem 1.

3 Even 1-factors

Next we prove Theorem 2. We must determine the numbers of the fixed points of each permutation ρ_i and σ_i to prove the Theorem by applying Burnside's Lemma.

Lemma 8. *The number of the even 1-factors of K_n is zero if $n \equiv 2 \pmod 4$ and is $(n/2-1)!! \times (n/2-1)!!$ if $n \equiv 0 \pmod 4$. This is the number of the fixed points of ρ_0.*

Proof. In order to partitition $\{v_0, v_2, v_4, \cdots, v_{n-2}\}$ into $n/4$ subsets which consist two elements $n/2$ must be even. Then, if $n \equiv 2 \pmod 4$ then the number of the even 1-factors of K_n is zero. Therefore we assume that $n \equiv 0 \pmod 4$. The number of ways to partitation $\{v_0, v_2, v_4, \cdots, v_{n-2}\}$ into $n/4$ pairs is equal to $(n/2-1)!!$ and the number of ways to partitation $\{v_1, v_3, v_5, \cdots, v_{n-1}\}$ into $n/4$ pairs is equal to $(n/2-1)!!$. Therefore if $n \equiv 0 \pmod 4$ then the number of the even 1-factors of K_n is $(n/2-1)!! \times (n/2-1)!!$. We have the results. □

By Lemma 8 we have that if $n \equiv 2 \pmod 4$ then the number of the non-equivarent even 1-factors of K_n is zero.

Therefore we assume from now on that $n \equiv 0 \pmod 4$.

Lemma 9. *If $(n, i) = 1$ then the number of the fixed points of ρ_i is one.*

Proof. Let M_n be the 1-factor $\{v_\alpha v_{n/2+\alpha} | 0 \leq \alpha \leq n/2-1\}$ of K_n. Since $n/2$ is even, M_n is the even 1-factor of K_n and $\rho_i(M_n) = M_n$. Conversely, let H be an even 1-factor of K_n which is fixed by ρ_i and let $v_0 v_m$ be an edge of H. By the essentially the same methods, we have $H = \{v_\alpha v_{n/2+\alpha} | 0 \leq \alpha \leq n/2-1\}$. Then the number of the fixed points of ρ_i is one. □

Lemma 10. *Let* $(n,i) = d$ *be greater than one. The number of the fixed points of* ρ_i *is given by the following formula:*

1. *If* d *is odd then*

$$\sum_{\substack{d=2s+t\\ s,t\geq 0}} \frac{d!}{2^s s!t!}\left(\frac{n}{2d}\right)^s.$$

2. *In the case that* d *is even:*

   (a) *If* $n/d \equiv 0 \pmod 2$ *then*

$$\sum_{\substack{d/2=2s_1+t_1\\ d/2=2s_2+t_2\\ s_1,s_2,t_1,t_2\geq 0}} \frac{(d/2)!\times(d/2)!}{2^{s_1+s_2}s_1!s_2!t_1!t_2!}\left(\frac{n}{d}\right)^{s_1+s_2}.$$

   (b) *If* $n/d \equiv 1 \pmod 2$ *then*

$$\left(\frac{d-2}{2}\right)!!\times\left(\frac{d-2}{2}\right)!!\times\left(\frac{n}{d}\right)^{\frac{d}{2}}.$$

Proof. Let $V_0 = \{v_0, v_d, v_{2d}, \cdots, v_{n-d}\}, V_1 = \{v_1, v_{d+1}, v_{2d+1}, \cdots, v_{n-d+1}\}$, $V_2 = \{v_2, v_{d+2}, v_{2d+2}, \cdots, v_{n-d+2}\}$, $\cdots$, $V_{d-1} = \{v_{d-1}, v_{2d-1}, v_{3d-1}, \cdots, v_{n-1}\}$. Then we have $\rho_i(V_k) = V_k \ for\ 0 \leq k \leq d-1$.

We first consider the case that d is odd. Each V_k, for $0 \leq k \leq d-1$, contains $\frac{n}{2d}$ vertices whose suffix is even and $\frac{n}{2d}$ vertices whose suffixes are odd. Since $\frac{n}{2d}$ is even, each V_k is able to make unique even 1-factor fixed by ρ_i alone, Furthermore any V_k and V_l are able to make $\frac{n}{2d}$ even 1-factors fixed by ρ_i. Then the number of the fixed points of ρ_i is

$$\sum_{\substack{d=2s+t\\ s,t\geq 0}} \frac{d!}{2^s s!t!}\left(\frac{n}{2d}\right)^s.$$

Next we consider the case that d is even. We first consider the case that n/d is even. $V_0, V_2, V_4, \cdots, V_{d-2}$ contain n/d vertices whose suffixes are even and $V_1, V_3, V_5, \cdots, V_{d-1}$ contain n/d vertices whose suffixes are odd. Since n/d is even, each V_k is able to make unique even 1-factor fixed by ρ_i alone. Furthermore any two elements of $\{V_0, V_2, V_4, \cdots, V_{d-2}\}$ and any two elements of $\{V_1, V_3, V_5, \cdots, V_{d-1}\}$ are able to make $\frac{n}{d}$ even 1-factors fixed by ρ_i, respectively. Then the number of the fixed points of ρ_i is

$$\sum_{\substack{d/2=2s_1+t_1\\ d/2=2s_2+t_2\\ s_1,s_2,t_1,t_2\geq 0}} \frac{(d/2)!\times(d/2)!}{2^{s_1+s_2}s_1!s_2!t_1!t_2!}\left(\frac{n}{d}\right)^{s_1+s_2}.$$

Next we consider the case that n/d is odd. Since n/d is odd, V_k is not able to make even 1-factor fixed by ρ_i alone. But any two elements of $\{V_0, V_2, V_4, \cdots, V_{d-2}\}$ and any two elements of $\{V_1, V_3, V_5, \cdots, V_{d-1}\}$ are able to make $\frac{n}{d}$ even 1-factors fixed by ρ_i, respectively. Since $n \equiv 0 \pmod 4$ and n/d is odd, $d \equiv 0 \pmod 4$. Then the number of the fixed points of ρ_i is

$$\left(\frac{d-2}{2}\right)!!\times\left(\frac{d-2}{2}\right)!!\times\left(\frac{n}{d}\right)^{\frac{d}{2}}.$$

We have the results. □

The next two Lemmas are proved by the essentially the same methods as Lemma 4.

Lemma 11. *The number of the fixed points of* σ_0 *is equal to the number of the fixed points of* σ_{2d} *for all* $1 \leq d \leq n/2-1$.

Lemma 12. *The number of the fixed points of* σ_1 *is equal to the number of the fixed points of* σ_{2d+1} *for all* $1 \leq d \leq n/2-1$.

Lemma 13. *The number of the fixed points of* σ_0 *is*

$$\sum_{\substack{\frac{n}{4}=2s_1+t_1 \\ \frac{n-4}{4}=2s_2+t_2 \\ s_1,s_2,t_1,t_2\geq 0}} \frac{(\frac{n}{4})! \times (\frac{n-4}{4})!}{2^{s_1+s_2}s_1!t_1!s_2!t_2!} \times 2^{s_1+s_2}.$$

Proof. Let $V_1 = \{v_1, v_{n-1}\}, V_2 = \{v_2, v_{n-2}\}, V_3 = \{v_3, v_{n-3}\}, \cdots, V_{n/2-1} = \{v_{n/2-1}, v_{n/2+1}\}$. Then we have $\sigma_0(V_k) = V_k$ for $1 \leq k \leq n/2-1$. Each of $V_1, V_3, V_5, \cdots, V_{n/2-1}$ contains two vertices whose suffixes are odd and each of $V_2, V_4, V_6, \cdots, V_{n/2-2}$ contains two vertices whose suffixes are odd. Then each V_k is able to make unique even 1-factor fixed by σ_0 alone. Furthermore any two elements of $\{V_2, V_4, \cdots, V_{n/2-2}\}$ and any two elements of $\{V_1, V_3, V_5, \cdots, V_{n/2-1}\}$ are able to make two even 1-factors fixed by σ_0, respectively. Then the number of the fixed points of ρ_i is

$$\sum_{\substack{\frac{n}{4}=2s_1+t_1 \\ \frac{n-4}{4}=2s_2+t_2 \\ s_1,s_2,t_1,t_2\geq 0}} \frac{(\frac{n}{4})! \times (\frac{n-4}{4})!}{2^{s_1+s_2}s_1!t_1!s_2!t_2!} \times 2^{s_1+s_2}.$$

□

Lemma 14. *The number of the fixed points of* σ_1 *is* $(n/2-1)!!$.

Proof. Let $V_1 = \{v_1, v_0\}, V_2 = \{v_2, v_{n-1}\}, V_3 = \{v_3, v_{n-2}\}, \cdots, V_{n/2} = \{v_{n/2}, v_{n/2+1}\}$. Then we have $\sigma_1(V_k) = V_k$ for $1 \leq k \leq n/2$. Each V_k contains one vertex whose suffix is even and one vertex whose suffix is odd. Then V_k is not able to make even 1-factor fixed by σ_1 alone. But any two elements of $\{V_1, V_2, V_3, \cdots, V_{n/2}\}$ are able to make unique even 1-factor fixed by σ_1. Then the number of the fixed points of σ_1 is $(n/2-1)!!$.
We have the results. □

Then we completely proved Theorem 2.

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