INTUITIONISTIC FUZZY α-IDEALS OF IS-ALGEBRAS

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Abstract. In this paper, we introduce the notion of intuitionistic fuzzy α-ideals of IS-algebras and investigate some of their properties.

1.Introduction and Preliminaries In 1996, Iseki introduced the notion of BCK/BCI-algebras. For the general development of BCK/BCI-algebras, the ideal theory plays an important role. In 1993, Jun et al [3] introduced a new class of algebras related to BCI-algebras and semigroups, called a BCI-semigroup. In 1998, for the convenience of study, Jun et al [4] renamed the BCI-semigroups as the IS-algebra and studied further properties. In this paper, we consider the fuzzification of α-ideals of IS-algebras and study their properties.

By a BCI-algebra we mean $(X;*,0)$ of type $(2,0)$ satisfying the following conditions:

(I) $((x*y)*(x*z))*(z*y)=0$

(II) $(x*(x*y))*y=0$

(III) $x*x=0$

(IV) $x*y=0$ and $y*x=0$ imply $x=y$ for all $x,y,z\in X$.

In any BCI-algebra X one can define a partial order $\leq$ by putting $x\leq y$ if and only if $x*y=0$.

By an IS-algebra we mean a nonempty set X with two binary operation "*" and " $\cdot$ " and constant 0 satisfying the axioms:

(I) $I(X)=(X;*,0)$ is a BCI-algebra

(II) $S(X)=(X;\cdot)$ is a semigroup.

(III) The operation "$\cdot$" is distribute over the operation "$*$", that is, $x\cdot(y*z)=(y\cdot z)*(x\cdot z)$ and $(x*y)\cdot z=(x\cdot z)*(y\cdot z)$ for all $x,y,z\in X$. For the convience, we use xy instead for $x\cdot y$.

A nonempty subset A of a semigroup $S(X)=(X;\cdot)$ is said to be stable if $xa\in A$ whenever $x\in S(X)$ and $a\in A$.

We now review some fuzzy logical concepts. A fuzzy set in a set X is a function $\mu:X\to[0,1]$, and the complement of μ, denoted by the $\overline{\mu}$, is the fuzzy set in X given by $\overline{\mu}(x)=1-\mu(x)$.

A fuzzy set μ in a semigroup $S(X)=(X,\cdot)$ is said to be fuzzy stable if $\mu(xy)\geq\mu(y)$ for all $x,y\in X$. For $t\in[0,1]$, the set $U(\mu;t)=(x\in X\mid\mu(x)\geq t)$ is called an upper t-level cut μ, and the set $L(\mu;t)=\{x\in X\mid\mu(x)\leq t\}$ is called a lower t-level cut of μ. We shall write $a\wedge b$ for $\min\{a,b\}$ and $a\vee b$ for $\max\{a,b\}$, where a and b are any real numbers.

An intuitionistic fuzzy set (briefly, IFSA) in a nonempty set X is an object having the form

$$A=\{(x,\alpha_A(x),\beta_A(x))\mid x\in X\}$$

where the function $\alpha_A:X\to[0,1]$ and $\beta_A:X\to[0,1]$ denote the degree of membership and the degree of nonmembership respectively, and $0\leq\alpha_A(x)+\beta_A(x)\leq 1,\forall x\in X$.

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An intuitionistic fuzzy set $A=\{(x,\alpha_A(x),\beta_A(x)) \mid x\in X\}$ in X can be identified to an ordered pair (α_A,β_A) in $I^X\times I^X$. For the sake of simplicity, we shall use the symbol symbol $A=(\alpha_A,\beta_A)$ for the IFSA=$\{(x,\alpha_A(x),\beta_A(x)) \mid x\in X\}$.

Definition 1.1 ([8]) A nonempty subset I of a BCI-algebra X is called an α-ideal of X if it satisfies (I1) $0\in I$, (I2) $(x*z)*(0*y)\in I$ and $z\in I$ imply $y*x\in I$ for all $x,y,z\in X$.

Definition 1.2([7]) A fuzzy set μ of a BCI-algebra X is called a fuzzy α-ideal (briefly, Fα-ideal) of X if it satisfies (F1) $\mu(0)\geq\mu(x)$, (F2) $\mu(y*x)\geq\mu((x*z)*(0*y))\wedge\mu(z)$ for all $x,y,z\in X$.

2.Intuitionistic Fuzzy α-ideals

Definition 2.1([1]) A nonempty subset I of an IS-algebra X is called an α-ideal of X if it satisfies $(\alpha 1)xa\in I$ for any $x\in S(X)$ and $a\in I$;

$(\alpha 2)(x*z)*(0*y)\in I$ and $z\in I$ imply $y*x\in I$ for all $x,y,z\in I(X)$.

Definition 2.2([1]) A fuzzy set μ in an IS-algebra X is called a fuzzy α-ideal of X if it setisfies

(F1) μ is a fuzzy stable set in $S(X)$.

(F2) $\mu(y*x)\geq\mu((x*z)*(0*y))\wedge\mu(z)$ for all $x,y,z\in I(X)$.

Definition 2.3 An IFSA=(α_A,β_A) in an IS-algebra X is called an intuitionistic fuzzy α-ideal (briefly, IFα-ideal) of X if it satisfies

(IF1) $\alpha_A(xy)\geq\alpha_A(y)$

(IF2) $\beta_A(xy)\leq\beta_A(y)$

(IF3) $\alpha_A(y*x)\geq\alpha_A((x*z)*(0*y))\wedge\alpha_A(z)$

(IF4) $\beta_A(y*x)\leq\beta_A((x*z)*(0*y))\vee\beta_A(z)$ for all $x,y,z\in X$.

Example 2.4 Consider an IS-algebra $X=\{0,a,b,c\}$ with Cayley table as follows

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	0	0	0

Define an IFSA=(α_A,β_A) in X by $\alpha_A(0)=\alpha_A(a)=1,\alpha_A(b)=\alpha_A(c)=t,\beta_A(0)=\beta_A(a)=0,\beta_A(b)=\beta_A(c)=s$, where $t,s\in[0,1]$ and $t+s\leq 1$. It's easy to check that $A=(\alpha_A,\beta_A)$ is an IFα-ideal of X.

Theorem 2.5 An IFSA=(α_A,β_A) is an IFα-ideal of an IS-algebra X if and only if for all $s,t\in[0,1]$, the nonempty sets $U(\alpha_A;t)$ and $L(\beta_A;s)$ are α-ideals of X.

Proof. Suppose that $A=(\alpha_A,\beta_A)$ is an IF α-idela of X. Let $x\in S(X)$ and $y\in U(\alpha_A;t)$, then $\alpha_A(y)\geq t$, and that $\alpha_A(xy)\geq\alpha_A(y)\geq t$ since $A=(\alpha_A,\beta_A)$ is an IFα-ideal of X, which implies that $xy\in U(\alpha_A;t)$. Let $x,y,z\in I(X)$ be such that $(x*z)*(0*y)\in U(\alpha_A;t)$ and $z\in U(\alpha_A;t)$. Then $\alpha_A((x*z)*(0*y))\geq t$ and $\alpha_A(z)\geq t$. It follows that $\alpha_A(y*x)\geq\alpha_A((x*z)*(0*y))\wedge\alpha_A(z)\geq t$, and that $y*x\in U(\alpha_A;t)$. Hence $U(\alpha_A;t)$ is an α-ideal of X. Now, let $x\in S(X)$ and $y\in L(\beta_A;s)$, then $\beta_A(y)\leq s$, and that $\beta_A(xy)\leq\beta_A(y)\leq s$, which implies that $xy\in L(\beta_A;s)$. Let $x,y,z\in I(X)$ be such that $(x*z)*(0*y)\in L(\beta_A;s)$ and $z\in L(\beta_A;s)$. Then $\beta_A((x*z)*(0*y))\leq s$ and $\beta_A(z)\leq s$. It follows that $\beta_A(y*x)\leq\beta_A((x*z)*(0*y))\vee\beta_A(z)\leq s$, which implies that $y*x\in L(\beta_A;s)$. Hence $(\beta_A;s)$ is an α-ideal of X.

Conversely, assume that for each $s, t \in [0,1]$, the nonempty sets $U(\alpha_A; t)$ and $L(\beta_A; s)$ are α-ideals of X. If there exist $x_0, y_0 \in S(X)$ be such that $\alpha_A(x_0y_0) < \alpha_A(y_0)$, then taking to $(\alpha_A(x_0y_0) + \alpha_A(y_0))/2$, we have $\alpha_A(x_0y_0) < t_0 < \alpha_A(y_0)$. It follows that $y_0 \in\notin U(\alpha_A; t_0)$ and $x_0y_0U(\alpha_A; t_0)$. This is a contradiction. If there exist $x_0, y_0 \in S(X)$ be such $\beta_A(x_0y_0) > \beta_A(y_0)$, then taking $s_0 = (\beta_A(x_0y_0) + \beta_A(y_0))/2$, we have $\beta_A(y_0) < s_0 < \beta_A(x_0y_0)$, it follows that $y_0 \in L(\beta_A; s_0)$ and $x_0y_0 \notin L(\beta_A; s_0)$. This is impossible. Now, let $x_0, y_0, z_0 \in X$, be such that $\alpha_A(y_0 * x_0) < \alpha_A((x_0 * z_0) * (0 * y_0)) \wedge \alpha_A(z_0)$. Putting $t_0 = (\alpha_A(y_0 * x_0) + \alpha_A((x_0 * z_0) * (0 * y_0)) \wedge \alpha_A(z_0))/2$, we have $\alpha_A(y_0 * x_0) < t_0 < \alpha_A((x_0 * z_0) * (0 * y_0)) \wedge \alpha_A(z_0)$. It follows that $(x_0 * z_0) * (0 * y_0) \in U(\alpha_A; t_0)$ and $z_0 \in U(\alpha_A; t_0)$, but $y_0 * x_0 \notin U(\alpha_A; t_0)$. This is a contradiction. Finally, assume that $x_0, y_0, z_0 \in X$ be such that $\beta_A(y_0 * x_0) > \beta_A((x_0 * z_0) * (0 * y_0)) \vee \beta_A(z_0)$. Taking $s_0 = (\beta_A(y_0*x_0)+\beta_A((x_0*z_0)+(0*y_0))\vee\beta_A(z_0))/2$, it follows that $(x_0*z_0)*(0*y_0) \in L(\beta_A; s)$ and $z_0 \in L(\beta_A; s_0)$, but $y_0 * x_0 \notin L(\beta_A; s_0)$, which is a contradiction. This completes the proof.

Theorem 2.6 Let A be an α-ideal of an IS-algebra X and let IFSA=$(\alpha_A; \beta_A)$ in X defined by

$$\alpha_A(x) = \begin{cases} t_0 & \text{if} \quad x \in A \\ t_1 & \text{otherwise} \end{cases} \qquad \beta_A(x) = \begin{cases} s_0 & \text{if} \quad x \in A \\ s_1 & \text{otherwise} \end{cases}$$

where $t_0 > t_1$ and $s_0 < s_1$ in $[0,1]$. Then IFSA=(α_A, β_A) is an IFα-ideal of X and $U(\alpha_A; t_0) = A = L(\beta_A; s_0)$.

Proof. Note that

$$U(\alpha_A; t) = \begin{cases} \phi & \text{if} \quad t_0 < t \\ A & \text{if} \quad t_1 < t \leq t_0 \\ X & \text{if} \quad t \leq t_1 \end{cases} \qquad L(\beta_A; s) = \begin{cases} \phi & \text{if} \quad s < s_0 \\ A & \text{if} \quad s_0 \leq s < s_1 \\ X & \text{if} \quad s_1 \leq s \end{cases}$$

It follows from Theorem 2.5 that IFSA=(α_A, β_A) is an IFα-ideal of X. Clearly, we have $U(\alpha_A; t_0) = A = L(\beta_A; s_0)$

Now, we consider the converse of Theorem 2.6.

Throrem 2.7 For a nonempty subset A an IS-algebra X, let IFSA=(α_A, β_A) in X which is given in Theorem 2.6. If IFSA=(α_A, β_A) is an IFα-ideal of X, then A is an α-ideal of X.

Proof. Assume that IFSA=(α_A, β_A) is an IFα-ideal of X and let $x \in S(X)$ and $y \in A$. Then $\alpha_A(xy) \geq \alpha_A(y) = t_0$ and so $xy \in U(\alpha_A; t_0) = A; \beta_A(xy) \leq \beta_A(y) = s_0$ and so $xy \in L(\beta_A; s_0) = A$. Let $x, y, z \in I(X)$ be such that $(x*z)*(0*y) \in A$ and $z \in A$. It follows that $\alpha_A(y*x) \geq \alpha_A((x*z)*(0*y))\wedge\alpha_A(z) = t_0$ and $\beta_A(y*x) \leq \beta_A((x*z)*(0*y))\vee\beta_A(z) = s_0$. Hence $y * z \in U(\alpha_A; t_0) = A$ and $y * x \in L(\beta_A; s_0) = A$. This completes the proof.

For a subset A of X, we call

$$\chi_A(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{otherwise} \end{cases}$$

the characterization function of A.

Theorem 2.8 Let A be a subset of an IS-algebra X and let IFSA=(α_A, α_A) in X. Then $\chi_A(x)$ is an IFα-ideal of X if and only if A is an α-ideal of X.

The proof is straightforward by using Theorem 2.6 and Theorem 2.7.

3. On homomorphism of IS-algebras

Definition 3.1 ([2]) A mapping $f : X \to Y$ of IS-algebras is called a homomorphism if
(i) $f(x * y) = f(x) * f(y)$ for all $x, y \in I(X)$
(ii) $f(xy) = f(x)f(y)$ for all $x, y \in S(X)$.

For any IFSA=(α_A, β_A) in Y, we define a new IFSA^f = (α^f_A, β^f_A) in X by $\alpha^f_A(x) = \alpha_A(f(x)), \beta^f_A(x) = \beta_A(f(x)), \forall x \in X$.

Theorem 3.2 Let $f : X \to Y$ be a homomorphism of IS-algebras. If IFSA=(α_A, β_A) is an IFα-ideal of Y, then IFSA^f = (α^f_A, β^f_A) in X ia an IFα-ideal of X.

Proof. For any $x, y \in Y$, there exist $a, b \in X$ such that $f(a) = x$ and $f(b) = y$. Then $\alpha_A(xy) = \alpha_A(f(a)f(b)) = \alpha_A(f(ab)) = \alpha^f_A(ab) \geq \alpha^f_A(b) = \alpha_A(f(b)) = \alpha_A(y), \beta_A(xy) = \beta(f(a)f(b)) = \beta_A(f(ab)) = \beta^f_A(ab) \leq \beta^f_A(b) = \beta_A(f(b)) = \beta_A(y)$. Suppose that IFAS$(\alpha_A, \beta_A)$ is an IFα-ideal of Y, then $\alpha^f_A(xy) = \alpha_A(f(xy)) = \alpha_A(f(x)f(y)) \geq \alpha_A(f(y)) = \alpha^f_A(y)$ and $\beta^f_A(xy) = \beta_A(f(xy)) = \beta_A(f(x)f(y)) \leq \beta_A(f(y)) = \beta^f_A(y)$. Now, let $x, y, z \in X$, then $\alpha^f_A(y * x) = \alpha_A(f(y * x)) = \alpha_A(f(y) * f(x)) \geq \alpha_A((f(x) * f(z)) * (f(0) * f(y))) \wedge \alpha_A(f(z)) = \alpha_A(f((x * z) * (0 * y))) \wedge \alpha_A(f(z)) = \alpha^f_A((x * z) * (0 * y)) \wedge \alpha^f_A(z)$ and $\beta^f_A(y * x) = \beta_A(f(y * x)) = \beta_A(f(y) * f(x)) \leq \beta_A((f(x) * f(z)) * (f(0) * f(y))) \vee \beta_A(f(z)) = \beta_A(f((x * z) * (0 * y))) \vee \beta_A(f(z)) = \beta^f_A((x * z) * (0 * y)) \vee \beta^f_A(z)$. This completes the proof.

If we strengthen the condition f, then the converse of Theorem 3.2 is obtained as follows.

Theorem 3.3 Let $f : X \to Y$ be an epimorphism of IS-algebras and let IFSA=(α_A, β_A) be in Y. If IFSA^f = (α^f_A, β^f_A) is an IFα-ideal of X, then IFSA=(α_A, β_A) is an IFα-ideal of Y.

Proof. For any $x, y \in Y$, there exist $a, b \in X$ such that $f(a) = x$ and $f(b) = y$. Then $\alpha_A(xy) = \alpha_A(xy) = \alpha_A(f(a)f(b)) = \alpha_A(f(ab)) = \alpha^f_A(ab) \geq \alpha^f_A(b) = \alpha_A(f(b)) = \alpha_A(y), \beta_A(xy) = \beta_A(f(a)f(b)) = \beta_A(f(ab)) = \beta^f_A(ab) \leq \beta^f_A(b) = \beta_A(f(b)) = \beta_A(y)$. Now, let $x, y, z \in Y$, then $f(a) = x, f(b) = y$ and $f(c) = z$ for some $a, b, c \in X. Hence \alpha_A(y * x) = \alpha_A(f(b) * f(a)) = \alpha^f_A(b * a) \geq \alpha^f_A((a * c) * (0 * b)) \wedge \alpha^f_A(c) = \alpha_A(f((a * c) * (0 * b))) \wedge \alpha_A(f(c)) = \alpha_A((f(a) * f(c)) * (f(0) * f(b))) \wedge \alpha_A(f(c)) = \alpha_A((x * z) * (0 * y)) \wedge \alpha_A(z)$ and $\beta_A(y * x) = \beta_A(f(b) * f(a)) = \beta^f_A(b * c) \leq \beta^f_A((a * c) * (0 * b)) \vee \beta_A(f(c)) = \beta_A((f(a) * f(c)) * (f(0) * f(b))) \vee \beta_A(f(c)) = \beta_A((x * z) * (0 * y)) \vee \beta_A(z)$. This completes the proof.

References

[1] Zhan Jianming & Tan Zhisong, Fuzzy α-ideals of IS-algebras, Sci. Math. Japon online 8(2003),25-27.

[2] Zhan Jianming & Tan Zhisong, Intuitionistic fuzzy K-ideals of IS-algebras, Sci.Math.Japon,57(2003),419-422.

[3] Y.B.Jun, S.M.Hong & E.H.Roh, BCI-semigroups, Honam Math. J. 15(1993),59-64.

[4] Y.B.Jun, X.L.Xin & E.H.Roh, A class of algebras related to BCI-algebras and semigroups. Soochow J. Math. 24(1998), 309-321.

[5] Y.B.Jun & K.H.Kim, Intuitionistic fuzzy ideals of BCK-algebras, Int. J.Math. & Math. Sci,24(12)(2000),839-849.

[6] E.H.Roh, Y.B,Jun & W.H.Shim, Fuzzy associative φ-ideals of IS-algebras, Int.J.Math. & Math.Sci.24(11)(2000),729-735.

[7] Liu Yonglin & Zhang Xiaohong, Fuzzy α-ideals of BCI-algebras, adv.in Math.31(2002),65-73(in Chinese).

[8] Liu Yonglin, Meng Jie etc, q-ideals and α-ideals in BCI-algebras, SEA Bull Math 24(2000),243-253.

[9] Xi Ougen, Fuzzy BCI-algebras, Math. Japon 36(1991),935-942.

[10] L.A.Zadeh, Fuzzy sets, Inform & Control, 8(1965), 338-353.

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