MINIMIZATION THEOREM IN A BANACH SPACE AND ITS APPLICATIONS

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Abstract. In this paper, we prove a minimization theorem for a proper lower semicontinuous convex function in a real Banach space, applying Takahashi's nonconvex minimization theorem. Then we give another proof of Bishop-Phelps' theorem.

1 Introduction In 1965, Brøndsted and Rockafellar [5] proved the following theorem: Let E be a real Banach space and let $f : E \to (-\infty, \infty]$ be a proper lower semicontinuous convex function. Then for all $\varepsilon > 0$ and $(x_0, x_0^*) \in \partial_\varepsilon f$, there exists $(x, x^*) \in \partial f$ such that $\|x - x_0\| \leq \sqrt{\varepsilon}$ and $\|x^* - x_0^*\| \leq \sqrt{\varepsilon}$. This is a generalization of Bishop-Phelps' theorem and dual Bishop-Phelps' theorem [2]; see also Phelps [13]. Applying Brøndsted-Rockafellar's theorem, Rockafellar [16] proved that the subdifferential of a proper lower semicontinuous convex function on a Banach space is maximal monotone. Later, Borwein [3] obtained a generalization of Brøndsted-Rockafellar's theorem by applying Ekeland's variational principle [8], and gave another proof of Rockafellar's theorem; see also Simons [18] for another proof of Rockafellar's theorem.

On the other hand, in 1976, Caristi [6] proved a fixed point theorem in a complete metric space which is a generalization of the Banach contraction principle. Ekeland [8] also proved a nonconvex minimization theorem for a proper lower semicontinuous function, bounded from below. Takahashi [21] proved the following nonconvex minimization theorem: Let (X, d) be a complete metric space and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function which is bounded from below. Suppose that, for each $u \in X$ with $f(u) > \inf_{x\in X} f(x)$, there exists $v \in X$ such that $v \neq u$ and $f(v) + d(u, v) \leq f(u)$. Then there exists $x_0 \in X$ such that $f(x_0) = \inf_{x\in X} f(x)$. This theorem was used to obtain Caristi's fixed point theorem [6], Ekeland's variational principle [8] and Nadler's fixed point theorem [12].

In this paper, applying Takahashi's nonconvex minimization theorem, we prove a minimization theorem in a Banach space. Further, using this, we give another proof of dual Bishop-Phelps' theorem [2]. We also study the metric completeness of a normed linear space.

2 Preliminaries Throughout this paper, we denote by $\mathbb{R}$ and $\mathbb{N}$ the set of all real numbers and the set of all positive integers, respectively. Let (X, d) be a metric space. Then a mapping $f : X \to (-\infty, \infty]$ $(= \mathbb{R} \cup \{\infty\})$ is said to be *proper* if there exists $a \in X$ such that $f(a) \in \mathbb{R}$. The *domain* of f is defined by $D(f) = \{x \in X : f(x) \in \mathbb{R}\}$. Also f is said to be *lower semicontinuous* if the set $\{x \in X : f(x) \leq r\}$ is closed in X for all $r \in \mathbb{R}$. Let E be a (real) normed linear space and let E^* be the dual space of E. Then a mapping $f : E \to (-\infty, \infty]$ is said to be *convex* if

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$$

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for all $x, y \in E$ and $\alpha \in (0,1)$. Let $f : E \to (-\infty, \infty]$ be a proper and convex function. Then the *subdifferential* ∂f of f is defined as follows:

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle y - x, x^* \rangle \leq f(y) \text{ for all } y \in E\}$$

for all $x \in E$. It is easy to prove that $0 \in \partial f(x_0)$ if and only if $f(x_0) = \min_{x \in E} f(x)$. For $\varepsilon > 0$, the *approximate subdifferential* $\partial_\varepsilon f$ of f is defined as follows:

$$\partial_\varepsilon f(x) = \{x^* \in E^* : f(x) + \langle y - x, x^* \rangle \leq f(y) + \varepsilon \text{ for all } y \in E\}$$

for all $x \in E$. The *domain* of ∂f and the *range* of ∂f are defined by $D(\partial f) = \{x \in E : \partial f(x) \neq \emptyset\}$ and $R(\partial f) = \{x^* \in E^* : x^* \in \partial f(x) \text{ for some } x \in D(\partial f)\}$, respectively. The *duality mapping* $J : E \to 2^{E^*}$ is defined as follows:

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in E$. It is known that if $j(x) = 2^{-1}\|x\|^2$ for all $x \in E$, then $\partial j(x) = J(x)$ for all $x \in E$. A function $f : E \to \mathbb{R}$ is said to be *affine* if

$$f(\alpha x + (1-\alpha)y) = \alpha f(x) + (1-\alpha) f(y)$$

for all $x, y \in E$ and $\alpha \in [0,1]$. If $f : E \to \mathbb{R}$ is affine continuous, then there exist $x^* \in E^*$ and $\mu \in \mathbb{R}$ such that $f(x) = \langle x, x^* \rangle + \mu$ for all $x \in E$. We know the following theorems; see [13, 20]:

Theorem 2.1. *Let E be a normed linear space, let $f : E \to (-\infty, \infty]$ be a proper lower semicontinuous convex function and let $g : E \to \mathbb{R}$ be a continuous convex function. Then*

$$\partial(f + g)(x) = \partial f(x) + \partial g(x)$$

for all $x \in E$.

Theorem 2.2. *Let E be a normed linear space and let $f : E \to (-\infty, \infty]$ be a proper lower semicontinuous convex function. Then there exist $x^* \in E^*$ and $\mu \in \mathbb{R}$ such that*

$$f(x) \geq \langle x, x^* \rangle + \mu$$

for all $x \in E$.

We also know the following theorem; see [7]:

Theorem 2.3. *Let E be a Banach space, let $f_1, f_2, \ldots, f_m : E \to (-\infty, \infty]$ be proper lower semicontinuous convex functions and let f be a function defined by*

$$f(x) = \max_{i=1,2,\ldots,m} f_i(x)$$

for all $x \in E$. If $D(f)$ has a nonempty interior, then

$$\partial f(x) = co\Big(\bigcup \{\partial f_i(x) : i \in I(x)\}\Big)$$

for all x in the interior of $D(f)$, where $I(x) = \{i = 1, 2, \ldots, m : f(x) = f_i(x)\}$.

3 Minimization Theorem and its Applications Applying Takahashi's nonconvex minimization theorem, we prove the following theorem in a Banach space:

Theorem 3.1. *Let E be a Banach space and let $f : E \to (-\infty, \infty]$ be a proper lower semicontinuous convex function which is bounded from below. Suppose that there exists $\delta > 0$ such that $(x, x^*) \in \partial f$ and $f(x) > \inf_{w \in E} f(w)$ imply $\|x^*\| \geq \delta$. Then there exists $x_0 \in E$ such that $f(x_0) = \inf_{w \in E} f(w)$.*

Proof. Suppose the existence of $\delta > 0$ such that $(x, x^*) \in \partial f$ and $f(x) > \inf_{w \in E} f(w)$ imply $\|x^*\| \geq \delta$. For $u \in E$ satisfying $f(u) > \inf_{w \in E} f(w)$, we define a proper lower semicontinuous convex function F from E into $(-\infty, \infty]$ as follows:

$$F(x) = f(x) + \frac{\delta}{2} \|x - u\|$$

for all $x \in E$. Let $k(x) = \|x - u\|$ for all $x \in X$. Then, it holds from Theorem 2.1 that

$$\begin{aligned} \partial F(u) &= \partial\Big(f + \frac{\delta}{2} k\Big)(u) \\ &= \partial f(u) + \frac{\delta}{2} \partial k(u) \\ &= \partial f(u) + \Big\{x^* \in E^* : \|x^*\| \leq \frac{\delta}{2}\Big\}. \end{aligned}$$

If $0 \in \partial F(u)$, then we have $u^* \in E^*$ and $v^* \in E^*$ such that

$$u^* \in \partial f(u),\ \|v^*\| \leq \frac{\delta}{2} \text{ and } 0 = u^* + v^*.$$

Hence we have $\|u^*\| \leq \delta/2$. By assumption, we have $f(u) = \inf_{w \in E} f(w)$. This contradicts to $f(u) > \inf_{w \in E} f(w)$. Hence we have

$$0 \notin \partial F(u).$$

Hence there exists $v \in E$ such that $F(v) < F(u)$, that is,

$$f(v) + \frac{\delta}{2} \|u - v\| < f(u).$$

Since E is complete and f is bounded from below, by Takahashi's minimization theorem, there exists $x_0 \in E$ such that $f(x_0) = \inf_{w \in E} f(w)$. This completes the proof. □

Applying Theorem 3.1, we first prove the following dual Bishop-Phelps' theorem [2]:

Theorem 3.2 (Bishop-Phelps [2]). *Let E be a Banach space, let C be a nonempty bounded closed convex subset of E and let A be the set of all continuous linear functionals $x^* \in E^*$ such that*

$$x^*(x_0) = \max_{x \in C} x^*(x)$$

for some $x_0 \in C$. Then A is norm dense in E^*.

Proof. Assume that there exists $a^* \in E^*$ such that $a^* \notin \overline{A}$. Then there exists $\delta > 0$ such that $S_\delta(a^*) \cap A = \emptyset$, where $S_\delta(a^*) = \{x^* \in E^* : \|x^* - a^*\| < \delta\}$. Let g be the indicator function of C, that is, $g(x) = 0$ if $x \in C$ and $g(x) = \infty$ if $x \notin C$. Then we have

$$R(\partial g) = A.$$

So, it holds that

$$x^* \in R(\partial g) \Longrightarrow \|x^* - a^*\| \geq \delta. \tag{1}$$

Define a proper lower semicontinuous convex function $f : E \to (-\infty, \infty]$ as follows:

$$f(x) = g(x) - a^*(x)$$

for all $x \in E$. Then f is bounded from below. Indeed, since C is bounded, there exists $M > 0$ such that $\|x\| \leq M$ for all $x \in C$. This implies that

$$\inf_{x\in E} f(x) = \inf_{x\in E}\{g(x) - a^*(x)\} = \inf_{x\in C}\{-a^*(x)\} \geq -M\|a^*\|.$$

Hence f is bounded from below.

Let $(z, z^*) \in \partial f$ be given. Since $\partial f(z) = \partial g(z) - a^*$, there exists $x^* \in \partial g(z)$ such that $z^* = x^* - a^*$. Since $x^* \in R(\partial g)$, by (1), we have $\|z^*\| = \|x^* - a^*\| \geq \delta$. Hence we have

$$(z, z^*) \in \partial f \Longrightarrow \|z^*\| \geq \delta. \tag{2}$$

Applying Theorem 3.1, we have $x_0 \in E$ such that $f(x_0) = \inf_{x\in E} f(x)$. This implies $0 \in \partial f(x_0)$ and this contradicts to (2). Therefore we have $\overline{A} = E^*$. This completes the proof. □

Next, applying Theorem 3.1, we prove that if $f : E \to (-\infty, \infty]$ is a proper lower semicontinuous convex function which is coercive, then $\overline{R(\partial f)} = E^*$. Before proving it, we prove the following lemma:

Lemma 3.3. *Let E be a normed linear space and let $f : E \to (-\infty, \infty]$ be a proper lower semicontinuous convex function satisfying*

$$\|x_n\| \to \infty \Longrightarrow f(x_n) \to \infty.$$

Then f is bounded from below.

Proof. Suppose that f is not bounded from below. Then there exists a sequence $\{x_n\}$ in E such that $f(x_n) \to -\infty$. This sequence $\{x_n\}$ is bounded. In fact, if $\{x_n\}$ is unbounded, then we have a subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}$ such that $\|x_{n_i}\| \to \infty$. By assumption, we have $f(x_{n_i}) \to \infty$. This contradicts to $f(x_{n_i}) \to -\infty$, and hence $\{x_n\}$ is bounded. Thus we have $M > 0$ satisfying $\|x_n\| \leq M$ for all $n \in \mathbb{N}$.

Applying Theorem 2.2, we have $x^* \in E^*$ and $\mu \in \mathbb{R}$ such that $f(x) \geq \langle x, x^*\rangle + \mu$ for all $x \in E$. Thus we have

$$f(x_n) \geq \langle x_n, x^*\rangle + \mu \geq -M\|x^*\| + \mu$$

for all $n \in \mathbb{N}$. Thus $\{f(x_n)\}$ is bounded from below. This contradicts to $f(x_n) \to -\infty$. Therefore f is bounded from below. □

Let E be a normed linear space and let $f : E \to (-\infty, \infty]$ be a proper lower semicontinuous convex function. Then f is said to be *coercive* [4] if

$$\|x_n\| \to \infty \Longrightarrow \frac{f(x_n)}{\|x_n\|} \to \infty.$$

Theorem 3.4. *Let E be a Banach space and let $f : E \to (-\infty, \infty]$ be a proper lower semicontinuous convex function which is coercive. Then*

$$\overline{R(\partial f)} = E^*.$$

Proof. Assume that there exists $a^* \in E^*$ such that $a^* \notin \overline{R(\partial f)}$. Then there exists $\delta > 0$ such that $S_\delta(a^*) \cap R(\partial f) = \emptyset$. Define a proper lower semicontinuous convex function $g : E \to (-\infty, \infty]$ as follows: $g(x) = f(x) - a^*(x)$ for all $x \in E$. Then g is coercive. In fact, let $\{x_n\}$ be a sequence in E such that $\|x_n\| \to \infty$. Then since we have

$$\frac{g(x_n)}{\|x_n\|} = \frac{f(x_n)}{\|x_n\|} - \frac{\langle x_n, a^* \rangle}{\|x_n\|} \geq \frac{f(x_n)}{\|x_n\|} - \|a^*\|$$

and f is coercive, we have that $g(x_n)/\|x_n\| \to \infty$. Thus g is coercive. Then it follows that $\|x_n\| \to \infty \Longrightarrow g(x_n) \to \infty$. So, by Lemma 3.3, g is bounded from below.

Let $(z, z^*) \in \partial g$ be given. Since $\partial g(z) = \partial f(z) - a^*$, there exists $x^* \in \partial f(z)$ such that $z^* = x^* - a^*$. Since $x^* \in R(\partial f)$, we have $\|z^*\| = \|x^* - a^*\| \geq \delta$. Hence we have

$$(z, z^*) \in \partial g \Longrightarrow \|z^*\| \geq \delta. \tag{3}$$

Applying Theorem 3.1, we have $x_0 \in E$ such that $g(x_0) = \inf_{x \in E} g(x)$. This implies $0 \in \partial g(x_0)$. This contradicts to (3). Therefore $\overline{R(\partial f)} = E^*$. This completes the proof. □

Corollary 3.5. *Let E be a Banach space, let J be the duality mapping of E and let $f : E \to (-\infty, \infty]$ be a proper lower semicontinuous convex function. Then*

$$\overline{R(J + r\partial f)} = E^*$$

for all $r > 0$.

Proof. Let $r > 0$ be given and let $j(x) = 2^{-1}\|x\|^2$ for all $x \in E$. Then it holds from Theorem 2.1 that $\partial(j + rf)(x) = J(x) + r\partial f(x)$ for all $x \in E$. By Theorem 2.2, there exist $x^* \in E^*$ and $\mu \in \mathbb{R}$ such that $rf(x) \geq \langle x, x^* \rangle + \mu$ for all $x \in E$. Let $\{x_n\}$ be a sequence in E such that $\|x_n\| \to \infty$. Then since

$$\frac{j(x_n) + rf(x_n)}{\|x_n\|} \geq \frac{1}{2}\|x_n\| + \frac{\langle x_n, x^* \rangle}{\|x_n\|} + \frac{\mu}{\|x_n\|} \geq \frac{1}{2}\|x_n\| - \|x^*\| + \frac{\mu}{\|x_n\|},$$

we have

$$\frac{j(x_n) + rf(x_n)}{\|x_n\|} \longrightarrow \infty.$$

Hence the function $j+rf$ is coercive. By Theorem 3.4, we have $\overline{R(J + r\partial f)} = \overline{R(\partial(j + rf))} = E^*$. □

4 The Metric Completeness of a Normed Linear Space In this section, we study the metric completeness of a normed linear space. The following theorem was proved by Takahashi [21].

Theorem 4.1 (Takahashi [21]). *Let X be a metric space. Then the following are equivalent:*

1. *X is complete;*
2. *for each Lipschitz continuous function $f : X \to [0, \infty)$, if for every $u \in X$ with $f(u) > \inf_{w \in X} f(w)$, there exists $v \in X$ such that $v \neq u$ and $f(v) + d(u, v) \leq f(u)$, then there exists $x_0 \in X$ such that $f(x_0) = \inf_{w \in X} f(w)$.*

In the case where the space X is a normed linear space, the mapping $f : X \to (-\infty, \infty]$ defined in the proof of Theorem 4.1 is a convex function. So, we have the following theorem:

Theorem 4.2. *Let E be a normed linear space. Then the following are equivalent:*

1. *E is complete;*
2. *for each Lipschitz continuous convex function $f : E \to [0,\infty)$, if there exists $\delta > 0$ such that $(x, x^*) \in \partial f$ and $f(x) > \inf_{w\in E} f(w)$ imply $\|x^*\| \geq \delta$, then there exists $x_0 \in E$ such that $f(x_0) = \inf_{w\in E} f(w)$.*

Proof. It is immediate from Theorem 3.1 that (1) implies (2). We prove that (2) implies (1). Let $f : E \to [0,\infty)$ be a Lipschitz continuous convex function such that, for each $u \in E$ with $f(u) > \inf_{w\in E} f(w)$, there exists $v \in E$ such that $v \neq u$ and $f(v)+\|u-v\| \leq f(u)$. Fix any $(x, x^*) \in E\times E^*$ such that $f(x) > \inf_{w\in E} f(w)$ and $\|x^*\| < 1$. We show $x^* \notin \partial f(x)$. Indeed, if $f(x) = \infty$, we have $\partial f(x) = \emptyset$. In the case of $f(x) < \infty$, since $f(x) > \inf_{w\in E} f(w)$, there exists $y \in E$ such that $y \neq x$ and $f(y)+\|x-y\| \leq f(x)$. Since $f(x) < \infty$, we have $f(y) < \infty$. Thus we have

$$\begin{aligned}\langle x - y, x^*\rangle &\leq \|x - y\|\|x^*\| \\ &< \|x - y\| \leq f(x) - f(y)\end{aligned}$$

and hence

$$f(y) < f(x) + \langle y - x, x^*\rangle.$$

This implies $x^* \notin \partial f(x)$. Thus it holds that

$$(x, x^*) \in \partial f \text{ and } f(x) > \inf_{w\in E} f(w) \Longrightarrow \|x^*\| \geq 1.$$

By assumption, there exists $x_0 \in E$ such that $f(x_0) = \inf_{w\in E} f(w)$. From Theorem 4.1, E is complete. □

5 Example In this section, we study an example of convex functions satisfying the assumption in Theorem 3.1. We first prove the following lemma:

Lemma 5.1. *Let E be a Banach space and let $f : E \to (-\infty,\infty]$ be a proper lower semicontinuous convex function which is bounded from below. Suppose that the set*

$$\{\partial f(x) : x \in E\}$$

is finite. Then there exists $x_0 \in E$ such that $f(x_0) = \inf_{w\in E} f(w)$.

Proof. By assumption, we have points $x_1, x_2, \ldots, x_r$ in E such that

$$\{\partial f(x_1), \partial f(x_2), \ldots, \partial f(x_r)\} = \{\partial f(x) : x \in E\}.$$

Put

$$I_0 = \{i = 1, 2, \ldots, r : f(x_i) > \inf_{w\in E} f(w)\}.$$

If I_0 is empty, then $f(x) = \inf_{w\in E} f(w)$ for all $x \in E$. So, we may assume that I_0 is nonempty. Fix any $i \in I_0$. Then, there exists $\delta_i > 0$ such that

$$x^* \in \partial f(x_i) \Longrightarrow \|x^*\| \geq \delta_i.$$

In fact, if not, then there exists a sequence $\{x^*_n\}$ in $\partial f(x_i)$ such that $\|x^*_n\| \to 0$. Since $\partial f(x_i)$ is closed, we have $0 \in \partial f(x_i)$. This implies $f(x_i) = \inf_{w\in E} f(w)$. This contradicts to $i \in I_0$.

Put $\delta = \min_{i\in I_0} \delta_i (> 0)$. If $(x, x^*) \in \partial f$ and $f(x) > \inf_{w\in E} f(w)$, then there exists $i \in I_0$ such that $\partial f(x) = \partial f(x_i)$. Hence we have $\|x^*\| \geq \delta_i \geq \delta$. Therefore, by Theorem 3.1, there exists $x_0 \in E$ such that $f(x_0) = \inf_{w\in E} f(w)$. □

Using Theorem 2.3 and Lemma 5.1, we can prove the following:

Theorem 5.2. *Let E be a Banach space and let $f_1, f_2, \dots, f_m : E \to \mathbb{R}$ be affine continuous functions. Suppose that the function f defined by*

$$f(x) = \max_{i=1,2,\dots,m} f_i(x)$$

for all $x \in E$ is bounded from below. Then there exists $x_0 \in E$ such that $f(x_0) = \inf_{w \in E} f(w)$.

Proof. Since each f_i is affine continuous, we have $x_i^* \in E^*$ and $\mu_i \in \mathbb{R}$ such that

$$f_i(x) = \langle x, x_i^* \rangle + \mu_i$$

for all $x \in E$. Hence we have $\partial f_i(x) = x_i^*$ for all $x \in E$ and $i = 1, 2, \dots, m$. Put $I(x) = \{i = 1, 2, \dots m : f(x) = f_i(x)\}$ for all $x \in E$. Since $D(f) = E$, by Theorem 2.3, we have

$$\partial f(x) = co\Big(\bigcup \{\partial f_i(x) : i \in I(x)\}\Big) = co\{x_i^* : i \in I(x)\}$$

for all $x \in E$. Since $\{I(x) : x \in E\}$ is finite, the set

$$\{\partial f(x) : x \in E\} = \{co\{x_i^* : i \in I(x)\} : x \in E\}$$

is also finite. Therefore, by Lemma 5.1, there exists $x_0 \in E$ such that $f(x_0) = \inf_{w \in E} f(w)$. □

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