SHAPES OF PLANAR CUBIC CURVES

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Abstract. We derive a value to determine the shape of a cubic curve segment. It can be easily calculated from the Hermite data at two points.

1 Introduction and Description of Method Walton & Meek have examined the shapes of the whole parametric cubic curves ([3]). Their paper presents results on the number and location of curvature extrema of the whole cubic segments.

With help of *Mathematica*, we derive a value to characterize the shapes of the cubic curves which is easily computed from given Hermite data at two specified points. The value enables us to determine in advance the number and location of the curvature extrema of the resulting curve without its practical computation.

We consider a cubic curve: $\boldsymbol{z}(t), -\infty < t < \infty$ satisfying $\boldsymbol{z}(0) = \boldsymbol{z}_0$ and $\boldsymbol{z}(1) = \boldsymbol{z}_1$. Its signed curvature $\kappa(t)$ is given by

$$\kappa(t) = (\boldsymbol{z}' \times \boldsymbol{z}'')(t) / \|\boldsymbol{z}'(t)\|^3 \tag{1}$$

where " $\times$ " and $\|\bullet\|$ mean the cross product of two vectors and the Euclidean norm, respectively. We assume that $\boldsymbol{z}'(0)(= \boldsymbol{z}'_0)$ and $\boldsymbol{z}'(1)(= \boldsymbol{z}'_1)$ are linearly independent, i.e., $\boldsymbol{z}'_0 \times \boldsymbol{z}'_1 (= D) \neq 0$. Then, $\Delta\boldsymbol{z}(= \boldsymbol{z}_1 - \boldsymbol{z}_0)$ can be represented in terms of $\boldsymbol{z}'_0$ and $\boldsymbol{z}'_1$:

$$\Delta\boldsymbol{z} = \lambda\boldsymbol{z}'_0 + \mu\boldsymbol{z}'_1 \tag{2}$$

where $D(\lambda, \mu) = (\Delta\boldsymbol{z} \times \boldsymbol{z}'_1, \boldsymbol{z}'_0 \times \Delta\boldsymbol{z})$. Note the identity

$$\begin{aligned} \boldsymbol{z}(t) &= f(t)\boldsymbol{z}_0 + f(1-t)\boldsymbol{z}_1 + g(t)\boldsymbol{z}'_0 - g(1-t)\boldsymbol{z}'_1 \\ &= \{f(t) + f(1-t)\}\,\boldsymbol{z}_0 + \{\lambda f(1-t) + g(t)\}\,\boldsymbol{z}'_0 + \{\mu f(1-t) - g(1-t)\}\,\boldsymbol{z}'_1 \end{aligned} \tag{3}$$

with $f(t) = (1-t)^2(1+2t),\ g(t) = (1-t)^2 t$. A simple calculation gives

Lemma 1 $\boldsymbol{z}'(t) \times \boldsymbol{z}''(t)(= \phi(t))$ *reduces to*

$$-2D\{3(1-\lambda-\mu)t^2 - 3(1-2\mu)t + 1 - 3\mu\} \tag{4}$$

The following theorem provides an alternative derivation of the results presented in Walton & Meek ([3]) on the shapes (cusp, loop, inflections points) of the cubic curves without use of translation, rotation, uniform scaling and reflection.

Theorem 1 *The presence of a singularity and inflection points on the cubic curve is characterized by the sign of* I $(=1 - 4\lambda - 4\mu + 12\lambda\mu)$*:*
Case 1 (Cusp): $I = 0$ $((\lambda, \mu) \neq (1/2, 1/2))$ *a cusp, no inflection point*
Case 2 (Loop): $I > 0$ *a loop, no inflection point*
Case 3 (Two or one inflection point): $I < 0$ *two inflection points* $(\lambda + \mu \neq 1)$ *or one inflection point* $(\lambda + \mu = 1)$*, no singularity*
Case 4 (Quadratic): $I = 0$ $((\lambda, \mu) = (1/2, 1/2))$ *no singularity, no inflection point*

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Dependent on the sign of I, we give a simple proof of the above four cases.
Case 1: Note that a cusp occurs if and only if the quadratic polynomials $\boldsymbol{z}'(t)(=(x'(t),y'(t))$ have the common zero(s). Sylvester's resultant of the above quadratic ones is equal to $-3D^2I$ and at least one of $\boldsymbol{z}'(t)$ is really quadratic for $(\lambda,\mu)\neq(1/2,1/2)$ since its coefficient of t^2 is $3(1-2\lambda)\boldsymbol{z}_0'+3(1-2\mu)\boldsymbol{z}_1'$. Hence, a cusp occurs if $I=0$ and $(\lambda,\mu)\neq(1/2,1/2)$. The common zero is $p=1/(3-6\lambda)$.
Case 2: $(\boldsymbol{z}(p)-\boldsymbol{z}(q))/(p-q)=(0,0)$ $(p\neq q)$ gives a homogeneous system of equations in $A(=(1-2\lambda)(p^2+pq+q^2)+(3\lambda-2)(p+q)+1)$ and $B(=(1-2\mu)(p^2+pq+q^2)+(3\mu-1)(p+q))$ whose coefficient matrix is $(\boldsymbol{z}_0',\boldsymbol{z}_1')$. Since the matrix is nonsingular, we obtain $A=B=0$, i.e., if $I>0$

$$p,q=\frac{1-2\mu\pm\sqrt{I}}{2(1-\lambda-\mu)} \tag{5}$$

Case 3: The discriminant of the quadratic (4) is $-12I$ if $\lambda+\mu\neq1$.
Case 4: Note that (4) is constant.

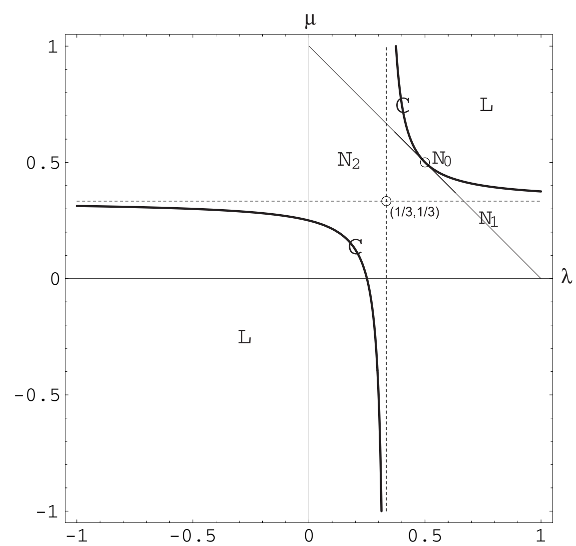

Figure 1: Singularities and inflection points on whole cubic segments.

Remark 1: In Case 3 ($\lambda+\mu=1$), a transformation makes a special case of a cubic function ([3]) since with the coefficient $\boldsymbol{z}_3$ of t^3 of $\boldsymbol{z}(t)$,

$$(\boldsymbol{z}_3\times\boldsymbol{z})(t)=(-1+2\mu)(tD+\boldsymbol{z}_0\times\Delta\boldsymbol{z}') \tag{6}$$

In Figures 1-2, $N_i(i=0,1,2),L$ and C mean the whole and restricted cubic curves have i inflection points, a loop and a cusp, respectively. Here we note the similar results on the restricted (not whole) cubic segment $\boldsymbol{z}(t),0\leq t\leq1$ ([1]). Since our analysis does not use any algebraic manipulation, Cases 1-2 require the conditions so that the common zero p $\in(0,1)$ and the both $(p,q)\in(0,1)$, respectively. Case 3 requires to count the number of the zeros of (4) $\in(0,1)$. As a consequence of these results, for example, we see that a cusp occurs in (or out of) the restricted segment if (λ,μ) lies on the lower (or upper) branch of the hyperbolic $I = 0$. The following lemma helps us examine the curvature extrema where " $\cdot$ " means the dot product of two vectors.

Lemma 2 *For* $v(t) = \kappa'(t)\|\mathbf{z}'(t)\|^5$,

$$v(t) = -3\phi(t)\mathbf{z}'(t)\cdot\mathbf{z}''(t) + \phi'(t)\,\|\mathbf{z}'(t)\|^2$$
$$v'(t) = -\phi(t)\left\{3\,\|\mathbf{z}''(t)\|^2 + 4\mathbf{z}'(t)\cdot\mathbf{z}^{(3)}(t)\right\}$$

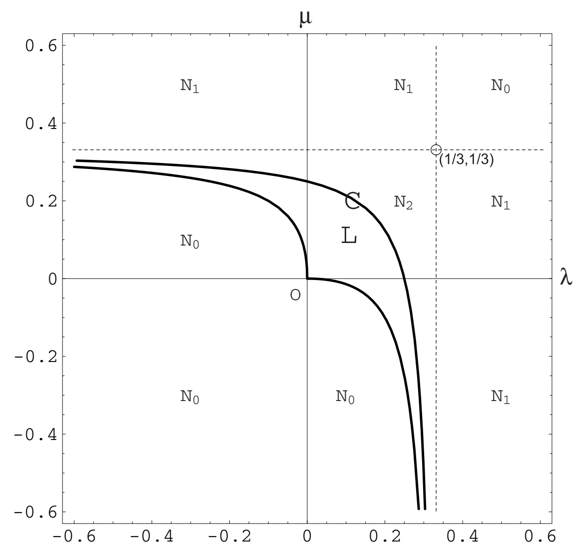

Figure 2: Singularities and inflection points on restricted cubic segments.

For the above Cases 1-4, we consider the curvature extrema where $(\alpha,\beta,\gamma) = (\|\boldsymbol{z}_0'\|, \|\boldsymbol{z}_1'\|, \boldsymbol{z}_0'\cdot\boldsymbol{z}_1')$.
Case 1: Letting $\lambda = 1/3 - m/6, \mu = 1/3 - 1/(6m)$ $(m \neq -1)$,

$$m^3 v(t) = D(-1+t+mt)^3 Q(t) \tag{7}$$

where quadratic $Q(t) (= a_1t^2 - b_1t + c_1)$ satisfies

$$\begin{aligned} a_1 &= 4(1+m)(\alpha^2m^2 + 2\gamma m + \beta^2)\left(= 4(1+m)\,\|m\boldsymbol{z}_0' + \boldsymbol{z}_1'\|^2\right)\\ b_1 &= 5\alpha^2m^3 + (8\alpha^2 + 5\gamma)m^2 + 11\gamma m + 3\beta^2, \quad c_1 = m\left\{m(m+4)\alpha^2 + 3\gamma\right\}\\ Q\left(\frac{1}{1+m}\right) &= \frac{\alpha^2m^4 - 2\gamma m^2 + \beta^2}{1+m}\left(= \frac{\|m^2\boldsymbol{z}_0' - \boldsymbol{z}_1'\|^2}{1+m}\right)\end{aligned} \tag{8}$$

Note that Q is a linear combination of $\alpha^2, \beta^2, \gamma$ to make the above derivation of $Q(1/(1+m))$ easier. Here we note that $t = 1/(1+m)$ does not give the curvature extrema since then the denominator of $\kappa'(t)$ vanishes. Since $a_1Q(1/(1+m)) > 0$, no or two curvature extrema occur and the two extrema (if exist) are on one side of the cusp.
Case 2: Since $\phi(t)$ of $v'(t)$ has no zero, the curve has one zero or three curvature extrema.

Mathematica helps us obtain the following relation with (p, q) by (5)

$$v(p) + v(q) = -48v\left(\frac{p+q}{2}\right) \tag{9}$$

which shows there is at least one curvature extremum on the same side of the loop.
Case 3: If $\lambda + \mu \neq 1$, unlike a cusp or loop case, $\phi(t)$ has two distinct zeros:

$$p, q = \frac{3(1-2\mu) \mp \sqrt{-3I}}{6(1-\lambda-\mu)} \tag{10}$$

Note $\phi(t) = 0$ for $t = p, q$ and $\phi'(t) = -6D\left\{(1-\lambda-\mu)t - (1-2\mu)\right\}$ to obtain

$$v(p) = 2D\left\|\boldsymbol{z}'(p)\right\|^2\sqrt{-3I}, \quad v(q) = -2D\left\|\boldsymbol{z}'(q)\right\|^2\sqrt{-3I} \tag{11}$$

If the quadratic factor $\psi(t)(= a_2t^2 + b_2t + c_2)$ in braces of $v'(t)$ of Lemma 2 has no zero or a double zero, there exists a single curvature extremum in the loop, and two extrema are on the opposite sides of the loop. Next, assume that $\psi(t)$ has two distinct zeros, i.e., $b_2^2 - 4a_2c_2 > 0$. Then,

$$\begin{aligned} &a_2^2\left(-\frac{b_2}{2a_2} - p\right)\left(-\frac{b_2}{2a_2} - q\right) - \frac{5(b_2^2 - 4a_2c_2)}{8} \\ &\left(= 5400(1-\lambda-\mu)^2(\alpha^2\beta^2 - \gamma^2)\right) = 5400\left\{D(1-\lambda-\mu)\right\}^2 \end{aligned} \tag{12}$$

where coefficients (a_2, b_2, c_2) are given by

$$\begin{aligned} &a_2 = 180\left\{(1-2\lambda)^2\alpha^2 + 2(1-2\lambda)(1-2\mu)\gamma + (1-2\mu)^2\beta^2\right\} \\ &\left(= 180\left\|(1-2\lambda)\boldsymbol{z}_0' + (1-2\mu)\boldsymbol{z}_1'\right\|^2\right) \\ &b_2 = -120\left\{(2-7\lambda+6\lambda^2)\alpha^2 + (3-5\lambda-7\mu+12\lambda\mu)\gamma + (1-5\mu+6\mu^2)\beta^2\right\} \\ &c_2 = 12\left\{(6-16\lambda+9\lambda^2)\alpha^2 + 2(3-3\lambda-8\mu+9\lambda\mu)\gamma + (1-3\mu)^2\beta^2\right\} \end{aligned} \tag{13}$$

Note the position of the symmetric axis of $\psi(t)$ to see that the two zeros do not lie in the interval (p, q) (or (q, p)). Therefore, $v(p)v(q) < 0$ shows that a single extremum is on the curve segment between the two inflection points corresponding to $t = p, q$. If there are five curvature extrema, there are one and three extrema in the opposite sides of the curve segment, respectively.

If $\lambda + \mu = 1 (\Leftrightarrow (\lambda, \mu) = (1/2 + s, 1/2 - s), s \neq 0)$, $\phi(t)$ has a single zero $p = (6s-1)/(12s)$. *Mathematica* helps us get $v(t) = D\left\{a_3(t-p)^4 + b_3(t-p)^2 + c_3\right\}$:

$$\begin{aligned} &a_3 = -2160s^3\left\|\boldsymbol{z}_0' - \boldsymbol{z}_1'\right\|^2, \quad b_3 = 12s\left\{\left\|(1+6s)\boldsymbol{z}_0' + (1-6s)\boldsymbol{z}_1'\right\|^2 - 4\gamma\right\} \\ &c_3 = \frac{1}{48s}\left\|(1+6s)^2\boldsymbol{z}_0' - (1-6s)^2\boldsymbol{z}_1'\right\|^2 \end{aligned} \tag{14}$$

Since $a_3c_3 < 0$, two curvature extrema exist on the opposite sides of the inflection point.
Case 4: Note

$$v(t) = 3D\left(\alpha^2 - \gamma - t\left\|\boldsymbol{z}_0' - \boldsymbol{z}_1'\right\|^2\right) \tag{15}$$

which shows there is a single curvature extremum on the quadratic curve.

The following theorem presents the number and positions of the curvature extrema:

Theorem 2 *Let M (=the number of the curvature extrema). Then, for Cases 1-4 of Theorem 1, we obtain*
Case 1 (Cusp): $M = 0, 2$. *If $M = 2$, curvature extrema are on the same side of the cusp.*
Case 2 (Loop): $M = 1, 3$. *At least one curvature extremum is in the loop.*
Case 3 (Two or one inflection point): *If $\lambda + \mu \neq 1$, $M = 3, 5$. One curvature extremum is on the curve segment connecting the two inflection points. On the (exterior) opposite sides of the connecting curve segment one extremum on each side for $M = 3$ or one and three extrema for $M = 5$ exist. If $\lambda + \mu = 1$, two curvature extrema exist on the opposite sides of the inflection point.*
Case 4 (quadratic): $M = 1$.

Finally we give a remark for $D(= \boldsymbol{z}_0' \times \boldsymbol{z}_1') = 0$, for example, $\boldsymbol{z}_1' = r\boldsymbol{z}_0'$. Assume $\boldsymbol{z}_0' \times \Delta\boldsymbol{z}(= \bar{D}) \neq 0$; otherwise $\boldsymbol{z}(t)$ reduces to a linear segment and we omit this case. Then, linearly independent $\Delta\boldsymbol{z}$ and $\boldsymbol{z}_0'$ are used in (3) instead of $\boldsymbol{z}_0'$ and $\boldsymbol{z}_1'$. Note the identity

$$\boldsymbol{z}(t) = f(t)\boldsymbol{z}_0 + f(1-t)\boldsymbol{z}_1 + g(t)\boldsymbol{z}_0' - g(1-t)\boldsymbol{z}_1' \tag{16}$$

First, note $\phi(t)(= \boldsymbol{z}'(t) \times \boldsymbol{z}''(t)) = 6\bar{D}\left\{(t-1)^2 - rt^2\right\}$. Next, (i) Sylvester's resultant of quadratic $\boldsymbol{z}'(t)$ is $36\bar{D}^2 r$ (note that a cusp occurs if $\boldsymbol{z}'(t)$ has common roots), and their coefficients of t^2 are $3\{(1+r)\boldsymbol{z}_0' - 2\Delta\boldsymbol{z}\}$. Therefore, a cusp occurs for $r = 0$ at $t = 1$. (ii) $(\boldsymbol{z}(p) - \boldsymbol{z}(q))/(p-q) = \boldsymbol{0}, p \neq q$ gives a system of homogeneous equations in $A(= (1+r)(p^2 + pq + q^2) - (2+r)(p+q) + 1)$ and $B(= 2(p^2 + pq + q^2) - 3(p+q))$ whose coefficient matrix is $(\boldsymbol{z}_0', -\Delta\boldsymbol{z})$. Note $\bar{D} \neq 0$ to obtain $A = B = 0$, i.e., $p, q = (1 \pm \sqrt{-3r})/(1-r)$ $(r < 0)$. Hence, a loop exists for $r < 0$. (iii) For $r > 0$ $(r \neq 1)$, $\phi(t)$ has two zeros $p, q = 1/(1 \mp \sqrt{r})$ where the curve has two inflection points, and for $r = 1$, $\phi(t)$ has one zero $t = 1/2$ where an inflection point occurs.

For the curvature extrema, we require to check the following results:
(i) for $r = 0$, $v(t) = -12\bar{D}s^3(18s^2\|\boldsymbol{c}\|^2 + 15s\boldsymbol{c}\cdot\boldsymbol{d} + 2\|\boldsymbol{d}\|^2)$, $t = s+1$ with $\boldsymbol{c} = \boldsymbol{z}_0' - 2\Delta\boldsymbol{z}$ and $\boldsymbol{d} = \boldsymbol{z}_0' - 3\Delta\boldsymbol{z}$. Here $s = 0$ makes the denominator of the derivative of the curvature, and so it does not give the curvature extrema. In addition, the signs of the coefficient of s^2 and the constant term are of the same. Therefore, if the two roots exist, they are of the same sign (with respect to s), i.e., both of them are greater or less than one (with respect to t) where the cusp occurs. (ii) for $r < 0$, $p, q = (1 \pm \sqrt{-3r})/(1-r)$ and (9) is valid. (iii) for $r > 0$ $(r \neq 1)$, with $(p, q) = (1/(1-\sqrt{r}), 1/(1+\sqrt{r}))$, $v(p) = 12\bar{D}\|\boldsymbol{z}'(p)\|^2\sqrt{r}, v(q) = -12\bar{D}\|\boldsymbol{z}'(q)\|^2\sqrt{r}$ and (12) is $5400\left\{\bar{D}(1-r)\right\}^2$; for $r = 1$, $v(t) = 3\bar{D}(720s^4\|\boldsymbol{c}\|^2 - 48s^2\boldsymbol{c}\cdot\boldsymbol{d} - \|\boldsymbol{d}\|^2)$, $t = s + 1/2$ with $\boldsymbol{c} = \boldsymbol{z}_0' - \Delta\boldsymbol{z}$ and $\boldsymbol{d} = \boldsymbol{z}_0' - 3\Delta\boldsymbol{z}$. Hence, $v(t)$ has a zero on each side of $t = 1/2$ where the inflection point occurs. (iv) Since the coefficient of t^3 of $\boldsymbol{z}(t)$ is $(1+r)\boldsymbol{z}_0' - 2\Delta\boldsymbol{z}$, $\boldsymbol{z}$ can not be quadratic. Hence we have the following result:
Remark 2 $(\boldsymbol{z}_1' = r\boldsymbol{z}_0', \boldsymbol{z}_0' \times \Delta\boldsymbol{z} \neq 0)$. For $r = 0$, $r < 0$, $r > 0$, we have exactly the same results in the above Cases 1-3 of Theorem 2, respectively where $r = 1$ corresponds to $\lambda + \mu = 1$.

References

[1] M. Sakai and R. Usmani. On fair parametric cubic splines. *BIT*, 36:359–377, 1996.
[2] B. Su and D. Liu. *Computaional Geometry: Curve and Surface Modeling.* Academic Press, New York, 1989.
[3] D. J. Walton and D. S. Meek. Curvature extrema of planar parametric polynomial cubic curves. *Computational and Applied Mathematics*, 134:69–83, 2001.

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