NORMAL MAPS AND HYPERBOLICITY

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Received April 21, 2003

Abstract. Criteria for a complex space to be hyperbolic, hyperbolically imbedded, taut or tautly imbedded are presented. In particular, the following generalization of theorems by Eastwood and Kobayashi is produced by replacing the requirement of hyperbolicity of the image space by normality of the mapping: Let $f : X \to Z$ be a normal map between complex spaces X and Z. If either (1) there is an open cover $\{V_\alpha\}$ of Z such that each connected component of $f^{-1}(V_\alpha)$ is hyperbolic or (2) for every $z \in Z$ each connected component of $f^{-1}(z)$ is compact hyperbolic, then X is hyperbolic. The following common generalization of results of Zaidenberg and Abate is also established: A complex subspace X of a complex space Y is hyperbolically imbedded in Y if the inclusion map from X to Y is normal.

§1. This paper begins with Theorem A, which cites a number of recent results of Kobayashi and Eastman.

Theorem A. [9] *Let $f : X \to Z$ be a holomorphic map between complex spaces X and Z. Then*

(1) [Kobayashi] X is hyperbolic if Z is hyperbolic and $f : X \to Z$ is a spread.

(2) [Kobayashi] X is hyperbolic if Z is hyperbolic and f is finite-to-one.

(3) [Eastman] X is hyperbolic if Z is hyperbolic and there is an open cover $\{V_\alpha\}$ of Z such that each $f^{-1}(V_\alpha)$ is hyperbolic.

(4) [Kobayashi] X is hyperbolic if Z is hyperbolic and $f : X \to Z$ is a complex fiber space with compact hyperbolic fibers.

(5) [Kobayashi] X is taut if Z is taut, f is proper and there is an open cover $\{V_\alpha\}$ of Z such that each $f^{-1}(V_\alpha)$ is taut.

(6) [Kobayashi] X is complete hyperbolic if Z is complete hyperbolic, f is proper and finite-to-one.

Zaidenberg and Kobayashi obtained results similar to those in Theorem A using normal maps (see §2 below for definition) with relative compact images instead of hyperbolicity of the image space. These are cited in Theorem B.

Theorem B[9]. *Let $f : X \to Z$ be a normal map of complex spaces with a relatively compact image. Then*

(1) [Zaidenburg] X is hyperbolic if there is an open cover $\{V_\alpha\}$ of Z such that each $f^{-1}(V_\alpha)$ is hyperbolic.

(2) [Kobayashi] X is complete hyperbolic if f is proper and for every $z \in Z$, each connected component of $f^{-1}(z)$ is hyperbolic.

In this article, common generalizations of Theorem A(2) and A(4), and of Theorem A(3) and Theorem B(1), and other generalizations of Theorems A and B are established. Abate

2000 *Mathematics Subject Classification.* Primary 32A10, 32A17, 32H20.
Key words and phrases. normal, hyperbolic, taut and proper..

[2] has proved that a complex space X is hyperbolic if the identity map of X is normal and Zaidenberg [12] has discovered that a relatively compact complex subspace X of a complex space Y is hyperbolically imbedded in Y if the inclusion map from X to Y is normal. These two results are unified and extended.

Some of the contributions of this note to the study of normal maps and hyperbolicity are listed below. The notation $\mathcal{H}(X,Y)$ is used for the space of holomorphic maps between complex spaces X and Y endowed with the compact-open topology.

- A complex subspace X of a complex space Y is hyperbolically imbedded in Y if the inclusion $i \in \mathcal{H}(X,Y)$ is normal, or more generally if there is a normal map $f \in \mathcal{H}(Z,Y)$ from a complex space Z such that $f(Z) = X$ and $f : Z \to X$ is a covering map.

- Let X and Z be complex spaces and let $f \in \mathcal{H}(X,Z)$ be a normal map. Then X is hyperbolic if either
(1) there is an open cover $\{V_\alpha\}$ of Z such that each connected component of $f^{-1}(V_\alpha)$ is hyperbolic or
(2) for every $z \in Z$, each connected component of $f^{-1}(z)$ is compact hyperbolic.
In addition if f is proper and has a relatively compact image in Z, then X is compact hyperbolic.

- Let $f \in \mathcal{H}(X,Z)$ be a proper map from a complex space X to a taut (complete hyperbolic) space Z. Then X is taut (complete hyperbolic) if
either (1) there is an open cover $\{V_\alpha\}$ of Z such that each connected component of $f^{-1}(V_\alpha)$ is hyperbolic
or (2) for each $z \in Z$, each connected component of $f^{-1}(z)$ is hyperbolic.

- Let X be a complex subspace of a complex space of Y and let $f \in \mathcal{H}(X,Z)$ be a normal map from X to a complex space Z with a relatively compact image. Then X is hyperbolically imbedded (tautly imbedded) in Y if there is an open cover $\{V_\alpha\}$ of Z such that each $f^{-1}(V_\alpha)$ is hyperbolically imbedded (tautly imbedded) in Y.

§2. The authors [8], extending Hayman's notion of uniformly normal family of functions, introduced the notion of uniformly normal families of holomorphic maps between complex spaces calling a map f normal if the singleton set $\{f\}$ is uniformly normal. This definition encompasses the concepts of normal maps defined by various authors in various settings and generalizations of results for normal maps including classical theorems by Lehto and Virtanen, Schottky, Hayman, Pommerenke and Lappan and generalizations of more recent theorems by Hahn, Järvi, and Zaidenberg have been obtained. It should be pointed out that holomorphic maps into hyperbolic spaces as well as into hyperbolically imbedded (not necessarily relatively compact) subspaces are normal maps and that a hyperbolic space is hyperbolically imbedded in itself. The definition of a uniformly normal family is now stated for the sake of completeness. Let X and Y be topological (complex) spaces. The notation Y^∞ will denote the one-point compactification of the space Y and $\mathcal{C}(X,Y)$ ($\mathcal{H}(X,Y)$) will represent the space of continuous (holomorphic) maps from X to Y endowed with the compact-open topology. The following notations will also be used: C is the complex plane, $\Delta = \{z \in C : |z| < 1\}$ and $F \circ G = \{f \circ g : f \in F, g \in G\}$ where F, G are function spaces.

Definition. *Let X, Y be complex spaces. A family $F \subset \mathcal{H}(X,Y)$ is uniformly normal if $F \circ \mathcal{H}(\Delta, X)$ is relatively compact in $\mathcal{C}(\Delta, Y^\infty)$. A map $f \in \mathcal{H}(X,Y)$ is said to be normal if a singleton set $\{f\}$ is uniformly normal.*

A semi-length function [10] on a complex space X is a upper-semi-continuous non-negative function H on the tangent cone, $T(X)$, such that $H(av) = |a|H(v)$ for $a \in$

C, v, $av \in T(X)$. A length function is a semi-length function which is continuous and $H(v) > 0$ for all nonzero $v \in T(X)$. We denote by d_H the distance function generated on X by H, that is,

$$d_H(x, y) = \inf_{\gamma} \int_a^b H(\gamma'(t))dt,$$

where $\gamma : [a, b] \to X$ is a C^1 curve joining x to y. The distance function d_H is known to generate the topology on X. If X, Y are complex spaces with length functions H, E respectively and $f \in \mathcal{H}(X, Y)$, the norm $|df|_{H,E}$ of the tangent map for f with respect to H, E is defined by

$$|df|_{H,E} = |df| = \sup\{|df_p| : p \in X\}$$

where

$$|df_p|_{H,E} = |df_p| = \sup\{E(df_p(v)) : v \in T(X)_p, H(v) = 1\}.$$

If X is a complex space, k_X (K_X) will denote the Kobayashi hyperbolic pseudo-distance (Kobayashi-Royden's semi-length function [10]) on X.

Propositions 1 and 2 for uniformly normal families are proved in [6].

Proposition 1 . *Let X be a complex space and H a semi-length function on X such that $f^*(H) \leq K_\Delta$ for all $f \in \mathcal{H}(\Delta, X)$. Let Y be a complex space. Then the following statements are equivalent for $F \subset \mathcal{H}(X, Y)$;*
(1) F is uniformly normal.
(2) $F \circ \mathcal{H}(\Delta, X)$ is an evenly continuous subset of $\mathcal{H}(\Delta, Y)$.
(3) There is a length function E on Y such that $|df|_{H,E} \leq 1$ for each $f \in F$.

Proposition 2. *Let X, Y be complex spaces. If $F \subset \mathcal{H}(X, Y)$ is uniformly normal, then F is relatively compact in $\mathcal{C}(X, Y^\infty)$.*

Proposition 3, due to Brody [9], will be used extensively.

Proposition 3. *Let Y be a compact subspace of a complex space Z. If Y is hyperbolic, there is a relatively compact neighborhood U of Y which is hyperbolically imbedded in Z.*

§3. Zaidenberg [12] has observed that a relatively compact complex subspace X of a complex space Y is hyperbolically imbedded in Y if the inclusion map from X to Y is normal. The following theorem extends this result by dropping the requirement of relative compactness as well as the result of Abate [2] that a complex space X is hyperbolic if the identity map of X is normal.

Theorem 1. *Let X be a complex subspace of a complex space Y. Then X is hyperbolically imbedded in Y if either of the following conditions is satisfied.*
(1) The inclusion $i \in \mathcal{H}(X, Y)$ is a normal map.
(2) There is a normal map $f \in \mathcal{H}(Z, Y)$ from a complex space Z such that $f(Z) \subset X$ and $f : Z \to X$ is a covering projection.

Proof. (1) By Proposition 1, choose a length function E on Y such that

$$d_E(f(p), f(q)) \leq k_X(p, q) \text{ for } p, q \in X.$$

(2) Let E be a length function on Y such that

$$d_E(f(a), f(b)) \leq k_Z(a, b) \text{ for } a, b \in Z.$$

Let $p, q \in X$ and $\tilde{p} \in f^{-1}(p)$. Then

$$k_X(p, q) = \inf_{\tilde{q} \in f^{-1}(q)} k_Z(\tilde{p}, \tilde{q}) \geq d_E(p, q). \quad \square$$

To be a hyperbolic space a weaker condition than Proposition 1(1) is sufficient as the following result shows. Theorem 2(3) extends Theorem A(2).

Theorem 2. *Let $f \in \mathcal{H}(X, Y)$ be a normal map between complex spaces. Then X is hyperbolic if either of the following conditions is satisfied.*

(1) Given $x, x' \in X$ with $x \neq x'$ and $f(x) = f(x')$, there is a neighborhood V of $f(x)$ in Y such that x and x' are in different components of $f^{-1}(V)$.

(2) Every $x \in X$ has a neighborhood U such that f is a homeomorphism from U onto an open set $f(U)$.

(3) For every $y \in Y$ the inverse image $f^{-1}(y)$ is finite.

Proof. By Proposition 1, choose a length function E on Y such that

$$d_E(f(p), f(q)) \leq k_X(p, q) \quad \text{for} \quad p, q \in X.$$

Since k_X is an inner pseudo-distance and d_E is an inner distance, the theorem follows from a result of Kobayashi [9]. $\square$

The following corollaries extend Theorem A(1) and other results of Kobayashi [9] where maps involved are into hyperbolic spaces.

Corollary 1. *Let $\pi \in \mathcal{H}(X, Z)$ be a spread of complex spaces. If π is a normal map, then X is hyperbolic.*

Corollary 2. *Let $\pi \in \mathcal{H}(X, Z)$ be a covering map of complex spaces. If π is normal, then X is hyperbolic.*

The following theorem extends Theorem A(2), A(3) and A(4) where the image space Z is assumed to be hyperbolic and Theorem B(1) where the map f is assumed to be a normal map with a relatively compact image. If a is a point in a topological space, the notation $\mathcal{N}(a)$ denotes the set of open neighborhoods of a.

Theorem 3. *Let $f \in \mathcal{H}(X, Z)$ be a normal map between complex spaces X and Z. Then X is hyperbolic if either of the following conditions holds.*

(1) There is an open cover $\{V_\alpha\}$ of Z such that each connected component of $f^{-1}(V_\alpha)$ is hyperbolic,

(2) For every $z \in Z$, each connected component of $f^{-1}(z)$ is compact hyperbolic.

Proof. Let ϕ_n be a sequence in $\mathcal{H}(\Delta, X)$ and let

$$D = \Delta - \{a \in \Delta : \phi_n(a_n) \to \infty \in X^\infty \text{ for every sequence } a_n \to a\}.$$

If $D = \emptyset$, the sequence ϕ_n diverges to ∞. It is now shown that if $a \in D$, with either of the assumptions (1) or (2), there is a $W(a) \in \mathcal{N}(a)$ and a subsequence $\phi_{n_k}|_{W(a)}$ of the sequence of restrictions of ϕ_n to $W(a)$ which is relatively compact in $\mathcal{H}(W(a), X)$, i.e., it is shown that D is open.

Suppose $a \in D$. Choose a subsequence μ_n of ϕ_n such that $\mu_n(a_n) \to x \in X$ for a sequence $a_n \to a$. Since $f \circ \mathcal{H}(\Delta, X)$ is relatively compact in $\mathcal{C}(\Delta, Z^\infty)$, it may be assumed that $f \circ \mu_n$ converges to $h \in \mathcal{C}(\Delta, Z^\infty)$. Assuming (1), a member V of the given open covering, $W(a) \in \mathcal{N}(a)$, and a subsequence $\phi_{n_k}|_{W(a)}$ of the sequence of restrictions of ϕ_n to $W(a)$ may be chosen such that $f \circ \phi_{n_k}(W(a)) \subset V$, and hence $\phi_{n_k}(W'(a)) \subset M$ where M is the connected component of $f^{-1}(V)$ satisfying $\phi_{n_k}(a_k) \to x \in M$. Since M is hyperbolic, $\{\phi_{n_k}|_{W(a)}\}$, the sequence of restrictions of ϕ_{n_k} to $W(a)$, is relatively compact in $\mathcal{C}(W(a), M^\infty)$. It follows that $\{\phi_{n_k}|_{W(a)}\}$ is relatively compact in $\mathcal{H}(W(a), X)$ for some $W(a) \in \mathcal{N}(a)$.

Assuming (2), let A be the connected component of $f^{-1}(h(a))$ containing x. From Proposition 3 choose a relatively compact neighborhood U of A hyperbolically imbedded in X. Since the hyperbolic distance k_U defines the topology of U, choose a relatively compact neighborhood $K \subset U$ of A such that $\partial K \cap f^{-1}(h(a)) = \emptyset$. Suppose that for each $W(a) \in \mathcal{N}(a)$, $\liminf \phi_{n_k}(W_a) \cap (X - K) \neq \emptyset$. Choose a subsequence $\phi^{(1)}_{n_k}$ of ϕ_{n_k} such that the subsequence $\phi^{(1)}_{n_k}(b_k) \notin K$ for a sequence $b_k \to a$. Let γ_k be the line segment $\overline{a_k b_k}$. Then $\phi_{n_k}(\gamma_k) \cap \partial K \neq \emptyset$. Choose $x_{n_k} \in \phi^{(1)}_{n_k}(\gamma_k) \cap \partial K$. A subsequence of the sequence x_{n_k} converges to a point $x' \in \partial K$ and $f(x') \neq f(x)$. Choose a neighborhood V of $f(x)$ such that $f(x') \notin V$. Now $f \circ \phi_{n_k}(W'(a)) \subset V$ ultimately for an element $W'(a) \in \mathcal{N}(a)$ since $f \circ \phi_{n_k} \to h \in \mathcal{C}(\Delta, Z^\infty)$. This is a contradiction as $\lim f(x_{n_k}) = f(x')$. It follows that there is $W(a) \in \mathcal{N}(a)$ such that ultimately $\phi_{n_k}(W_a) \subset K$ and then that the sequence of restrictions $\phi_{n_k}|_{W(a)}$ converges in $\mathcal{H}(W(a), X)$ since K is relatively compact and hyperbolically imbedded in X.

To complete the proof of the theorem, let

$$\mu = \{W_a : a \in D, W_a \in \mathcal{N}(a), \phi_n|_{W_a} \text{ has a convergent subsequence in } \mathcal{H}(W_a, X)\}.$$

Since D is open in Δ, there is a countable subcover $\{W_i\} \subset \mu$ of D. First choose a subsequence $\phi^{(1)}_n$ of ϕ_n converging in $\mathcal{H}(W_1, X)$. Then choose a subsequence $\phi^{(2)}_n$ of $\phi^{(1)}_n$ converging in $\mathcal{H}(W_2, X)$. Continuing, sequences are obtained satisfying $\phi^{(k)}_n$ is a subsequence of $\phi^{(k+1)}_n$ such that $\phi^{(k)}_n$ converges in $\mathcal{H}(\cup_{i=1}^k W_i, X)$. The diagonal sequence $\phi^{(k)}_k$ converges to a map $\phi \in \mathcal{H}(D, X)$. Define $\tilde{\phi} \in \mathcal{C}(\Delta, Y^\infty)$ by $\tilde{\phi}(a) = \phi(a)$ for $a \in D$ and $\tilde{\phi}(a) = \infty$ if $a \in \Delta - D$. Then ϕ^k_k converges to $\tilde{\phi}$, i.e., $\mathcal{H}(\Delta, X)$ is relatively compact in $\mathcal{C}(\Delta, X^\infty)$. □

Similar criteria provide sufficient conditions for a complex space to be taut or compact hyperbolic.

Theorem 4. *Let $f \in \mathcal{H}(X, Z)$ be a proper map between complex spaces X and Z such that $f \circ \mathcal{H}(\Delta, X)$ is relatively compact in $\mathcal{H}(\Delta, Z) \cup \{\infty\}$ ($\mathcal{H}(\Delta, Z)$). Then X is taut (compact hyperbolic) if either of the following conditions holds.*

(1) There is an open cover $\{V_\alpha\}$ of Z such that each connected component of $f^{-1}(V_\alpha)$ is hyperbolic.

(2) For each $z \in Z$, each $f^{-1}(z)$ is hyperbolic.

Proof. Let ϕ_n be a sequence in $\mathcal{H}(\Delta, X)$ and let

$$D = \Delta - \{a \in \Delta : \phi_n(a_n) \to \infty \in X^\infty \text{ for every sequence } a_n \to a\}.$$

If $D = \emptyset$, the sequence ϕ_n diverges to ∞ and we are finished. So assume $D \neq \emptyset$ and we may suppose $f \circ \phi_n$ converges to $h \in \mathcal{H}(\Delta, Z)$. We will show that $D = \Delta$ and then the proof can proceed as those given for Theorem 3.

Let $a \in \Delta$ and let L be a compact neighborhood of $h(a)$. There is a $W(a) \in \mathcal{N}(a)$ such that ultimately $f \circ \phi_n(W(a)) \subset L$ and so $\phi_n(W(a)) \subset f^{-1}(L)$. There is, then, a subsequence ϕ_{n_j} such that $\phi_{n_j}(a)$ converges to a point $x \in f^{-1}(L)$ since $f^{-1}(L)$ is compact and so $a \in D$.

To show X is compact when $f \circ \mathcal{H}(\Delta, X)$ is relatively compact in $\mathcal{H}(\Delta, Z)$, let x_n be a sequence in X and define $\phi_n \in \mathcal{H}(\Delta, X)$ by $\phi_n(x) = x_n$. Since $f \circ \phi_n$ is relatively compact in $\mathcal{H}(\Delta, Z)$, it follows from the properness of f that a subsequence x_{n_j} converges to a point $x \in X$. $\square$

Generalizations of A(5) and A(6) are presented in Corollary 3.

Corollary 3. *Let $f \in \mathcal{H}(X, Z)$ be a proper map from a complex space X to a complete hyperbolic (taut) space Z. Then X is complete hyperbolic (taut) if either of the following conditions is satisfied.*

(1) There is an open cover $\{V_\alpha\}$ of Z such that each connected component of $f^{-1}(V_\alpha)$ is hyperbolic.

(2) For each $z \in Z$, each connected component of $f^{-1}(z)$ is hyperbolic.

Proof. Only the statement concerning completeness needs to be checked. Let x_n be a Cauchy sequence with respect to k_X. Since f is distance decreasing with respect to the hyperbolic distances k_X and k_Z, the sequence $f(x_n)$ is Cauchy and converges to a point, say $p \in Z$. Let K be a compact neighborhood of p. ultimately $x_n \in f^{-1}(K)$. Since $f^{-1}(K)$ is compact, x_n, being a Cauchy sequence, must converge. $\square$

Corollary 4 extends Theorem B(2) under the same assumption. It should be cautioned that this corollary applies neither to the identity map $id : \Delta \to \Delta$ since it does not have a relative compact image , nor to the inclusion map $i : \Delta \to C$ since it is not proper.

Corollary 4. *Let $f \in \mathcal{H}(X, Z)$ be a proper normal map between complex spaces X and Z with a relatively compact image. Then X is compact and hyperbolic if either of the following conditions holds.*

(1) There is an open cover $\{V_\alpha\}$ of Z such that each connected component of $f^{-1}(V_\alpha)$ is hyperbolic.

(2) For each $z \in Z$, each connected component of $f^{-1}(z)$ is hyperbolic.

Theorem 5 is similar to Theorem B(1).

Theorem 5. *Let X be a complex subspace of a complex space of Y. Let $f \in \mathcal{H}(X, Z)$ be a normal map from X to a complex space Z with a relatively compact image. Then*

X is hyperbolically imbedded (tautly imbedded) in Y if there is an open cover $\{V_\alpha\}$ of Z such that each $f^{-1}(V_\alpha)$ is hyperbolically imbedded (tautly imbedded) in Y .

Proof. (1) Let ϕ_n be a sequence $\mathcal{H}(\Delta, X) \subset \mathcal{C}(\Delta, Y^\infty)$ and without loss suppose $f \circ \phi_n$ converges to $h \in \mathcal{H}(\Delta, Z)$. Let

$$E = \Delta - \{a \in \Delta : \phi_n(a_n) \to \infty \in Y^\infty \text{ for every sequence } a_n \to a\}.$$

If $E = \emptyset$, the sequence ϕ_n diverges to ∞ . On the other hand let $a \in E$ and let $V \in \mathcal{N}(h(a))$ such that $f^{-1}(V)$ is hyperbolically imbedded in Y. There is a subsequence ϕ_{n_k} such that

$\phi_{n_k}(a_k) \to y \in Y$ for a sequence $a_n \to a$. There is a $W'(a) \in \mathcal{N}(a)$ of a such that ultimately $f \circ \phi_n(W'(a)) \subset V$ and hence $\phi_n(W'(a)) \subset f^{-1}(V)$. It follows that $\{\phi_{n_k}|_{W'(a)}\}$ is relatively compact in $\mathcal{C}(W'(a), Y^\infty)$. Since $\phi_{n_k}(a_k) \to y \in Y$, we have that $\{\phi_{n_k}|_{W(a)}\}$ is relatively compact in $\mathcal{H}(W(a), Y)$ for a $W(a) \in \mathcal{N}(a)$ i.e., $a \in E$ iff a subsequence of $\phi_n|_{W(a)}$ converges in $\mathcal{H}(W(a), Y)$ for $W(a) \in \mathcal{N}(a)$. As in Theorem 3, it follows that $\mathcal{H}(\Delta, X)$ is relatively compact in $\mathcal{C}(\Delta, Y^\infty)$.

It will now be shown that $E = \Delta$ when $E \neq \emptyset$ and the proof X is tautly imbedded in Y can then proceed as in the proof of Theorem 3. Let $a \in \Delta$ and let $U \in \mathcal{N}(h(a))$ such that $f^{-1}(U)$ is tautly imbedded in Y. Choose a $W(a) \in \mathcal{N}(a)$ such that ultimately $f \circ \phi_n(W(a)) \subset U$ and hence $\phi_n(W(a)) \subset f^{-1}(U)$. It follows that there is a $W(a) \in \mathcal{N}(a)$ such that $\phi_n|_{W(a)}$ has a convergent sequence in $\mathcal{H}(W(a), Y)$. Therefore $a \in E$. $\square$

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