STABLE RANK OF RESIDUALLY FINITE DIMENSIONAL C^*-ALGEBRAS

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Abstract. We estimate the stable rank of residually finite dimensional C^*-algebras by that of the C^*-algebras of continuous fields associated with their (continuous) separating, finite dimensional irreducible representations. Moreover, as the main application we estimate the stable rank, connected stable rank and real rank of reduced group C^*-algebras of residually finite, discrete groups with the property (T).

0. Introduction

The theory of the stable rank for C^*-algebras was first studied by Rieffel [Rf1], and the stable rank for some concrete C^*-algebras were computed by some other works (cf. References). In particular, the stable rank and connected stable rank of group C^*-algebras of some connected Lie groups were computed ([Sh], [Sd1-4], [ST1,2]), and the stable rank and real rank of reduced group C^*-algebras of some discrete groups without Kazhdan's property (T) such as free groups, free product groups and some amalgamated free product groups were also computed ([DHR], [Dk], [DH]) (See [BP] for the real rank of C^*-algebras).

In this paper as the main result we estimate the stable rank of residually finite dimensional C^*-algebras in terms of their spectrums, i.e. spaces of all unitary equivalence classes of their irreducible representations. For the proof we use some techniques of [Sd5] for the stable rank estimate of C^*-algebras of continuous fields. As a highly non-trivial consequence of the result we estimate the stable rank, connected stable rank, general stable rank and real rank of reduced group C^*-algebras of residually finite, countable discrete groups with the property (T) such as $SL_n(\mathbb{Z})$ $(n \geq 3)$ (cf. [HV]), and also estimate those of group C^*-algebras of some amenable discrete subgroups of $GL_n(\mathbb{C})$.

Notations and facts. We now set up some notations and review some facts used later.

Let $\mathfrak{A}$ be a C^*-algebra. We denote by $\mathrm{sr}(\mathfrak{A})$, $\mathrm{csr}(\mathfrak{A})$, $\mathrm{gsr}(\mathfrak{A})$ and $\mathrm{RR}(\mathfrak{A})$ the stable rank, connected stable rank, general stable rank and the real rank of $\mathfrak{A}$ respectively ([Rf1], [BP]). By definition, $\mathrm{sr}(\mathfrak{A}), \mathrm{csr}(\mathfrak{A}), \mathrm{gsr}(\mathfrak{A}) \in \{1, 2, \cdots, \infty\}$ and $\mathrm{RR}(\mathfrak{A}) \in \{0, 1, 2, \cdots, \infty\}$. In particular, $\mathrm{sr}(\mathfrak{A}) \leq n$ if and only if the open space $L_n(\mathfrak{A})$ is dense in $\mathfrak{A}^n$, where $(a_j) \in L_n(\mathfrak{A})$ if and only if $\sum_{j=1}^n a_j^* a_j$ is invertible in $\mathfrak{A}$. If $\mathfrak{A}$ is nonunital, we define its ranks by those of its unitization $\mathfrak{A}^+$.

(F0): $\mathrm{gsr}(\mathfrak{A}) \leq \mathrm{csr}(\mathfrak{A}) \leq \mathrm{sr}(\mathfrak{A}) + 1$ and $\mathrm{RR}(\mathfrak{A}) \leq 2\,\mathrm{sr}(\mathfrak{A}) - 1$ for any C^*-algebra $\mathfrak{A}$ ([Rf1, Corollary 4.10 and p.328], [BP, Proposition 1.2]).

(F1): $\max\{\mathrm{sr}(\mathfrak{I}), \mathrm{sr}(\mathfrak{A}/\mathfrak{I})\} \leq \mathrm{sr}(\mathfrak{A})$ for an exact sequence of C^*-algebras: $0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{I} \to 0$ ([Rf1, Theorems 4.3 and 4.4]).

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(F2): Let $C_0(X)$ be the C^*-algebra of all continuous functions on a locally compact Hausdorff space X vanishing at infinity. If X is compact, set $C_0(X) = C(X)$. Then $\mathrm{sr}(C(X)) = [\dim X/2] + 1$, where $\dim X$ is the covering dimension of X and $[x]$ means the maximal integer $\leq x$ ([Rf1, Proposition 1.7]).

(F3): $\mathrm{sr}(M_n(\mathfrak{A})) = \{(\mathrm{sr}(\mathfrak{A}) - 1)/n\} + 1$ for $M_n(\mathfrak{A})$ the $n \times n$ matrix algebra over a C^*-algebra $\mathfrak{A}$, where $\{x\}$ means the least integer $\geq x$ ([Rf1, Theorem 6.1]).

Let $\Pi_{j\in J}\mathfrak{B}_j$ denote the direct product of C^*-algebras $\{\mathfrak{B}_j\}_{j\in J}$ indexed by a set J consisting of all elements $a = (a_j)_{j\in J}$ with $a_j \in \mathfrak{B}_j$ and the norm $\|a\| = \sup_{j\in J} \|a_j\|$ finite. For a C^*-algebra $\mathfrak{A}$, denote by $C_0(X, \mathfrak{A})$ the C^*-algebra of all $\mathfrak{A}$-valued continuous functions on X vanishing at infinity. It is known that $C_0(X, \mathfrak{A})$ is isomorphic to the C^*-tensor product $C_0(X) \otimes \mathfrak{A}$ (cf.[Mp, Theorem 6.4.17]). Let $\hat{\mathfrak{A}} = \mathfrak{A}^\wedge$ be the spectrum of all irreducible representations of a C^*-algebra $\mathfrak{A}$ up to unitary equivalence. The space $\hat{\mathfrak{A}}$ is locally compact under the hull kernel topology (cf.[Dx, Chapter 3], [Pd, Chapter 4]). For $1 \leq n < \infty$, let $\hat{\mathfrak{A}}_n$ be the space of all n-dimensional irreducible representations of $\mathfrak{A}$ having the relative topology of $\hat{\mathfrak{A}}$. Denote by $\hat{G}$ the unitary dual of all irreducible unitary representations of a locally compact group G up to unitary equivalence.

1. Residually finite dimensional C^*-algebras

First of all, we recall that

Definition. A C^*-algebra $\mathfrak{A}$ is residually finite dimensional (RFD) if there exists a family $\{\pi_j\}_{j\in J}$ of finite dimensional representations of $\mathfrak{A}$ such that the intersection of their kernels is zero, i.e. $\{\pi_j\}_{j\in J}$ is separating $\mathfrak{A}$ in the sense that for any nonzero $a \in \mathfrak{A}$, there exists π_j with $\pi_j(a) \neq 0$. If necessary we may assume that each π_j is irreducible by considering its decomposition into irreducible representations (cf. [GM], [BK], [Kb1,2], [Ln]). When the set J is a locally compact Hausdorff space, we say that $\{\pi_j\}_{j\in J}$ is continuous on J if all the functions $J \ni j \mapsto \|\pi_j(a)\|$ (norm-valued) for any $a \in \mathfrak{A}$ are continuous on J.

As a key result we have the following:

Lemma 1.1. *Let $\mathfrak{A}$ be a residually finite dimensional C^*-algebra with respect to $\{\pi_j\}_{j\in J}$ of $\hat{\mathfrak{A}}$. Suppose that J is a locally compact Hausdorff space and $\{\pi_j\}_{j\in J}$ is continuous on J. Then $\mathfrak{A}$ is a quotient of the C^*-algebra of a continuous field on J contained in $\Pi_{j\in J} C_0(\hat{\mathfrak{A}}_{n_j}, \pi_j(\mathfrak{A}))$ where $n_j = \dim \pi_j$.*

Proof. Following the idea of [Sd5] we define $\mathfrak{B}$ to be the C^*-algebra of a continuous field on J, $\Gamma_0(J, \{C_0(\hat{\mathfrak{A}}_{n_j}, \pi_j(\mathfrak{A}))\}_{j\in J})$, with $C_0(\hat{\mathfrak{A}}_{n_j}, \pi_j(\mathfrak{A}))$ fibers such that for $f = \{f(j, \cdot)\}_{j\in J} \in \mathfrak{B}$ with $f(j, \cdot) = f(j) \in C_0(\hat{\mathfrak{A}}_{n_j}, \pi_j(\mathfrak{A}))$, there exists $x \in \mathfrak{A}$ such that for any $j \in J$, $f(j, \pi_j) = \pi_j(x) \in \pi_j(\mathfrak{A})$, that is, $f(j)$ is an arbitrary element of $C_0(\hat{\mathfrak{A}}_{n_j}, \pi_j(\mathfrak{A}))$ taking the value $\pi_j(x)$ at π_j. In fact, such an element may be defined by $f(j) = h_j\pi_j(x)$ for certain $h_j \in C_0(\hat{\mathfrak{A}}_{n_j})$ such that the function: $J \ni j \mapsto \|f(j)\|$ is continuous on J. It is clear that $\mathfrak{B}$ is a C^*-subalgebra of $\Pi_{j\in J} C_0(\hat{\mathfrak{A}}_{n_j}, \pi_j(\mathfrak{A}))$. Then we have a quotient map Φ from $\mathfrak{B}$ to $\mathfrak{A}$ defined by $\Phi(f) = x$, where the following function on $\{\pi_j\}_{j\in J}$: $\hat{x} : \{\pi_j\}_{j\in J} \ni \pi_j \mapsto \pi_j(x) \in \pi_j(\mathfrak{A})$ is identified with $x \in \mathfrak{A}$ since $\{\pi_j\}_{j\in J}$ is separating. □

Remark. In the above situation, note that

$$\Gamma(\hat{\mathfrak{A}}_n, \{\pi(\mathfrak{A})\}_{\pi\in\hat{\mathfrak{A}}_n}) \equiv \{\hat{\mathfrak{A}}_n \ni \pi \mapsto \pi(x) \in \pi(\mathfrak{A}) \,|\, x \in \mathfrak{A}\}$$

is the C^*-algebra of a continuous field on $\hat{\mathfrak{A}}_n$ with fibers $\{\pi(\mathfrak{A})\}_{\pi\in\hat{\mathfrak{A}}_n}$ (cf.[Dx, Chapter 10]). The assumption of $\{\pi_j\}_{j\in J}$ being continuous on J might be unnecessary because the function: $J \ni j \mapsto \|f(j)\|$ could be continuous without the function: $J \ni j \mapsto \|\pi_j(a)\|$ being continuous.

As the main result we have the following:

Theorem 1.2. *Let $\mathfrak{A}$ be a residually finite dimensional C^*-algebra with a separating family $\{\pi_j\}_{j\in J}$ of $\hat{\mathfrak{A}}$. Suppose that J is a locally compact Hausdorff space and $\{\pi_j\}_{j\in J}$ is continuous on J, and $n_j = \dim \pi_j$. Then*

$$\mathrm{sr}(\mathfrak{A}) = \sup_{j\in J} \mathrm{sr}(C_0(\hat{\mathfrak{A}}_{n_j}, \pi_j(\mathfrak{A}))).$$

Proof. First note that $\hat{\mathfrak{A}}_n$ $(1 \le n < \infty)$ are Hausdorff spaces, and there exist subquotients $\mathfrak{D}_n$ of $\mathfrak{A}$ such that $\hat{\mathfrak{D}}_n = \hat{\mathfrak{A}}_n$ by [Dx, 3.2 and 3.6] or [Pd, Proposition 4.4.10]. Then by (F1) and from that $\mathfrak{D}_n$ is homogeneous we have

$$\mathrm{sr}(\mathfrak{A}) \ge \mathrm{sr}(\mathfrak{D}_{n_j}) = \mathrm{sr}(C_0(\hat{\mathfrak{A}}_{n_j}, \pi_j(\mathfrak{A}))) \quad \text{for any } j \in J.$$

By Lemma 1.1 and (F1), we have $\mathrm{sr}(\mathfrak{A}) \le \mathrm{sr}(\mathfrak{B})$. We define $\mathfrak{C}$ to be the C^*-algebra of all elements $\{(f(j,\cdot), \lambda_j)\}_{j\in J}$ with $(f(j,\cdot),\lambda_j) \in C_0(\hat{\mathfrak{A}}_{n_j}, \pi_j(\mathfrak{A}))^+$, $\lambda_j \in \mathbb{C}$ and $f \in \mathfrak{B} = \Gamma_0(J, \{C_0(\hat{\mathfrak{A}}_{n_j}, \pi_j(\mathfrak{A}))\}_{j\in J})$ as in the proof of Lemma 1.1. By (F1), we have $\mathrm{sr}(\mathfrak{B}) \le \mathrm{sr}(\mathfrak{C})$. Moreover, we may replace $\mathfrak{C}$ with $\Gamma(J^+, \{C_0(\hat{\mathfrak{A}}_{n_j}, \pi_j(\mathfrak{A}))\}_{j\in J})$ when J is noncompact.

Now suppose that

$$M = \sup_{j\in J} \mathrm{sr}(C_0(\hat{\mathfrak{A}}_{n_j}, \pi_j(\mathfrak{A}))) < \infty.$$

For any $\varepsilon > 0$ and $(c_k)_{k=1}^M \in \mathfrak{C}^M$ with $c_k = (c_k(j,\cdot))_{j\in J}$ and $c_k(j,\cdot) \in C_0(\hat{\mathfrak{A}}_{n_j}, \pi_j(\mathfrak{A}))^+$, we can find $d_k(j,\cdot) \in C_0(\hat{\mathfrak{A}}_{n_j}, \pi_j(\mathfrak{A}))^+$ for all k,j such that $\|c_k(j,\cdot) - d_k(j,\cdot)\| < \varepsilon_{k,j} < \varepsilon$, and $e_j \equiv \sum_{k=1}^M d_k(j,\cdot)^* d_k(j,\cdot)$ is invertible in $C_0(\hat{\mathfrak{A}}_{n_j}, \pi_j(\mathfrak{A}))^+$. For a large constant $L > 0$, we may assume that $e_j \ge \varepsilon/L > 0$ if necessary, by taking $\varepsilon_{k,j}$ small enough, and replacing $d_{k,j}$ with its suitable purturbation, and $\varepsilon_{k,j}$ with $\varepsilon' < \varepsilon$, when $e_j \ge \delta_j 1 > 0$ and $\delta_j < \varepsilon/L$ for some j.

In fact, for a unital C^*-algebra $\mathcal{A}$ we have a continuous map Φ from $L_n(\mathcal{A})$ to the positive part $\mathcal{A}_+$ of $\mathcal{A}$ by $(a_j) \mapsto \sum_{j=1}^n a_j^* a_j$. Moreover, the quotient topology induced by Φ is stronger than the relative topology of $\mathcal{A}^{-1}$ in $\mathcal{A}$, which is proved by a usual argument of the general topology about inclusions of open neighborhoods. Let $\mathcal{S} = \{b \in \mathcal{A}_+ \mid \|\sum_{j=1}^n a_j^* a_j - b\| < \eta$, and $b - (\sum_{j=1}^n a_j^* a_j + \eta' 1) > 0\}$ for some $\eta, \eta' > 0$, where the second inequality " $>$ " means being invertible. Then $\mathcal{S}$ is open in $\mathcal{A}_+$ since for $b' \in \mathcal{A}_+$ with $\|b - b'\|$ small, we can make the distance of their spectrums small. Taking η, η' suitably, we make the distance between $\sum_{j=1}^n a_j^* a_j$ and $\mathcal{S}$ small enough. Then we can find a small open neighborhood of (a_j) such that its image under Φ has the nonzero intersection with $\mathcal{S}$.

Moreover, if necessary taking $\varepsilon_{k,j}$ small enough for each k,j, we may assume that $d_k(j,\cdot)$ is a suitable perturbation of $c_k(j,\cdot)$ such that the function $l \mapsto d_k(l)$ on an open neighborhood of j belongs to the corresponding restriction of $\mathfrak{C}$ by using [Dx, Propositions 10.1.10 and 10.2.2] (the local density and continuity of continuous fields of C^*-algebras). Also note that if $\sum_{k=1}^M d_k(j,\cdot)^* d_k(j,\cdot)$ is invertible, then $\sum_{k=1}^M d_k(l,\cdot)^* d_k(l,\cdot)$ is also invertible for l in an open neighborhood of j, which is deduced from a direct computation using continuity of the norm on fibers. Use this argument inductively for a suitable open covering of J. Therefore, we have $d_k \in \mathfrak{C}$. Thus, $\mathrm{sr}(\mathfrak{C}) \le M$. $\square$

Remark. The space J could be taken as a countable set in some cases as in Section 2. In fact, the space J can be a sequence for separable, residually finite dimensional C^*-algebras (cf. [Ln, Definition 1]). However, it would be not easy to know the spaces $\hat{\mathfrak{A}}_{n_j}$ and their dimensions in general. Also, subhomogeneous C^*-algebras are clearly residually finite dimensional but be not always of continuous trace (cf. [Dx, 10.10.4]).

Note that $\pi_j(\mathfrak{A}) \cong M_{n_j}(\mathbb{C})$ in the above theorem. Applying (F2) and (F3) to the stable rank estimate of Theorem 1.2, we have

Corollary 1.3. *Let $\mathfrak{A}$ be an RFD C^*-algebra with a continuous separating family $\{\pi_j\}_{j\in J}$ of $\hat{\mathfrak{A}}$ for J a locally compact Hausdorff space, and $n_j = \dim \pi_j$. Then*

$$\mathrm{sr}(\mathfrak{A}) = \sup_{j\in J}(\{[\dim \hat{\mathfrak{A}}_{n_j}/2]/n_j\} + 1).$$

Moreover, we have the following product formula of the stable rank, which partially answers a question by Rieffel [Rf1, Question 7.3].

Corollary 1.4. *Let $\mathfrak{A}$, $\mathfrak{B}$ be residually finite dimensional C^*-algebras. Suppose that a C^*-tensor product $\mathfrak{A} \otimes \mathfrak{B}$ with a suitable C^*-norm is residually finite dimensional by a continuous separating family $\{\pi'_j\}_{j\in J}$ of $(\mathfrak{A}\otimes\mathfrak{B})^\wedge$ for J a locally compact Hausdorff space. Then*

$$\mathrm{sr}(\mathfrak{A}\otimes\mathfrak{B}) \le \mathrm{sr}(\mathfrak{A}) + \mathrm{sr}(\mathfrak{B}).$$

Proof. Since each π'_j is finite dimensional, we may assume that $\pi'_j = \pi_j \otimes \rho_j$ for some $\pi_j \in \hat{\mathfrak{A}}_{n_j}$ and $\rho_j \in \hat{\mathfrak{B}}_{m_j}$, where $n_j = \dim \pi_j$ and $m_j = \dim \rho_j$. Then Corollary 1.3 implies

$$\mathrm{sr}(\mathfrak{A}\otimes\mathfrak{B}) = \sup_{j\in J}\{[\dim(\hat{\mathfrak{A}}_{n_j} \times \hat{\mathfrak{B}}_{m_j})/2]/n_j m_j\} + 1.$$

Then the product formula of the covering dimension (cf.[Ng]) implies

$$\begin{aligned}
&\sup_{j\in J}(\{[\dim(\hat{\mathfrak{A}}_{n_j} \times \hat{\mathfrak{B}}_{m_j})/2]/n_j m_j\} + 1) \\
&\le \sup_{j\in J}\{[(\dim \hat{\mathfrak{A}}_{n_j} + \dim \hat{\mathfrak{B}}_{m_j})/2]/n_j m_j\} + 1 \\
&\le \sup_{j\in J}(\{[\dim \hat{\mathfrak{A}}_{n_j}/2]/n_j\} + 1) + \sup_{j\in J}(\{[\dim \hat{\mathfrak{B}}_{m_j}/2]/m_j\} + 1) \le \mathrm{sr}(\mathfrak{A}) + \mathrm{sr}(\mathfrak{B}).
\end{aligned}$$

See the first part of the proof of Theorem 1.2 for the last estimate. □

Remark. Let F_2 be the free group with two generators, and $C^*(F_2)$ its full group C^*-algebra. By [Ch], $C^*(F_2)$ is residually finite dimensional. By [Rf1, Theorem 6.7] or [Ngs] we get $\mathrm{sr}(C^*(F_2)\otimes C^*(F_2)) = \infty = \mathrm{sr}(C^*(F_2))$, where $\otimes$ means any C^*-tensor norm. By [Ngs, p.378] we know that there is a quotient map from $C^*(F_2)$ to $C([0,1]^{n^2}) \otimes M_{n+1}(\mathbb{C})$ for any $n \in \mathbb{N}$, which implies that $\sup_{n\in\mathbb{N}} \dim C^*(F_2)^\wedge_n = \infty$.

2. Reduced group C^*-algebras of residually finite discrete groups

First recall that a group G is residually finite if for each $g \in G\setminus\{1_G\}$ with 1_G the identity of G, there exists a subgroup H of G with finite index and $g \notin H$. The group H can be taken as a normal subgroup of G (cf. [Sr, p.122]). Also, a group G has Kazhdan's property (T) if the trivial representation of G is an isolated point of the unitary dual $\hat{G}$ of G with the hull-kernel topology (cf. [Wg]).

As a remarkable application of Theorem 1.2, we have the following:

Theorem 2.1. *Let G be a residually finite, countable discrete group with Kazhdan's property* (T) *and $C_r^*(G)$ its reduced group C^*-algebra. Then*

$$\mathrm{sr}(C_r^*(G)) = 1, \quad \mathrm{csr}(C_r^*(G)) \le 2, \quad \mathrm{gsr}(C_r^*(G)) = 1$$

and in addition $\mathrm{RR}(C_r^*(G)) \le 1$.

Proof. Since G is residually finite, there exists a separating family $X = \{\pi_j\}_{j\in J}$ of finite dimensional unitary representations of G in $\hat{G}$ (cf.[Kb1, Remarks 7.4]). Hence we have the factorization (cf.[KW, Examples 4.4])

$$C^*(G) \to \mathfrak{A} = C^*((\oplus_{\pi\in X}\pi)(G)) \to C_r^*(G) \to 0$$

where $C^*(G)$ is the full group C^*-algebra of G, and $\mathfrak{A}$ is the C^*-algebra generated by $(\oplus_{\pi\in X}\pi)(G)$ under the direct sum representation $\oplus_{\pi\in X}\pi$ of G. We note that Note $\mathfrak{A} \cong (\oplus_{\pi\in X}\pi)(C^*(G))$ by the identification $\hat{G} = C^*(G)^\wedge$ (cf. [Dx, 13.9.3] and [Pd, 7.1.4]). In particular, $\hat{\mathfrak{A}}_n$ is a subspace of $\hat{G}_n = C^*(G)^\wedge_n$ for $n \in \mathbb{N}$. Then by (F0), (F1) and [Eh2, Theorem 1.4] we have

$$\begin{cases} \mathrm{sr}(C_r^*(G)) \le \mathrm{sr}(\mathfrak{A}), \quad \mathrm{csr}(C_r^*(G)) \le \mathrm{sr}(C_r^*(G)) + 1, \\ \mathrm{RR}(C_r^*(G)) \le \mathrm{RR}(\mathfrak{A}). \end{cases}$$

We note that since G has the property (T), any finite dimensional, irreducible unitary representation of G is an isolated point of $\hat{G}$ ([HV], [Wg, Theorem 2.1]). Hence, $X = \{\pi_j\}_{j\in J}$ is continuous automatically. Moreover, by [Wg, Theorem 2.6] or [Ws, Corollary 3], for H any countable discrete group with (T), the space $\hat{H}_n$ is finite for any $n \in \mathbb{N}$. By using Theorem 1.2, (F0), (F2) and (F3), we have $\mathrm{sr}(\mathfrak{A}) = 1$ and $\mathrm{RR}(\mathfrak{A}) \le 2\,\mathrm{sr}(\mathfrak{A}) - 1 = 1$. Hence $\mathrm{csr}(C_r^*(G)) \le 2$. Since $C_r^*(G)$ is finite, we obtain $\mathrm{gsr}(C_r^*(G)) = 1$ (cf.[Rf2, p.247]). $\square$

Remark. If G is a nonamenable locally compact group, then $C_r^*(G)$ is not residually finite dimensional because it has no finite dimensional irreducible representations (cf. [F, Theorem 3], [Dx, 18.3 and 18.9.5]). It is known by [Lf1,2] that there exists a residually finite, countable discrete group with (T) such that the Baum-Connes conjecture holds for its reduced group C^*-algebra so that the algebra has no nontrivial projections, which implies that the algebra does not have real rank zero.

Remark. Let G be as in Theorem 2.1. If $\mathrm{csr}(C_r^*(G)) = 1$, then $C_r^*(G)$ is stably finite, and the K_1-group $K_1(C_r^*(G))$ is trivial by using [Eh1, Proposition 1.15]. The equality $\mathrm{csr}(C_r^*(G)) = 1$ could be implied by the same method for a connected stable rank estimate as in [Sd5].

We now give a list of groups with (T) or without (T).

Examples 2.2.

$$\text{As groups with (T),} \quad \begin{cases} \text{Compact groups} \\ F_{4(-20)},\ Sp(n,1),\ Sp(n,1)_{\mathbb{Z}} \text{ for } n \ge 2 \\ SL_n(\mathbb{R}),\ SL_n(\mathbb{Z}),\ PSL_n(\mathbb{Z}) \text{ for } n \ge 3 \\ \mathbb{R}^n \rtimes_\alpha SL_n(\mathbb{R}),\ \mathbb{Z}^n \rtimes_\alpha SL_n(\mathbb{Z}) \text{ for } n \ge 3 \end{cases}$$

$$\text{As groups without (T),} \quad \begin{cases} \text{Noncompact amenable groups} \\ SO_0(n,1), SU(n,1) \text{ for } n \ge 2 \\ SL_2(\mathbb{R}), SL_2(\mathbb{Z}), PSL_2(\mathbb{Z}) \\ \text{Free groups } F_n \text{ with } n \text{ generators with } n \ge 2 \end{cases}$$

where $SO_0(n,1)$, $SU(n,1)$, $Sp(n,1)$ for $n \geq 2$ and $F_{4(-20)}$ are the connected real simple Lie groups of rank 1, and $Sp(n,1)_{\mathbb{Z}}$ is the discrete subgroup of $Sp(n,1)$ with its components integers, and the actions α of $SL_n(\mathbb{R})$ and $SL_n(\mathbb{Z})$ are the matrix multiplications on $\mathbb{R}^n$ and $\mathbb{Z}^n$ respectively. See [HV] and [Wg] for more details. Note that any lattice of simple Lie groups with rank greater than 2 has the property (T) (cf.[Mg]). Among the above examples, compact groups, $SL_n(\mathbb{Z})$, $PSL_n(\mathbb{Z})$, $Sp(n,1)_{\mathbb{Z}}$ for $n \geq 2$, and $\mathbb{Z}^n \rtimes_\alpha SL_n(\mathbb{Z})$, F_n for $n \geq 2$ are residually finite. Moreover, any finitely generated subgroup of $GL_n(\mathbb{C})$ is residually finite (cf.[Ap], [Kb1,2]). Also note that any free product of residually finite groups is residually finite (cf.[Sr]). In particular, $F_2 \cong \mathbb{Z} * \mathbb{Z}$. On the other hand, it is deduced by Lie's theorem that connected solvable Lie groups have no finite dimensional, irreducible unitary representations except one dimensional ones. See [Sd1] for the stable rank of reduced group C^*-algebras of those connected real semi-simple Lie groups above, and [Ka] for the real rank of some group C^*-algebras.

It is deduced from Theorem 2.1 and Examples 2.2 that

Corollary 2.3. *Let G be either $SL_n(\mathbb{Z})$, $PSL_n(\mathbb{Z})$, $\mathbb{Z}^n \rtimes_\alpha SL_n(\mathbb{Z})$ for $n \geq 3$, or $Sp(n,1)_{\mathbb{Z}}$ for $n \geq 2$. Then*

$$\mathrm{sr}(C_r^*(G)) = 1, \quad \mathrm{csr}(C_r^*(G)) \leq 2, \quad \mathrm{gsr}(C_r^*(G)) = 1,$$

and in addition $\mathrm{RR}(C_r^*(G)) \leq 1$.

Remark. The above rank estimates hold for any closed subgroup H of G in this theorem such that G/H has a finite volume (cf. [Wg, Theorem 3.7], [HV]). See [DHR] and [DH] for the ranks of reduced group C^*-algebras of F_n and $PSL_2(\mathbb{Z})$. Note that $K_1(C_r^*(F_n)) \cong \mathbb{Z}^n$ (cf.[Bl, 10.11.11]) and $\mathrm{csr}(C_r^*(F_n)) = 2$ by using [Eh1, Corollary 1.6] and (F0). On the other hand, $K_1(C_r^*(\mathbb{Z}_2 * \mathbb{Z}_3))$ and $K_1(C_r^*(SL_2(\mathbb{Z})))$ are trivial by K-theory of amalgamated free products since $\mathbb{Z}_2 * \mathbb{Z}_3$ and $SL_2(\mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ are K-amenable (cf.[Bl, 10.11.11 and 20.9]).

Using Theorem 2.1 we have the following product formula of the stable ranks:

Corollary 2.4. *Let G, H be two residually finite, countable discrete groups with the property* (T). *Then*

$$\begin{cases} \mathrm{sr}(C_r^*(G) \otimes_{\min} C_r^*(H)) = 1 < 2 = \mathrm{sr}(C_r^*(G)) + \mathrm{sr}(C_r^*(H)), \\ \mathrm{csr}(C_r^*(G) \otimes_{\min} C_r^*(H)) \leq 2 \leq \mathrm{csr}(C_r^*(G)) + \mathrm{csr}(C_r^*(H)), \\ \mathrm{gsr}(C_r^*(G) \otimes_{\min} C_r^*(H)) = 1 < 2 = \mathrm{gsr}(C_r^*(G)) + \mathrm{gsr}(C_r^*(H)), \end{cases}$$

and $\mathrm{RR}(C_r^*(G) \otimes_{\min} C_r^*(H)) \leq 1$, *where* $\otimes_{\min}$ *means the minimal C^*-tensor product.*

Proof. Note that the direct product $G \times H$ is a residually finite, countable discrete group with (T) from the assumption of G, H. Also note that $C_r^*(G \times H)$ is isomorphic to $C_r^*(G) \otimes_{\min} C_r^*(H)$. Let X, Y be (continuous) separating families of G, H respectively. Then, we have

$$C^*(G \times H) \to C^*((\oplus_{\pi \in X, \rho \in Y} \pi \otimes \rho)(G \times H)) \to C_r^*(G \times H) \to 0. \qquad \square$$

On the other hand, we obtain

Theorem 2.5. *Let G be an amenable, finitely generated, countable discrete subgroup of $GL_n(\mathbb{C})$, and $C^*(G)$ its group C^*-algebra. If $C^*(G)$ has a continuous separating family $\{\pi_j\}_{j \in J}$ of finite dimensional representations for J a locally compact Hausdorff space, then*

$$\begin{cases} \mathrm{sr}(C^*(G)) = \sup_{n \in \mathbb{N}}(\{[\dim \hat{G}_n/2]/n\} + 1), \\ \mathrm{csr}(C^*(G)) \leq 1 + \sup_{n \in \mathbb{N}}(\{[(\dim \hat{G}_n)/2]/n\} + 1), \end{cases}$$

and $\mathrm{RR}(C^*(G)) \leq \sup_{n\in\mathbb{N}}(2\{[\dim \hat{G}_n/2]/n\}+1)$, *where* $\hat{G}_n$ *means the space of all* n*-dimensional irreducible unitary representations of* G *up to unitary equivalence.*

Proof. By the assumption, G is residually finite, and $C^*(G) = C^*_r(G)$. □

Remark. Any lattice of simply connected, solvable Lie groups is finitely generated and regarded as a subgroup of $GL_n(\mathbb{Z})$ (cf.[Rg, Corollary of Proposition 3.8 and Theorem 4.34]).

Example 2.6. The semi-direct products $\mathbb{Z}^n \rtimes_\alpha \mathbb{Z}$ would be included in the case of Theorem 2.5. In fact, we have the identification

$$\mathbb{Z}^n \rtimes_\alpha \mathbb{Z} \ni (z,t) \leftrightarrow \begin{pmatrix} \alpha_t & z^t \\ 0 & 1 \end{pmatrix} \in GL_{n+1}(\mathbb{Z}) \subset GL_{n+1}(\mathbb{C})$$

where z^t is the transpose of $z \in \mathbb{Z}^n$. Note that the discrete Heisenberg group $H^{\mathbb{Z}}_{2n+1}$ of rank $(2n+1)$ is a finitely generated subgroup of $GL_{n+2}(\mathbb{Z})$, and isomorphic to the semi-direct product $\mathbb{Z}^{n+1} \rtimes_\alpha \mathbb{Z}^n$ with the action α defined by $\alpha_t(z_0, z) = (z_0 + \sum_{j=1}^n t_j z_j, z)$ for $t = (t_j), z = (z_j) \in \mathbb{Z}^n$. Moreover, it is well known that $C^*(H^{\mathbb{Z}}_3)$ is isomorphic to $\Gamma(\mathbb{T}, \{\mathfrak{A}_\theta\}_{\theta\in\mathbb{T}})$ the C^*-algebra of a continuous field on $\mathbb{T}$ with fibers $\mathfrak{A}_\theta = C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ the rotation algebras associated with the action by the multiplication on $\mathbb{T}$ by θ (cf. [AP]). In this case, the space J in Theorem 2.5 can be taken as the disjoint union $\sqcup_{t\in\mathbb{Q}\cap[0,1]} V_t$ of $V_t = \mathbb{T}^2$ (or $[0,1]^2 \cap \mathbb{Q}^2$, where $[0,1] = \mathbb{R}$ mod 1) since the spectrum of the rational rotation algebra $\mathfrak{A}_{\theta_t}$ for $\theta_t = e^{2\pi i t}$ and $t \in \mathbb{Q}\cap[0,1]$ consists of finite dimensional irreducible representations with a constant dimension, and it is identified with $\mathbb{T}^2$. Hence, we obtain $\dim C^*(H^{\mathbb{Z}}_3)^\wedge_n = 2$ for any $n \in \mathbb{Z}$. Therefore, $\mathrm{sr}(C^*(H^{\mathbb{Z}}_3)) = 2$. Note that when t is irrational, $\mathfrak{A}_{\theta_t}$ is simple so that it has no finite dimensional representations, and its spectrum is not computable at all since it is of non type I. Thus, our merit is that we can compute the stable rank of the C^*-algebras of this type by using only computable data of their finite dimensional irreducible representations. See also [Mo] for locally compact groups with only finite dimensional irreducible representations.

By the same way as the proof of Corollary 1.4 we obtain

Corollary 2.7. *Let* G, H *be two amenable, finitely generated, countable discrete subgroups of* $GL_n(\mathbb{C})$ *with the same assumption as Theorem* 2.5. *Then*

$$\mathrm{sr}(C^*(G) \otimes C^*(H)) \leq \mathrm{sr}(C^*(G)) + \mathrm{sr}(C^*(H)).$$

Proof. Note that $C^*(G), C^*(H)$ are nuclear since G, H are amenable (cf.[Bl, 15.8]). □

Remark. See [Sd3] for the product formula of the stable rank in the case of G, H two connected Lie groups of type I (cf. [Sd4]).

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