FOLDNESS OF SOME TYPES OF POSITIVE IMPLICATIVE HYPER BCK-IDEALS IN HYPER BCK-ALGEBRAS

YOUNG BAE JUN, SEOK ZUN SONG AND WOOK HWAN SHIM

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ABSTRACT. The foldness of $PI(\ll, \ll, \subseteq)_{BCK}$-ideals and $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideals is considered, and their properties are investigated. Hyper homomorphic inverse image of $PI(\ll, \ll, \subseteq)_{BCK}$-ideals, $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideals, $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideals, and $PI(\ll, \ll, \ll)_{BCK}$-ideals are discussed.

1. Introduction

The study of BCK-algebras was initiated by K. Iséki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Since then a great deal of literature has been produced on the theory of BCK-algebras, In particular, emphasis seems to have been put on the ideal theory of BCK-algebras. The hyperstructure theory (called also multialgebras) is introduced in 1934 by F. Marty [11] at the 8th congress of Scandinavian Mathematiciens. In [8], Y. B. Jun et al. applied the hyperstructures to BCK-algebras, and introduced the concept of a hyper BCK-algebra which is a generalization of a BCK-algebra, and investigated some related properties. They also introduced the notion of a hyper BCK-ideal and a weak hyper BCK-ideal, and gave relations between hyper BCK-ideals and weak hyper BCK-ideals. Y. B. Jun et al. [9] gave a condition for a hyper BCK-algebra to be a BCK-algebra, and introduced the notion of a strong hyper BCK-ideal, a weak hyper BCK-ideal and a reflexive hyper BCK-ideal. They showed that every strong hyper BCK-ideal is a hypersubalgebra, a weak hyper BCK-ideal and a hyper BCK-ideal; and every reflexive hyper BCK-ideal is a strong hyper BCK-ideal. In [6], Y. B. Jun and X. L. Xin introduced the notion of an implicative hyper BCK-ideal. They gave the relations among hyper BCK-ideals, implicative hyper BCK-ideals and positive implicative hyper BCK-ideals. They stated some characterizations of implicative hyper BCK-ideals. And they also introduced the notion of implicative hyper BCK-algebras and investigated the relation between implicative hyper BCK-ideals and implicative hyper BCK-algebras. In [7], Y. B. Jun and X. L. Xin introduced the notion of a positive implicative hyper BCK-ideal, and investigated some related properties. Y. B. Jun and W. H. Shim [2] discussed several types of positive implicative hyper BCK-ideals in hyper BCK-algberas, and investigated their relations. Y. B. Jun and W. H. Shim [1] considered the foldness of $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideals and $PI(\ll, \ll, \ll)_{BCK}$-ideals in hyper BCK-algebras, and discussed their fuzzy version. In this paper we deal with the foldness of $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideals and $PI(\ll, \ll, \subseteq)_{BCK}$-ideals. We investigate relations among such notions. We

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verify that every hyper homomorphic inverse images of $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal, $PI(\ll, \ll, \ll)_{BCK}$-ideals, $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideals and $PI(\ll, \ll, \subseteq)_{BCK}$-ideals are also $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal, $PI(\ll, \ll, \ll)_{BCK}$-ideals, $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideals and $PI(\ll, \ll, \subseteq)_{BCK}$-ideals.

2. Preliminaries

We include some elementary aspects of hyper BCK-algebras that are necessary for this paper, and for more details we refer to [5] and [10]. Let H be a nonempty set endowed with a hyper operation "$\circ$", that is, $\circ$ is a function from $H \times H$ to $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$. For two subsets A and B of H, denote by $A \circ B$ the set $\bigcup_{a \in A, b \in B} a \circ b$.

By a *hyper BCK-algebra* we mean a nonempty set H endowed with a hyper operation "$\circ$" and a constant 0 satisfying the following axioms:

(K1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(K2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(K3) $x \circ H \ll \{x\}$,
(K4) $x \ll y$ and $y \ll x$ imply $x = y$,

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$.

In any hyper BCK-algebra H, the following hold (see [5] and [10]):

(p1) $0 \circ 0 = \{0\}$, (p2) $0 \ll x$,
(p3) $x \ll x$, (p4) $A \ll A$,
(p5) $A \subseteq B$ implies $A \ll B$, (p6) $0 \circ x = \{0\}$,
(p7) $0 \circ A = \{0\}$, (p8) $x \circ 0 = \{x\}$ and $A \circ 0 = A$,
(p9) $A \ll \{0\}$ implies $A = \{0\}$, (p10) $A \subseteq B \ll C$ implies $A \ll C$

for all $x, y, z \in H$ and for all nonempty subsets A, B and C of H. In what follows let H denote a hyper BCK-algebra unless otherwise specified.

Definition 2.1. [10] A nonempty subset A of H is called a *hyper BCK-ideal* of H if it satisfies the following conditions:

(I1) $0 \in A$,
(I2) $\forall x, y \in H$ $(x \circ y \ll A,\ y \in A \Rightarrow x \in A)$.

Definition 2.2. [10] A nonempty subset A of H is called a *weak hyper BCK-ideal* of H if it satisfies (I1) and

(I3) $\forall x, y \in H$ $(x \circ y \subseteq A,\ y \in A \Rightarrow x \in A)$.

Definition 2.3. [2] A nonempty subset A of H is called a *$PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal* of H if it satisfies (I1) and

(I4) $\forall x, y, z \in H$ $((x \circ y) \circ z \ll A,\ y \circ z \subseteq A \Rightarrow x \circ z \subseteq A)$.

Proposition 2.4. [5] *Let A be a subset of a hyper BCK-algebra H. If I is a hyper BCK-ideal of H such that $A \ll I$, then A is contained in I.*

3. Foldness of Some Types of Positive Implicative Hyper BCK-ideals

For any $x, y \in H$ and any natural number n, denote

$$x \circ y^n = (\cdots((x \circ \underbrace{y) \circ y) \cdots) \circ y}_{n\text{-times}}$$

Definition 3.1. [1] Let k, m and n be natural numbers. A nonempty subset A of H is called a $(k, m; n)$*-fold* $PI(\ll, \subseteq, \subseteq)_{BCK}$*-ideal* of H if it satisfies (I1) and

(J1) $\forall x, y, z \in H\ ((x \circ y) \circ z^k \ll A,\ y \circ z^m \subseteq A\ \Rightarrow\ x \circ z^n \subseteq A)$.

Example 3.2. Let $H = \{0, a, b\}$ be a hyper BCK-algebra with the following Cayley table:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

Then $A = \{0, a\}$ is a $(k, m; n)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of H for natural numbers k, m and n.

Example 3.3. Let $H = \{0, a, b\}$ be a hyper BCK-algebra with the following Cayley table:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{a\}$	$\{0, a\}$

Then $A = \{0, b\}$ is a $(k, m; n)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of H for natural numbers k, m and $n > 2$. But it is not a $(2, 3; 1)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of H since $(b \circ a) \circ a^2 = \{0\} \ll A$ and $a \circ a^3 = \{0\} \subseteq A$, but $b \circ a^1 = \{a\} \nsubseteq A$.

Theorem 3.4. [1] *Every* $(k, m; n)$*-fold* $PI(\ll, \subseteq, \subseteq)_{BCK}$*-ideal is a hyper* BCK*-ideal for natural numbers* k, m *and* n.

The converse of Theorem 3.4 may not be true (see [1]).

Definition 3.5. [9] A nonempty subset A of H is said to be *reflexive* if $x \circ x \subseteq A$ for all $x \in H$.

Definition 3.6. Let k be a natural number. A nonempty subset A of H is said to be k*-reflexive* if $x \circ x^k \subseteq A$ for all $x \in H$.

Theorem 3.7. *Let* A *be a hyper* BCK*-ideal of* H*. If* A *is reflexive then it is* k*-reflexive.*

Proof. Let $x \circ x \subseteq A$, then $x \circ x^2 \subseteq A \circ x \ll A$, and so $x \circ x^2 \subseteq A$ since A is a hyper BCK-ideal. Continuing this process, we get $x \circ x^k \subseteq A$ for natural number k. □

The converse of Theorem 3.7 is not true as seen in the following example.

Example 3.8. Let $H = \{0, a, b\}$ be the hyper BCK-algebra in Example 3.3. Then the subset $A = \{0\}$ of H is 2-reflexive, but it is not reflexive since $b \circ b = \{0, a\} \nsubseteq A$.

Proposition 3.9. *Let* A *be an* m*-reflexive subset of* H*. If* A *is a* $(k, m; n)$*-fold* $PI(\ll, \subseteq, \subseteq)_{BCK}$*-ideal of* H*, then*

$$\forall x, y \in H\ (x \circ y^{k+1} \subseteq A\ \Rightarrow\ x \circ y^n \subseteq A).$$

Proof. Let $x, y \in H$ be such that $x \circ y^{k+1} \subseteq A$. Then $(x \circ y) \circ y^k = x \circ y^{k+1} \ll A$ by (p5). Since $y \circ y^m \subseteq A$ by hypothesis, it follows from (J1) that $x \circ y^n \subseteq A$. □

Definition 3.10. [1] Let k, m and n be natural numbers. A nonempty subset A of H is called a $(k, m; n)$*-fold* $PI(\ll, \ll, \ll)_{BCK}$*-ideal* of H if it satisfies (I1) and

(J2) $\forall x, y, z \in H\ ((x \circ y) \circ z^k \ll A,\ y \circ z^m \ll A\ \Rightarrow\ x \circ z^n \ll A)$.

The following example shows that there is a $(k, m; n)$-fold $PI(\ll, \ll, \ll)_{BCK}$-ideal which is not a hyper BCK-ideal.

Example 3.11. Let $H = \{0, a, b\}$ be a hyper BCK-algebra with the following Cayley table:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

Then $A = \{0, b\}$ is a $(k, m; n)$-fold $PI(\ll, \ll, \ll)_{BCK}$-ideal of H for natural numbers k, m and n, but not a hyper BCK-ideal of H since $a \circ b = \{0, a\} \ll A$ and $b \in A$, but $a \notin A$.

Definition 3.12. Let k, m and n be natural numbers. A nonempty subset A of H is called a *$(k, m; n)$-fold $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal* of H if it satisfies (I1) and

(J3) $\forall x, y, z \in H \ ((x \circ y) \circ z^k \subseteq A,\ y \circ z^m \subseteq A \ \Rightarrow \ x \circ z^n \subseteq A)$.

Example 3.13. Let $H = \{0, a, b\}$ be a hyper BCK-algebra with the following Cayley table.

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{b\}$	$\{0, a\}$

Then $A = \{0, b\}$ is a $(k, m; n)$-fold $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal of H for natural numbers k, m and n.

Theorem 3.14. *Let k, m and n be natural numbers. Every $(k, m; n)$-fold $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal is a weak hyper BCK-ideal.*

Proof. Let A be a $(k, m; n)$-fold $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal of H and let $x, y \in H$ be such that $x \circ y \subseteq A$ and $y \in A$. Taking $z = 0$ in (J3) and using (p8), we have $(x \circ y) \circ 0^k = x \circ y \subseteq A$ and $y \circ 0^m = \{y\} \subseteq A$. It follows from (J3) that $\{x\} = x \circ 0^n \subseteq A$, that is, $x \in A$. Therefore A is a weak hyper BCK-ideal of H. □

The converse of Theorem 3.14 may not be true as seen in the following example.

Example 3.15. Let $H = \{0, a, b, c\}$ be a hyper BCK-algebra with the following table:

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0\}$	$\{0\}$
c	$\{c\}$	$\{c\}$	$\{b\}$	$\{0\}$

Then $A = \{0, a\}$ is a (weak) hyper BCK-ideal but not a $(k, m; 1)$-fold $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal because $(c \circ b) \circ b^k = \{0\} \subseteq A$ and $b \circ b^m = \{0\} \subseteq A$ but $c \circ b = \{b\} \nsubseteq A$.

Definition 3.16. Let k, m and n be natural numbers. A nonempty subset A of H is called a *$(k, m; n)$-fold $PI(\ll, \ll, \subseteq)_{BCK}$-ideal* of H if it satisfies (I1) and

(J4) $\forall x, y, z \in H \ ((x \circ y) \circ z^k \ll A,\ y \circ z^m \ll A \ \Rightarrow \ x \circ z^n \subseteq A)$.

Example 3.17. In Example 3.13, $A = \{0, b\}$ is a $(k, m; n)$-fold $PI(\ll, \ll, \subseteq)_{BCK}$-ideal of H for natural numbers k, m and n.

Definition 3.18. A hyper BCK-algebra H is said to be *m-fold positive implicative* if $(x \circ y) \circ z^m = (x \circ z^m) \circ (y \circ z^m)$ for all $x, y, z \in H$.

Example 3.19. Let $H = \{0, a, b\}$ be a hyper BCK-algebra with the following Cayley table.

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$

It is routine to verify that H is m-fold positive implicative for every natural number m.

Lemma 3.20. *Let A be a weak hyper BCK-ideal of H and let B and C be subsets of H. If $B \circ C \subseteq A$ and $C \subseteq A$, then $B \subseteq A$.*

Proof. Assume that $B \circ C \subseteq A$ and $C \subseteq A$. Then $b \circ c \subseteq A$ for all $b \in B$ and $c \in C \subseteq A$, and thus $b \in A$ by (I3). Therefore $B \subseteq A$. □

Definition 3.21. A hyper BCK-algebra H is said to satisfy the *increasing* (resp. *decreasing*) *condition* if for every $x, y \in H$, $x \circ y^m \subseteq x \circ y^n$ (resp. $x \circ y^m \supseteq x \circ y^n$) whenever $m \leq n$ for natural numbers m and n.

Example 3.22. The hyper BCK-algebra $H = \{0, a, b\}$ in Example 3.11 satisfies the increasing condition, and the hyper BCK-algebra $H = \{0, a, b\}$ in Example 3.3 satisfies the decreasing condition.

Theorem 3.23. *If H satisfies the increasing condition and if H is r-fold positive implicative for some natural number r, then every weak hyper BCK-ideal is a $(k, m; n)$-fold $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal where k, m and n are natural numbers such that $r = \min\{k, m\} \geq n$.*

Proof. Let A be a weak hyper BCK-ideal of H and let $x, y, z \in H$ be such that $(x \circ y) \circ z^k \subseteq A$ and $y \circ z^m \subseteq A$. Since H is r-fold positive implicative, $(x \circ y) \circ z^r = (x \circ z^r) \circ (y \circ z^r)$. Note that $(x \circ y) \circ z^r \subseteq (x \circ y) \circ z^k \subseteq A$ and $y \circ z^r \subseteq y \circ z^m \subseteq A$. Thus, by Lemma 3.20, we have $x \circ z^n \subseteq x \circ z^r \subseteq A$. Consequently A is a $(k, m; n)$-fold hyper BCK-ideal of H. □

Theorem 3.24. *Let m be a natural number and let A be a nonempty subset of H. Then A is an $(m, m; m)$-fold $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal of H if and only if the set*

$$A_w := \{x \in H \mid x \circ w^m \subseteq A\}, \ w \in H,$$

is a weak hyper BCK-ideal of H.

Proof. Suppose that A is an $(m, m; m)$-fold $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal of H. Since $0 \circ w^m = \{0\} \subseteq A$ for all $w \in H$, we have $0 \in A_w$. Let $x, y, w \in H$ be such that $x \circ y \subseteq A_w$ and $y \in A_w$. Then $(x \circ y) \circ w^m \subseteq A$ and $y \circ w^m \subseteq A$. It follows from (J3) that $x \circ w^m \subseteq A$ or equivalently $x \in A_w$. Hence A_w is a weak hyper BCK-ideal of H. Conversely, assume that for $u \in H$, A_u is a weak hyper BCK-ideal of H. Obviously $0 \in A$. Let $x, y, z \in H$ be such that $(x \circ y) \circ z^m \subseteq A$ and $y \circ z^m \subseteq A$. Then $x \circ y \subseteq A_z$ and $y \in A_z$. Since A_z is a weak hyper BCK-ideal of H, it follows from (I3) that $x \in A_z$ or equivalently $x \circ z^m \subseteq A$. Hence A is an $(m, m; m)$-fold $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal of H. □

Theorem 3.25. *Assume that H satisfies the increasing condition. For any natural numbers k, m and n with $k \leq m \leq n$, if A is a $(k, m; n)$-fold $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal of H, then*

$$A_w := \{x \in H \mid x \circ w^m \subseteq A\}, \ w \in H,$$

is a weak hyper BCK-ideal of H.

Proof. Assume that A is a $(k, m; n)$-fold $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal of H. Obviously, $0 \in A_w$. Let $x, y, w \in H$ be such that $x \circ y \subseteq A_w$ and $y \in A_w$. Then $(x \circ y) \circ w^k \subseteq (x \circ y) \circ w^m \subseteq A$ and $y \circ w^m \subseteq A$. Hence, by (J3), we have $x \circ w^m \subseteq x \circ w^n \subseteq A$ or equivalently $x \in A_w$. Therefore A_w is a weak hyper BCK-ideal of H. □

Using (p5), we have the following result.

Theorem 3.26. *Every $(k, m; n)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal is a $(k, m; n)$-fold $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal.*

The converse of Theorem 3.26 may not be true as seen in the following example.

Example 3.27. In Example 3.13, the set $A = \{0, b\}$ is a $(k, m; n)$-fold $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal of H for natural numbers k, m and n, but not a $(k, m; n)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of H since $(a \circ 0) \circ 0^k = \{a\} \ll A$ and $0 \circ 0^m = \{0\} \subseteq A$, but $a \circ 0^n = \{a\} \not\subseteq A$.

Theorem 3.28. *Every $(k, m; n)$-fold $PI(\ll, \ll, \subseteq)_{BCK}$-ideal is both a $(k, m; n)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal and a $(k, m; n)$-fold $PI(\ll, \ll, \ll)_{BCK}$-ideal.*

Proof. Let A be a $(k, m; n)$-fold $PI(\ll, \ll, \subseteq)_{BCK}$-ideal of H. Using (p5), we know that A is a $(k, m; n)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of H. Let $x, y, z \in H$ be such that $(x \circ y) \circ z^k \ll A$ and $y \circ z^m \ll A$. Then $x \circ z^n \subseteq A$ by (J4), and so $x \circ z^n \ll A$ by (p5). Hence A is a $(k, m; n)$-fold $PI(\ll, \ll, \ll)_{BCK}$-ideal of H. □

Corollary 3.29. *Every $(k, m; n)$-fold $PI(\ll, \ll, \subseteq)_{BCK}$-ideal is a hyper BCK-ideal.*

Proof. It follows from Theorems 3.4 and 3.28. □

Definition 3.30. [3] Let G and H be hyper BCK-algebras. A mapping $\mathfrak{f} : G \to H$ is called a *hyper homomorphism* if it satisfies

- $\mathfrak{f}(0) = 0$,
- $\forall x, y \in G \ (\mathfrak{f}(x \circ y) = \mathfrak{f}(x) \circ \mathfrak{f}(y))$.

Lemma 3.31. *Let $\mathfrak{f} : G \to H$ be a hyper homomorphism of hyper BCK-algebras. For any subsets A and B of G, if $A \ll B$, then $\mathfrak{f}(A) \ll \mathfrak{f}(B)$.*

Proof. Let A and B be subsets of G such that $A \ll B$. Let $u \in \mathfrak{f}(A)$. Then $u = \mathfrak{f}(a)$ for some $a \in A$. Since $A \ll B$, for the $a \in A$ there exists $b \in B$ such that $a \ll b$, i.e., $0 \in a \circ b$. Hence $0 = \mathfrak{f}(0) \in \mathfrak{f}(a \circ b) = \mathfrak{f}(a) \circ \mathfrak{f}(b) = u \circ \mathfrak{f}(b)$ for $\mathfrak{f}(b) \in \mathfrak{f}(B)$, and so $\mathfrak{f}(A) \ll \mathfrak{f}(B)$. □

Theorem 3.32. *Let $\mathfrak{f} : G \to H$ be a hyper homomorphism of hyper BCK-algebras. If B is a $(k, m; n)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of H, then $\mathfrak{f}^{-1}(B)$ is a $(k, m; n)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of G.*

Proof. Obviously $0 \in \mathfrak{f}^{-1}(B)$. Let $x, y, z \in G$ be such that $(x \circ y) \circ z^k \ll \mathfrak{f}^{-1}(B)$ and $y \circ z^m \subseteq \mathfrak{f}^{-1}(B)$. Then $(\mathfrak{f}(x) \circ \mathfrak{f}(y)) \circ \mathfrak{f}(z)^k = \mathfrak{f}((x \circ y) \circ z^k) \ll \mathfrak{f}(\mathfrak{f}^{-1}(B)) \subseteq B$ by Lemma 3.31, and $\mathfrak{f}(y) \circ \mathfrak{f}(z)^m = \mathfrak{f}(y \circ z^m) \subseteq \mathfrak{f}(\mathfrak{f}^{-1}(B)) \subseteq B$. Since B is a $(k, m; n)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal, it follows from (J1) that $\mathfrak{f}(x \circ z^n) = \mathfrak{f}(x) \circ \mathfrak{f}(z)^n \subseteq B$ so that $x \circ z^n \subseteq \mathfrak{f}^{-1}(B)$. Therefore $\mathfrak{f}^{-1}(B)$ is a $(k, m; n)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of G. □

Lemma 3.33. *Let $\mathfrak{f} : G \to H$ be a hyper isomorphism of hyper BCK-algebras. Then*

(i) $\forall x, y \in H \ (x \ll y \ \Rightarrow \ \mathfrak{f}^{-1}(x) \ll \mathfrak{f}^{-1}(y))$.
(ii) $\forall B \subseteq H, \forall y \in H \ (B \ll y \ \Rightarrow \ \mathfrak{f}^{-1}(B) \ll \mathfrak{f}^{-1}(y))$.

Proof. (i) Let $x, y \in H$ be such that $x \ll y$. Then $\mathfrak{f}(a_x) = x \ll y = \mathfrak{f}(a_y)$ for some $a_x, a_y \in G$. Hence $\mathfrak{f}(0) = 0 \in x \circ y = \mathfrak{f}(a_x) \circ \mathfrak{f}(a_y) = \mathfrak{f}(a_x \circ a_y)$, and so

$$0 \in \mathfrak{f}^{-1}(\mathfrak{f}(0)) \in \mathfrak{f}^{-1}(\mathfrak{f}(a_x \circ a_y)) = a_x \circ a_y.$$

Hence $a_x \ll a_y$, i.e., $\mathfrak{f}^{-1}(x) \ll \mathfrak{f}^{-1}(y)$.

(ii) Assume that $B \ll y$ for $B \subseteq H$ and $y \in H$. Let $a \in \mathfrak{f}^{-1}(B)$. Then $\mathfrak{f}(a) \in B$, and so $\mathfrak{f}(a) \ll y$. Since $\mathfrak{f}$ is onto, we have $\mathfrak{f}(a_y) = y$ for some $a_y \in G$. Hence $0 \in \mathfrak{f}(a) \circ y = \mathfrak{f}(a) \circ \mathfrak{f}(a_y) = \mathfrak{f}(a \circ a_y)$, which implies that $0 = \mathfrak{f}^{-1}(0) \in \mathfrak{f}^{-1}(\mathfrak{f}(a \circ a_y)) = a \circ a_y$. This shows that $a \ll a_y$, that is, $\mathfrak{f}^{-1}(B) \ll \mathfrak{f}^{-1}(y)$. □

Theorem 3.34. *Let $\mathfrak{f} : G \to H$ be a hyper isomorphism of hyper BCK-algebras. If B is a $(k, m; n)$-fold $PI(\ll, \ll, \ll)_{BCK}$-ideal of H, then $\mathfrak{f}^{-1}(B)$ is a $(k, m; n)$-fold $PI(\ll, \ll, \ll)_{BCK}$-ideal of G.*

Proof. Obviously $0 \in \mathfrak{f}^{-1}(B)$. Let $x, y, z \in G$ be such that $(x \circ y) \circ z^k \ll \mathfrak{f}^{-1}(B)$ and $y \circ z^m \ll \mathfrak{f}^{-1}(B)$. Using Lemma 3.31, we get $(\mathfrak{f}(x) \circ \mathfrak{f}(y)) \circ \mathfrak{f}(z)^k \ll B$ and $\mathfrak{f}(y) \circ \mathfrak{f}(z)^m \ll B$. Since B is a $(k, m; n)$-fold $PI(\ll, \ll, \ll)_{BCK}$-ideal, it follows from (J2) that $\mathfrak{f}(x \circ z^n) = \mathfrak{f}(x) \circ \mathfrak{f}(z)^n \ll B$. Let $a \in x \circ z^n$. Then $\mathfrak{f}(a) \in \mathfrak{f}(x \circ z^n) \ll B$ and so $\mathfrak{f}(a) \ll b$ for some $b \in B$. Hence $a \in \mathfrak{f}^{-1}(\mathfrak{f}(a)) \ll \mathfrak{f}^{-1}(b) \subseteq \mathfrak{f}^{-1}(B)$ by Lemma 3.33, and thus $x \circ z^n \ll \mathfrak{f}^{-1}(B)$. Therefore $\mathfrak{f}^{-1}(B)$ is a $(k, m; n)$-fold $PI(\ll, \ll, \ll)_{BCK}$-ideal of G. □

Theorem 3.35. *Let $\mathfrak{f} : G \to H$ be a hyper homomorphism of hyper BCK-algebras. If B is a $(k, m; n)$-fold $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal of H, then $\mathfrak{f}^{-1}(B)$ is a $(k, m; n)$-fold $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal of G.*

Proof. Obviously $0 \in \mathfrak{f}^{-1}(B)$. Let $x, y, z \in G$ be such that $(x \circ y) \circ z^k \subseteq \mathfrak{f}^{-1}(B)$ and $y \circ z^m \subseteq \mathfrak{f}^{-1}(B)$. Then $(\mathfrak{f}(x) \circ \mathfrak{f}(y)) \circ \mathfrak{f}(z)^k = \mathfrak{f}((x \circ y) \circ z^k) \subseteq \mathfrak{f}(\mathfrak{f}^{-1}(B)) \subseteq B$ and $\mathfrak{f}(y) \circ \mathfrak{f}(z)^m = \mathfrak{f}(y \circ z^m) \subseteq \mathfrak{f}(\mathfrak{f}^{-1}(B)) \subseteq B$. Since B is a $(k, m; n)$-fold $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal, it follows from (J3) that $\mathfrak{f}(x \circ z^n) = \mathfrak{f}(x) \circ \mathfrak{f}(z)^n \subseteq B$ so that $x \circ z^n \subseteq \mathfrak{f}^{-1}(\mathfrak{f}(x \circ z^n)) \subseteq \mathfrak{f}^{-1}(B)$. Hence $\mathfrak{f}^{-1}(B)$ is a $(k, m; n)$-fold $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal of G. □

Theorem 3.36. *Let $\mathfrak{f} : G \to H$ be a hyper homomorphism of hyper BCK-algebras. If B is a $(k, m; n)$-fold $PI(\ll, \ll, \subseteq)_{BCK}$-ideal of H, then $\mathfrak{f}^{-1}(B)$ is a $(k, m; n)$-fold $PI(\ll, \ll, \subseteq)_{BCK}$-ideal of G.*

Proof. Obviously $0 \in \mathfrak{f}^{-1}(B)$. Let $x, y, z \in G$ be such that $(x \circ y) \circ z^k \ll \mathfrak{f}^{-1}(B)$ and $y \circ z^m \ll \mathfrak{f}^{-1}(B)$. Then by Lemma 3.31, we have $(\mathfrak{f}(x) \circ \mathfrak{f}(y)) \circ \mathfrak{f}(z)^k \ll B$ and $\mathfrak{f}(y) \circ \mathfrak{f}(z)^m \ll B$. Since B is a $(k, m; n)$-fold $PI(\ll, \ll, \subseteq)_{BCK}$-ideal, it follows from (J4) that $\mathfrak{f}(x \circ z^n) = \mathfrak{f}(x) \circ \mathfrak{f}(z)^n \subseteq B$ so that $x \circ z^n \subseteq \mathfrak{f}^{-1}(\mathfrak{f}(x \circ z^n)) \subseteq \mathfrak{f}^{-1}(B)$. Hence $\mathfrak{f}^{-1}(B)$ is a $(k, m; n)$-fold $PI(\ll, \ll, \subseteq)_{BCK}$-ideal of G. □

The following example shows that $\{0\}$ may not be a $(k, m; n)$-fold $PI(\ll, \ll, \ll)_{BCK}$-ideal (resp. $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal, $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal, $PI(\ll, \ll, \subseteq)_{BCK}$-ideal) of H.

Example 3.37. Let $H = \{0, a, b\}$ be a hyper BCK-algebra with the following Cayley table.

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0, a\}$

Then H satisfies the decreasing condition and is a 2-fold positive implicative hyper BCK-algebra but not a 1-fold positive implicative, since $\{0\} = (b \circ b) \circ b \neq (b \circ b) \circ (b \circ b) = \{0, a\}$. It is routine to verify that $\{0\}$ is none of $(2, 2; 1)$-fold $PI(\ll, \ll, \ll)_{BCK}$-ideal, $(2, 2; 1)$-fold $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal, $(2, 2; 1)$-fold $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal and $(2, 2; 1)$-fold $PI(\ll, \ll, \subseteq)_{BCK}$-ideal) of H.

Theorem 3.38. *If H satisfies the increasing condition and if H is r-fold positive implicative for some natural number r, then $\{0\}$ is a $(k, m; n)$-fold $PI(\ll, \ll, \ll)_{BCK}$-ideal (resp. $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal, $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal, $PI(\ll, \ll, \subseteq)_{BCK}$-ideal) of G, where k, m and n are natural numbers such that $r = \min\{k, m\} \geq n$.*

Proof. Assume that H satisfies the increasing condition and let k, m, n and r be natural numbers such that $r = \min\{k, m\} \geq n$ and H is r-fold positive implicative. Let $x, y, z \in H$ be such that $(x \circ y) \circ z^k \ll \{0\}$ and $y \circ z^m \ll \{0\}$. Then $(x \circ y) \circ z^r \subseteq (x \circ y) \circ z^k \ll \{0\}$ and $y \circ z^r \subseteq y \circ z^m \ll \{0\}$. Using (p9) and (p10), we have $(x \circ y) \circ z^r = \{0\}$ and $y \circ z^r = \{0\}$. Since H is r-fold positive implicative, we have

$$\{0\} = (x \circ y) \circ z^r = (x \circ z^r) \circ (y \circ z^r) = (x \circ z^r) \circ \{0\},$$

and so $x \circ z^r \ll \{0\}$. Since $n \leq r$, we get $x \circ z^n \subseteq x \circ z^r \ll \{0\}$, and thus $x \circ z^n \ll \{0\}$ by (p10). Therefore $\{0\}$ is a $(k, m; n)$-fold $PI(\ll, \ll, \ll)_{BCK}$-ideal of H. Similarly, we have the remaining results. □

Theorem 3.39. *Let G and H be r-fold positive implicative hyper BCK-algebras that satisfy the increasing condition. Let $\mathfrak{f} : G \to H$ be a hyper homomorphism. Then*

$$\mathrm{Ker}(\mathfrak{f}) := \{x \in G \mid \mathfrak{f}(x) = 0\}$$

is a $(k, m; n)$-fold $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal (resp. $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal, $PI(\ll, \ll, \subseteq)_{BCK}$-ideal) of G where k, m, n and r are natural numbers such that $r = \min\{k, m\} \geq n$. Moreover, if $\mathfrak{f}$ is a hyper isomorphism, then $\mathrm{Ker}(\mathfrak{f})$ *is a $(k, m; n)$-fold $PI(\ll, \ll, \ll)_{BCK}$-ideal of G.*

Proof. The proof is by Theorems 3.35, 3.32, 3.36, 3.34, and 3.38. □

We now pose an open problem.

Open Problem 3.40. Let $\mathfrak{f} : G \to H$ be a hyper homomorphism of hyper BCK-algebras and let A be a subset of G. Is $\mathfrak{f}(A)$ a $(k, m; n)$-fold $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal (resp. $PI(\ll, \ll, \ll)_{BCK}$-ideal, $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal, and $PI(\ll, \ll, \subseteq)_{BCK}$-ideal) of H under what condition(s)?

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Young Bae Jun and Wook Hwan Shim, Department of Mathematics Education, Gyeongsang National University, Chinju (Jinju) 660-701, Korea

E-mail address: ybjun@nongae.gsnu.ac.kr

Seok Zun Song, Department of Mathematics, Cheju National University, Cheju 690-756, Korea
E-mail address: szsong@cheju.ac.kr