Scientiae Mathematicae Japonicae Online, **Vol. 9**, (2003), 179–186

K-SET CONTRACTIVE RETRACTIONS IN SPACES OF CONTINUOUS FUNCTIONS

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Received April 11, 2003; revised May 7, 2003

Abstract. Let X be an infinite-dimensional Banach space, and let B_X and S_X be its closed unit ball and unit sphere, respectively. A continuous mapping $R : B_X \to S_X$ is said to be a retraction provided that $x = Rx$ for all $x \epsilon S_X$. It is well known that when X is finite-dimensional there is no retraction from B_X onto S_X. We prove that in some Banach spaces of continuous functions for every $\varepsilon > 0$ there exists a retraction of the closed unit ball onto the unit sphere being a $(1 + \varepsilon)$-set contraction.

1. Introduction

The Scottish Book [8] contains the following question (Problem 36) raised around 1935 by S. Ulam : " There exists a retraction of the closed unit ball of a Hilbert space onto the unit sphere ? " S. Kakutani [6] gave a positive answer to this question.V. Klee [7] proved that answer to Ulam's question is " yes " in the more general setting of infinite-dimensional Banach spaces. B. Nowak [9] using a complicated construction that was subsequentely somewhat simplified by Y. Benyamini and Y. Sternfeld [3] showed that, for any infinite-dimensional Banach space X, there is a retraction $R : B_X \to S_X$ satisfying the Lipschitz condition

$$(1)\, \|Rx - Ry\| \leq k \|x - y\|, \quad for\ all\ x, y\epsilon B_X.$$

Given an infinite-dimensional Banach space X, let $k_0(X)$ denote the infimum of the k's for which such retraction exists.

Then $k_0(X) \geq 3$ (See [5]). Recall that, if A is a bounded subset of a Banach space X, the Hausdorff measure of noncompactness of A is defined by

$$\chi(A) := \inf\{r > 0 : \text{A can be covered by a finite number of balls centered in X}\}.$$

A continuous mapping $T : D(T) \subset X \to X$ is said to be a k-set contraction if there exists a constant $k \geq 0$ such that

$$(2)\, \chi(T(A)) \leq k\chi(A), \quad for\ all\ bounded\ sets\ A \subset D(T).$$

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the maximum norm $|\cdot|_\infty$ .Throughout this paper we shall use the following notations. $E := (E, \|\cdot\|)$ will denote a finite-dimensional real normed space and K a compact convex subset of E with nonempty interior (Without loss of generality, we can assume that K contains the origin as an interior point). $C(K, \mathbb{R}^n)$ the space of continuous functions on K with values in $\mathbb{R}^n$ equipped with the sup norm $\|\cdot\|_\infty$. Let X be an infinite-dimensional Banach space. By $k_1(X)$ denote the infimum of the set of all numbers k for which there is a retraction $R : B_X \to S_X$ satisfying the above condition (2). In this context J. Wosko [10] proved that $k_1(C[0,1]) = 1$ and that for any infinite-dimensional Banach space X there is no a 1-set retraction $R : B_X \to S_X$ being lipschitzian

2000 *Mathematics Subject Classification.* 46E15, 47H09.
Key words and phrases. retraction, k-set contraction, Hausdorff measure of noncompactness.

with some constant k. Moreover, he posed the problem to estimate $k_1(X)$ for particular classical Banach spaces and to establish for which spaces is $k_1(X) < k_0(X)$. In this note we extend from $C[0,1]$ to $C(K, \mathbb{R}^n)$ the Wosko's result, i.e. we prove that $k_1(\, C(K, \mathbb{R}^n)) = 1$.

2. Preliminaries

Let Y be a real normed space. We write $B_{Y,r}$ to denote the closed ball of Y centered at the origin with radius r. For a set $A \subset Y$, $\overline{A}$ it is closure, $intA$ its interior, ∂A its boundary and $diamA$ its diameter. Further we set $S_{Y,r} := \partial\ B_{Y,r}$.

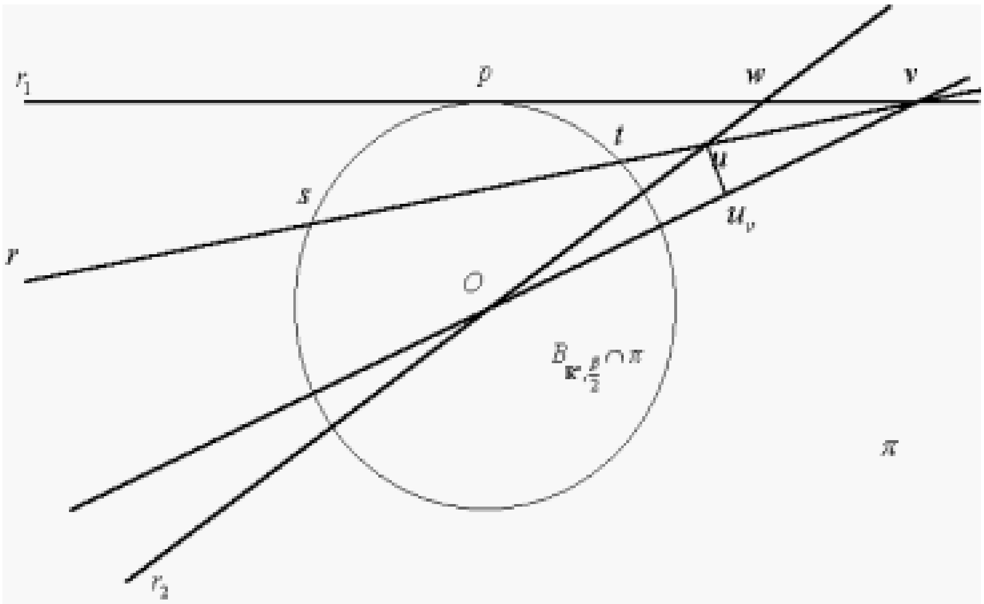
Consider the mapping $\varphi : K\backslash\{0\} \to \partial K$ defined by $\varphi(t) = w_t$,where w_t is the unique element of $\{\lambda t : \lambda \in [0, +\infty[\} \cap \partial K$. Let α be a positive real number such that $B_{E,\alpha} \subset K$. In this section we prove that φ satisfies the Lipschitz condition :

$$(2.1)\ \|w_t - w_s\| \leq L\,\|t - s\|\,, \quad for\ all\ s, t\epsilon K\backslash intB_{E,\alpha}.$$

Assume that $\mathbb{R}^n$ is the n-dimensional Euclidean space provided with the usual inner product $\langle u, v\rangle = \sum_{i=1}^n u_i v_i$ where $u = (u_1, ..., u_n)$ and $v = (v_1, ..., v_n)$. $|.|_n$ denotes the Euclidean norm on $\mathbb{R}^n$, $\theta(u, v)$ the angle between two non zero vectors u and v of $\mathbb{R}^n$ such that $0 \leq \theta(u, v) \leq \pi$ and u_v the orthogonal projection of u onto $\langle v\rangle := \{\lambda v : \lambda \in \mathbb{R}\}$. Let K be a compact convex set in $\mathbb{R}^n$ containing the origin as an interior point and let α be a positive real number such that $B_{\mathbb{R}^n,\alpha} \subset K$. In order to prove (2.1) it is sufficient to show that it is true for $\varphi : K\backslash intB_{\mathbb{R}^n,\alpha} \to \partial K$.

Lemma 1. *Let K be a compact convex set in* $\mathbb{R}^n$ *containing the origin as an interior point. Set* $\beta := \min\{|u|_n : u \in \partial K\}$ *and* $d := diamK$. *Then* $\inf\{\cos(\theta(v-u, u_v - u)) : u \neq v,\ |u|_n \leq |v|_n\,, |u-v|_n < \frac{\beta}{2}$ *and* $u, v \in \partial K\} \geq \frac{\beta}{2d}$.

Proof. Let $u, v \in \partial K$ with $u \neq v, |u|_n \leq |v|_n$ *and* $|u - v|_n < \frac{\beta}{2}$. We will prove that $sen(\theta(-v, u - v)) = sen\left(\frac{\pi}{2} - \theta(v - u, u_v - u)\right) \geq \frac{\beta}{2d}$. Therefore $\cos(\theta(v - u, u_v - u)) \geq \frac{\beta}{2d}$. Let r be the straight line through u and v. Then $r \cap intB_{\mathbb{R}^n,\frac{\beta}{2}} = \emptyset$. Suppose $r \cap S_{\mathbb{R}^n,\frac{\beta}{2}} = \{s, t\}$. We have two possible cases. The segment $[u, v]$ contains$\{s, t\}$. Then $\beta \leq |u|_n \leq |u - s|_n + |s|_n \leq |u - v|_n + |s|_n < \beta$, a contradiction! The segment $[u, v] \not\supseteq \{s, t\}$. Let π be the plane containing $0, u$ and v ; r_1 the straight line through v tangent to $B_{\mathbb{R}^n,\frac{\beta}{2}} \cap \pi$ in p, which lies in the half-plane determinted by the straight line through 0 and v that contains s and t ; r_2 the straight line through 0 and u (see figure below). Then the segment $[p, v] \cap r_2 = w \in K$. Hence $u \notin \partial K$, again a contradiction! Therefore $\cos(\theta(v - u, u_v - u)) = sen\left(\frac{\pi}{2} - \theta(v - u, u_v - u)\right) \geq sen(\theta(-v, p - v)) = \frac{|p|_n}{|v|_n} \geq \frac{\beta}{2d}$. ∎



Proposition 2. *Let K be a compact convex set in $\mathbb{R}^n$ containing the origin as an interior point and α be a positive real number such that $B_{\mathbb{R}^n,\alpha} \subset K$. Set $c := \max\{|u|_n : u \in \partial K\}$. Then the map φ is uniformly continuous on $K \backslash intB_{\mathbb{R}^n,\alpha}$.*

Proof. Since $K \backslash intB_{\mathbb{R}^n,\alpha}$ is compact, it is sufficient to show that φ is continuous on $K \backslash intB_{\mathbb{R}^n,\alpha}$. Our proof will be by way of contradiction. So suppose that there exists a sequence (t_n) of elements of $K \backslash intB_{\mathbb{R}^n,\alpha}$ such that $t_n \to t$ $(n \to +\infty)$ and $w_{t_n} \nrightarrow w_t$ $(n \to +\infty)$. By the compactness of ∂K we can find a subsequence $\left(w_{t_{n_k}}\right)$ of (w_{t_n}) convergent to $w \in \partial K \backslash \{w_t\}$. For all $k \in N$, since $t_{n_k} \in \left[\left(\alpha / \left|w_{t_{n_k}}\right|_n\right) w_{t_{n_k}}, w_{t_{n_k}}\right]$, there is $\lambda_{n_k} \in \left[1, \left|w_{t_{n_k}}\right|_n / \alpha\right] \subset [1, c/\alpha]$ such that $w_{t_{n_k}} = \lambda_{n_k}\, t_{n_k}$. Let $\left(\lambda_{n_{k_s}}\right)$ be a subsequence of (λ_{n_k}) convergent to $\lambda \in [1, c/\alpha]$. Then, since $w_{t_{n_{k_s}}} \to w$ $(s \to +\infty)$, $\lambda_{n_{k_s}} \to \lambda$ $(s \to +\infty)$ and $t_{n_{k_s}} \to t$ $(s \to +\infty)$, we have that $w = \lambda t$. Therefore $w = w_t$, a contradiction! ∎

Proposition 3. *Let K be a compact convex set in $\mathbb{R}^n$ containing the origin as an interior point and α be a positive real number such that $B_{\mathbb{R}^n,\alpha} \subset K$. Set $\beta := \min\{|u|_n : u \in \partial K\}$ and $d := diamK$. Then there exists L such that $|w_t - w_s|_n \leq L\,|t - s|_n$, for all $s, t \epsilon K \backslash intB_{\mathbb{R}^n,\alpha}$.*

Proof. By the Proposition 2 there exists a $\delta > 0$ such that $|t - s|_n < \delta \Rightarrow |w_t - w_s|_n < \frac{\beta}{2}$ for all $s, t \epsilon K \backslash intB_{\mathbb{R}^n,\alpha}$. Moreover $|t - s|_n \geq \delta \Rightarrow |w_t - w_s|_n \leq \frac{d}{\delta}|t - s|_n$, *for all* $s, t \epsilon K \backslash intB_{\mathbb{R}^n,\alpha}$. Now, suppose $s, t \epsilon K \backslash intB_{\mathbb{R}^n,\alpha}$, $|t - s|_n < \delta$, $s \neq t\,(\Rightarrow w_t \neq w_s)$ and $|w_s|_n \leq |w_t|_n$. By Lemma 1 it follows that $\cos\theta := \cos\left(\theta\left(w_t - w_s, w_{s_{(w_t)}} - w_s\right)\right) \geq \frac{\beta}{2d}$. On the other hand it is easy to see that $\left|w_{s_{(w_t)}} - w_s\right|_n \leq \frac{d}{\alpha}|t - s|_n$. Therefore $|w_t - w_s|_n = \frac{\left|w_{s_{(w_t)}} - w_s\right|_n}{\cos\theta} \leq \frac{d}{\alpha\cos\theta}|t - s|_n \leq \frac{2d^2}{\alpha\beta}|t - s|_n$. Set L:=max$\left\{\frac{d}{\delta}, \frac{2d^2}{\alpha\beta}\right\}$. It follows that $|w_t - w_s|_n \leq L\,|t - s|_n$, for all $s, t \epsilon K \backslash intB_{\mathbb{R}^n,\alpha}$. ∎

Corollary 4. *Let $K \subset E$ and let α be a positive real number such that $B_{E,\alpha} \subset K$. Then there exists L such that $\|w_t - w_s\| \leq L\,\|t - s\|$, for all $s, t \epsilon K \backslash intB_{E,\alpha}$*

We need the following proposition.

Proposition 5. *Let $K \subset E$ and $\alpha_0 \in \left]0,1\right[$. Set $\beta := \min\{\|u\| : u \in \partial K\}$ and $d :=$ diam$K$. Then there is a constant L such that $\forall \alpha \in [\alpha_0, 1[, \forall \varepsilon \in \left]0, \frac{\alpha_0\beta}{2}\right[, \forall\ s \in \alpha K$ and $\forall t \in K \backslash \alpha K$*

$$\|t-s\| \leq \varepsilon \Rightarrow \left\|\alpha^{-1}s - w_t\right\| \leq \left(2L + \frac{1}{\alpha_0} + \frac{4d}{\alpha_0\beta}\right)\varepsilon.$$

Proof. Fix $\alpha \in [\alpha_0, 1[, \varepsilon \in \left]0, \frac{\alpha_0\beta}{2}\right[, s \in \alpha K$ and $t \in K\backslash\alpha K$ with $\|t-s\| \leq \varepsilon$. Then $\|s\| > \frac{\alpha_0\beta}{2}$. Infact, if $\|s\| \leq \frac{\alpha_0\beta}{2}$, we have that $\alpha_0\beta < \|t\| \leq \|t-s\| + \|s\| \leq \varepsilon + \frac{\alpha_0\beta}{2} < \alpha_0\beta$, a contradiction! Therefore, by the Corollary 4, there exists a constant L such that $\|w_t - w_s\| \leq L\|t-s\|$ for all $s, t \epsilon K \backslash \frac{\alpha_0\beta}{2}K$.

Suppose $\|w_s\| \leq \|w_t\|$. We prove that $\left\|w_s - \alpha^{-1}s\right\| \leq \frac{\varepsilon}{\alpha_0}$. Hence $\left\|w_t - \alpha^{-1}s\right\| \leq \|w_t - w_s\| + \left\|w_s - \alpha^{-1}s\right\| \leq \left(L + \frac{1}{\alpha_0}\right)\varepsilon$. Clearly s$\in [0, \alpha w_s]$. Moreover, if $\|s\| < \left(\alpha - \frac{\varepsilon}{\|w_s\|}\right)\|w_s\|$, we have that $\alpha\|w_t\| < \|t\| \leq \|t-s\| + \|s\| < \alpha\|w_s\| \leq \alpha\|w_t\|$, a contradiction! Therefore $\left\|w_s - \alpha^{-1}s\right\| \leq \alpha^{-1}\|\alpha w_s - s\| \leq \frac{1}{\alpha_0}\left\|\left(\alpha - \frac{\varepsilon}{\|w_s\|}\right)w_s - \alpha w_s\right\| \leq \frac{\varepsilon}{\alpha_0}$.

Now assume $\|w_t\| \leq \|w_s\|$ and denote by w'_s the element of $[0, w_s]$ such that $\|w'_s\| = \|w_t\|$. Define the mapping $T : E \to B_{E, \frac{\alpha_0\beta}{2}}$ by $T(s) = \frac{\alpha_0\beta}{2}\frac{s}{\|s\|}$ if $s \in E \backslash B_{E,\frac{\alpha_0\beta}{2}}, T(s) = s$ if $s \in B_{E,\frac{\alpha_0\beta}{2}}$. T satisfies (see for instance [4, p. 88]) the Lipschitz condition: $\|T(t) - T(s)\| \leq 2\|t-s\|$ for all $s, t \in E$. Therefore, since $s, t \notin B_{E,\frac{\alpha_0\beta}{2}}$, we have that $\frac{\alpha_0\beta}{2}\frac{1}{\|w_t\|}\|w_t - w'_s\| = \|T(w_t) - T(w_s)\| = \|T(t) - T(s)\| \leq 2\|t-s\|$. Hence $\|w_t - w'_s\| \leq \frac{4}{\alpha_0\beta}\|w_t\|\|t-s\| \leq \frac{4d}{\alpha_0\beta}\varepsilon$. If $\|w'_s\| \leq \left\|\alpha^{-1}s\right\|$, then $\left\|w_t - \alpha^{-1}s\right\| \leq \|w_t - w_s\| + \left\|w_s - \alpha^{-1}s\right\| \leq \|w_t - w_s\| + \|w'_s - w_s\| \leq 2\|w_t - w_s\| + \|w_t - w'_s\| \leq (2L + \frac{4d}{\alpha_0\beta})\varepsilon$. If $\left\|\alpha^{-1}s\right\| < \|w'_s\|$, then $\left\|\alpha^{-1}s - w_t\right\| \leq \left\|\alpha^{-1}s - w'_s\right\| + \|w_t - w'_s\|$.

Now we prove that $\left\|\alpha^{-1}s - w'_s\right\| \leq \frac{\varepsilon}{\alpha_0}$. Therefore $\left\|\alpha^{-1}s - w_t\right\| \leq (\frac{1}{\alpha_0} + \frac{4d}{\alpha_0\beta})\varepsilon$. We show that $s \in \left[(\alpha - \frac{\varepsilon}{\|w'_s\|})w'_s, \alpha w'_s\right]$. Infact $\left\|\alpha^{-1}s\right\| < \|w'_s\| \Rightarrow \|s\| < \|\alpha w'_s\| \Rightarrow s \in [0, \alpha w'_s]$. Suppose $\|s\| < (\alpha - \frac{\varepsilon}{\|w'_s\|})w'_s = \alpha\|w'_s\| - \varepsilon$. Then $\alpha\|w_t\| < \|t\| \leq \|t-s\| + \|s\| < \alpha\|w'_s\|. = \alpha\|w_t\|$, a contradiction! Hence $\left\|\alpha^{-1}s - w'_s\right\| = \alpha^{-1}\|s - \alpha w'_s\| \leq \alpha^{-1}\left\|(\alpha - \frac{\varepsilon}{\|w'_s\|})w'_s, \alpha w'_s\right\| \leq \frac{\varepsilon}{\alpha_0}$. ∎

3. Main results

Set $C := C(K, \mathbb{R}^n)$. We start to define a mapping $Q : B_C \to B_C$ by

$$(Qf)(t) := \begin{cases} f(\frac{2}{1+\|f\|_\infty}t) & if \quad t \in K_f := \frac{1+\|f\|_\infty}{2}K \\ f(w_t) & if \quad t \in K\backslash K_f \end{cases}$$

By the continuity of f and by the Proposition 2 it is very simple to prove that Qf is continuous on K. Moreover we have that $\|f\|_\infty = \|Qf\|_\infty = \max\{|(Qf)(t)|_\infty : t \in K_f\}$ for all $f \in B_C$ and $Qf = f$ for all $f \in S_C$.

Proposition 6. *The mapping Q is continuous.*

Proof. Let (f_n) be a sequence in B_C such that $f_n \overset{\|\cdot\|_\infty}{\longrightarrow} f$ $(n \to +\infty)$.Fix ε. Then $\exists\ n_1 \in \mathbb{N} : \forall n \geq n_1$ $\|f_n - f\|_\infty \leq \frac{\varepsilon}{2}$ (1). Since f is uniformly continuous on K, we have that $\exists\ \delta > 0 : \forall s, t \in K$ $\|t-s\| \leq \delta \Rightarrow |f(t) - f(s)|_\infty \leq \frac{\varepsilon}{2}$ (2). Choose $n_2 \in \mathbb{N} : \forall n \geq n_2$

$$\left|\frac{2}{1+\|f_n\|_\infty} - \frac{2}{1+\|f\|_\infty}\right| \leq \frac{\delta}{c}\ (3),$$

where $c := \max_{t\in K} \|t\|$. Now we show that $\forall\ n \geq \overline{n} := \max\{n_1, n_2\}$ and $\forall t \in K$ we have that $|(Qf_n)(t) - (Qf)(t)|_\infty \leq \varepsilon$, so that $\|Qf_n - Qf\|_\infty \leq \varepsilon$. Let $t \in K_f \cap K_{f_n}$ and $n \geq \overline{n}$. By (1), (2) and (3) it follows that

$$|(Qf_n)(t) - (Qf)(t)|_\infty = \left| f_n(\frac{2}{1+\|f_n\|_\infty}t) - f(\frac{2}{1+\|f\|_\infty}t) \right|_\infty$$

$$\leq \left| f_n(\frac{2}{1+\|f_n\|_\infty}t) - f(\frac{2}{1+\|f_n\|_\infty}t) \right|_\infty + \left| f(\frac{2}{1+\|f_n\|_\infty}t) - f(\frac{2}{1+\|f\|_\infty}t) \right|_\infty \leq \varepsilon.$$

Let $t \in K_f \triangle K_{f_n}$ (where $\triangle$ denotes the symmetric difference) and $n \geq \overline{n}$. Then

$$|(Qf_n)(t) - (Qf)(t)|_\infty = \left| f_n(\frac{2}{1+\|f_n\|_\infty}t) - f(w_t) \right|_\infty \quad (4)$$

$$\text{or} \quad |(Qf_n)(t) - (Qf)(t)|_\infty = \left| f_n(w_t) - f(\frac{2}{1+\|f\|_\infty}t) \right|_\infty \quad (5)$$

If (4) holds. We have, by (1), (2) and (3), that

$$\left| f_n(\frac{2}{1+\|f_n\|_\infty}t) - f(w_t) \right|_\infty \leq \left| f_n(\frac{2}{1+\|f_n\|_\infty}t) - f(\frac{2}{1+\|f_n\|_\infty}t) \right|_\infty$$

$$+ \left| f(\frac{2}{1+\|f_n\|_\infty}t) - f(w_t) \right|_\infty \leq \varepsilon.$$

If (5) is true. Analogously we obtain $\left| f_n(w_t) - f(\frac{2}{1+\|f\|_\infty}t) \right|_\infty \leq \varepsilon$. Let $t \in K\backslash(K_f \cup K_{f_n})$ and $n \geq \overline{n}$. By (1) it follows

$$|(Qf_n)(t) - (Qf)(t)|_\infty = |f_n(w_t) - f(w_t)|_\infty \leq \frac{\varepsilon}{2}.$$

∎

Let us recall [2] that there is an explicite formula for the Hausdorff measure of noncompactness in C. For any bounded set $A \subset C$ we have

$$(*) \quad \chi(A) = \frac{1}{2}\omega_0(A) = \frac{1}{2}\lim_{\varepsilon\to 0^+} \omega(A, \varepsilon) = \frac{1}{2}\lim_{\varepsilon\to 0^+} \sup_{f\in A} \omega(f, \varepsilon),$$

where $\omega(f, \varepsilon) = \sup\{\ |f(t) - f(s)|_\infty : s, t \in K, \|t - s\| \leq \varepsilon\}$.

Proposition 7. *The mapping Q is a 1-set contraction.*

Proof. By Proposition 5 and Corollary 4 we can find a constant M such that $\forall\varepsilon \in \left[o, \frac{1}{4}\beta\right]$ (where $\beta := \min\{\|u\| : u \in \partial K\}$), $\forall f \in B_C$ and $\forall s, t \in K$ we have $\|t - s\| \leq \varepsilon \Rightarrow |(Qf)(t) - (Qf)(s)|_\infty \leq M\varepsilon$. Therefore for any $\varepsilon \in \left[o, \frac{1}{4}\beta\right]$ and any $f \in B_C$

$$\omega(Qf, \varepsilon) = \sup\{|(Qf)(t) - (Qf)(s)|_\infty : s, t \in K, \|t - s\| \leq \varepsilon\} \leq$$

$$\leq \sup\{\ |f(t) - f(s)|_\infty : s, t \in K, \|t - s\| \leq M\varepsilon\} \leq \omega(f, M\varepsilon).$$

In view of $(*)$ this implies $\omega_0(QA) \leq \omega_0(A)$ for any $A \subset B_C$. Therefore $\chi(QA) \leq \chi(A)$, i.e. Q is a 1-set contraction. ∎

For any $u \in]0, +\infty[$ define the mapping $P_u : f \in B_C \to P_u f \in C$ putting

$$(P_u f)_i(t) := \max\left\{0, \frac{u}{2}(2\frac{\|t\|}{\|w_t\|} - \|f\|_\infty - 1)\right\} (i = 1, ..., n).$$

Remark 8. *For all $f \in B_C$ and for all $t \in K_f$ we have that $(P_u f)_i(t) = 0$ for $i = 1, ..., n$.*

Proposition 9. *For any* $u \in]0,+\infty[$ *and any* $f \in B_C$: (i) $P_u f$ *is continuous,* (ii) P_u *is continuous,* (iii) P_u *is compact.*

Proof. (i) follows by the continuity of f and by the Proposition 2.

(ii) : Let (f_n) be a sequence in B_C such that $f_n \overset{\|\cdot\|_\infty}{\longrightarrow} f$ $(n \to +\infty)$.Fixε.Then $\exists\, \overline{n} \in \mathbb{N}$: $n \geq \overline{n}$ $\|f_n - f\|_\infty \leq \frac{2}{u}\varepsilon$ (1). Now we prove that $\forall\, n \geq \overline{n}$ and $\forall t \in K$ $|\ (P_u f_n)(t) - (P_u f)(t)|_\infty \leq \varepsilon$. Hence $\|P_u f_n - P_u f\|_\infty \leq \varepsilon$.

Let $t \in K_f \cap K_{f_n}$ and $n \geq \overline{n}$. Then $|\ (P_u f_n)(t) - (P_u f)(t)|_\infty = 0$. Let $t \in K_f \triangle K_{f_n}$ and $n \geq \overline{n}$. Then

$$|\ (P_u f_n)(t) - (P_u f)(t)|_\infty = \frac{u}{2}\left|2\frac{\|t\|}{\|w_t\|} - \|f_n\|_\infty - 1)\right| \ (2)$$

$$or\ |\ (P_u f_n)(t) - (P_u f)(t)|_\infty = \frac{u}{2}\left|2\frac{\|t\|}{\|w_t\|} - \|f\|_\infty - 1)\right| \ (3).$$

If (2) is true. We have, since $\|t\| \leq \frac{1+\|f\|_\infty}{2r}\|w_t\|$,

$$\frac{u}{2}\left|2\frac{\|t\|}{\|w_t\|} - \|f_n\|_\infty - 1)\right| \leq \frac{u}{2}\left|\|f_n\|_\infty - \|f\|_\infty\right| \leq \frac{u}{2}\|f_n - f\|_\infty \leq \varepsilon.$$

If (3) holds. Analogously we obtain $\frac{u}{2}\left|2\frac{\|t\|}{\|w_t\|} - \|f\|_\infty - 1)\right| \leq \varepsilon$.

Let $t \in K\backslash(K_f \cup K_{f_n})$ and $n \geq \overline{n}$. We have

$$|\ (P_u f_n)(t) - (P_u f)(t)|_\infty = \frac{u}{2}\|f_n - f\|_\infty \leq \varepsilon.$$

(iii) : Let $u \in]0,+\infty[$. Since C is a Banach space, it sufficient to show that $P_u(B_C)$ is totally bounded. We start to observe that $\|P_u f\|_\infty = \frac{u}{2}(1 - \|f\|_\infty)$ for any $f \in B_C$. Therefore, by $0 \leq \|f\|_\infty \leq 1$, it follows that $\|P_u f\|_\infty \in \left[0, \frac{u}{2}\right]$ for any $f \in B_C$. Now we prove that

$$\left|\|P_u f\|_\infty - \|P_u g\|_\infty\right| \leq \varepsilon \Rightarrow \|P_u f - P_u g\|_\infty \leq \varepsilon \quad (4).$$

Let $t \in K_f \cap K_g$. Then $|(P_u f)(t) - (P_u g)(t)|_\infty = 0$.

Let $t \in K\backslash(K_f \cup K_g)$. Then $|(P_u f)(t) - (P_u g)(t)|_\infty = \frac{u}{2}\left|\|f\|_\infty - \|g\|_\infty\right| = \left|\|P_u f\|_\infty - \|P_u g\|_\infty\right|$.

Let $t \in K_f \triangle K_{f_n}$. Then

$$|(P_u f)(t) - (P_u g)(t)|_\infty = |(P_u f)(t)|_\infty \ (5) \ \text{ or } \ |(P_u f)(t) - (P_u g)(t)|_\infty = |(P_u g)(t)|_\infty \ (6).$$

If (5) holds. For all $t \in K_g\backslash K_f$ we have $\|t\| \leq \frac{1+\|g\|_\infty}{2r}\|w_t\|$. Therefore

$$|(P_u f)(t)|_\infty = \frac{u}{2}\left|2\frac{\|t\|}{\|w_t\|} - \|f\|_\infty - 1\right| \leq \frac{u}{2}\left|\|f\|_\infty - \|g\|_\infty\right| = \left|\|P_u f\|_\infty - \|P_u g\|_\infty\right|$$

If (6) holds. Analogously we obtain $|(P_u g)(t)|_\infty \leq \left|\|P_u f\|_\infty - \|P_u g\|_\infty\right|$.

Hence the inequality (4) is true.

Let $\varepsilon > 0$. Fixed an ε-net $\{\alpha_1, ..., \alpha_m\}$ in $\left[0, \frac{u}{2}\right]$, choose $\{f_1, ..., f_m\} \subset B_C$ such that $\|P_u f_j\|_\infty = \alpha_j$ *for* $j = 1, ..., m$.Then $\{P_u f_1, ..., P_u f_m\}$ is an ε-net in $P_u(B_C)$. Infact *for any* $f \in B_C$ there exists $j \in \{1, ..., m\}$ such that $\left|\|P_u f\|_\infty - \|P_u f_j\|_\infty\right| \leq \varepsilon$. By (4) it follows that $\|P_u f - P_u f_j\|_\infty \leq \varepsilon$. Hence $P_u(B_C)$ is totally bounded. ∎

Now consider the mapping $T_u : B_C \to C$

$$T_u f = Qf + P_u f.$$

Clearly, the mapping T_u is a 1-set contraction, and $T_u f = f$ for any $f \in S_C$. Moreover, for any $f \in B_C$, we have that

$$\begin{aligned}
\|T_u f\|_\infty &= \|Qf + P_u f\|_\infty = \max\left\{|(Qf)(t) + (P_u f)(t)|_\infty : t \in K\right\} \\
&\geq \max\left\{\max_{t\in K_f} |(Qf)(t)|_\infty\,,\ \max_{t\in K\setminus K_f} |(Qf)(t) + (P_u f)(t)|_\infty\right\} \\
&\geq \max\left\{\|f\|_\infty\,,\ \max_{t\in K\setminus K_f} |(Qf)(w_t) + (P_u f)(w_t)|_\infty\right\} \\
&\geq \max\left\{\|f\|_\infty\,,\ \max_{t\in K\setminus K_f} |f(w_t) + (P_u f)(w_t)|_\infty\right\} \\
&= \max\left\{\|f\|_\infty\,,\ \max_{t\in K\setminus K_f} \max_{i=n}^{n} \left|f_i(w_t) + \frac{u}{2}(1 - \|f\|_\infty)\right|\right\} \\
&\geq \max\left\{\|f\|_\infty\,,\ \max_{t\in K\setminus K_f} \max_{i=n}^{n} f_i(w_t) + \frac{u}{2}(1 - \|f\|_\infty)\right\} \\
&\geq \max\left\{\|f\|_\infty\,,\ \frac{u}{2}(1 - \|f\|_\infty) - \|f\|_\infty\right\}.
\end{aligned}$$

The last term attains its minimum $\frac{u}{u+4}$ for functions f with $\|f\|_\infty = \frac{u}{u+4}$. Therefore $\|T_u f\|_\infty \geq \frac{u}{u+4}$ for all $f \in B_C$. Set

$$R_u f = \frac{1}{\|T_u f\|_\infty} T_u f$$

For all $f \in B_C$ we have

$$\omega(R_u f, \varepsilon) = \frac{1}{\|T_u f\|_\infty}\omega(T_u f, \varepsilon) \leq \frac{u+4}{u}\omega(T_u f, \varepsilon).$$

Hence for any set $A \subset B_C$

$$\omega_0(R_u A) \leq \frac{u+4}{u}\omega_0(A).$$

Therefore

$$\chi(R_u A) \leq \frac{u+4}{u}\chi(A).$$

Since $\lim_{u\to\infty} \frac{u+4}{u} = 1$, the following result holds.

Theorem 10. $k_1(C) = 1$.

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