Scientiae Mathematicae Japonicae Online, **Vol. 9**, (2003), 137–157

SOME WEAK LAWS ON BISEMILATTICE AND TRIPLE-SEMILATTICE

Kiyomitsu Horiuchi

Received April 16, 2003

Abstract. A bisemilattice, which satisfies absorption law, is a lattice. Because various laws are led from the absorption law, it can be said that absorption law is comparatively strong law on bisemilattice. As relations between laws on bisemilattice are delicate, we considered them. Then, we tried the introduction of those laws in triple-semilattice. We show that a state is a little different from the case of bisemilattice.

1 On bisemilattice

1.1 Some laws on bisemilattice. A semilattice $(S,\ *)$ is a set S with a single binary, idempotent, commutative and associative operation $*$.

(1) $$a * a = a \qquad (idempotent)$$

(2) $$a * b = b * a \qquad (commutative)$$

(3) $$a * (b * c) = (a * b) * c \qquad (associative)$$

Under the relation defined by $a \leq_* b \Longleftrightarrow a * b = b$, any semilattice $(S,\ *)$ is a partially ordered set $(S,\ \leq_*)$. A bisemilattice $(A,\ *_1, *_2)$ is an algebra which has two semilattice operations, that is, $(A, *_1)$ and $(A, *_2)$ are semilattices, respectively!ªHence, we can construct two ordered sets $(A, \leq_1)$ and $(A, \leq_2)$. A lattice is a bisemilattice which satisfy the following:

$[B]$ (absorption laws)

(4) $$a *_1 (a *_2 b) = a$$

(5) $$a *_2 (a *_1 b) = a$$

$[G]$

(6) $$a *_2 c \ \leq_1 \ a$$

(7) $$a *_1 c \ \leq_2 \ a$$

$[D]$

(8) $$a \leq_2 b \quad \Longrightarrow \quad b \leq_1 a$$

(9) $$a \leq_1 b \quad \Longrightarrow \quad b \leq_2 a$$

There is no special meaning in having made absorption laws the sign $[B]$, $[G]$ and $[D]$.

2000 *Mathematics Subject Classification.* 06A12, 20M10, 18B35.

Key words and phrases. bisemilattice, trple-seilattice, absorption laws, roundabout-absorption laws.

Proposition 1 (4) $\iff$ (6) $\iff$ (8) *and* (5) $\iff$ (7) $\iff$ (9), *therefore* $[B] \iff [D] \iff [G]$.

Proof To prove that (4) $\Longrightarrow$ (8), let $a \leq_2 b$. Then, $a *_2 b = b$. ¿From (4), $(a *_2 b) *_1 a = a$. Hence $b *_1 a = a$. That is, $b \leq_1 a$. Next, we show (8) $\Longrightarrow$ (6). It is clear that $a \leq_2 a *_2 c$. ¿From (8), $a *_2 c \leq_1 a$. It remains to prove that (6) $\Longrightarrow$ (4). If (6) is satisfied, then $(a *_2 c) *_1 a = a$ immediately. Consequently, (4) $\iff$ (6) $\iff$ (8). Similarly, (5) $\iff$ (7) $\iff$ (9).

Then, the following properties are satisfied from absorption laws. For the following laws, there may be a suitable name as well as absorption laws. But, it is shown with the sign ($[A]$ etc.) to avoid bringing preconception.

$[A]$

$$a \leq_1 b \text{ and } a \leq_2 b \implies a = b \tag{10}$$

$[E]$

$$a *_1 b = a *_2 b \implies a = b \tag{11}$$

Proposition 2 $[A] \iff [E]$.

Proof To prove that $[A] \Longrightarrow [E]$, let $a *_1 b = a *_2 b = c$. Then, $a \leq_1 c$, $b \leq_1 c$, $a \leq_2 c$ and $b \leq_2 c$. ¿From $[A]$, $a \leq_1 c$ and $a \leq_2 c$ imply $a = c$. Similarly, $b \leq_1 c$ and $b \leq_2 c$ imply $b = c$. Hence $a = c = b$.
Conversely, let $a \leq_1 b$ and $a \leq_2 b$. Then, $a *_1 b = b$ and $a *_2 b = b$. Hence, $a *_1 b = a *_2 b$. If $[E]$ is satisfied, $a = b$.

Proposition 3 (4) or (5) $\Longrightarrow$ (10), *therefore* $[B] \Longrightarrow [A]$.

Proof We show that (4) $\Longrightarrow$ (10). Let $a \leq_1 b$ and $a \leq_2 b$. Then, $a *_1 (a *_2 b) = a *_1 b = b$. ¿From (4), $a *_1 (a *_2 b) = a$. Hence $a = b$. Similarly, we can prove that (5) $\Longrightarrow$ (10).

This $[A]$ is the important law which influences anti-symmetry law of the axiom of the order. We think about some law that another fundamental character is expressed. The next laws make the transformation of the inequality possible.

$[F]$

$$a \leq_1 b \implies a *_2 c \leq_1 b *_2 c \tag{12}$$
$$a \leq_2 b \implies a *_1 c \leq_2 b *_1 c \tag{13}$$

$[I]$

$$a \leq_1 b \text{ and } c \leq_1 d \implies a *_2 c \leq_1 b *_2 d \tag{14}$$
$$a \leq_2 b \text{ and } c \leq_2 d \implies a *_1 c \leq_2 b *_1 d \tag{15}$$

[C]

$$a *_2 b \ \leq_1 \ a *_1 b \tag{16}$$

$$a *_1 b \ \leq_2 \ a *_2 b \tag{17}$$

Proposition 4 (12) $\Longleftrightarrow$ (14) *and* (13) $\Longleftrightarrow$ (15), *therefore* $[F] \Longleftrightarrow [I]$.

Proof We prove that (12) $\Longrightarrow$ (14). Let $a \leq_1 b$ and $c \leq_1 d$. ¿From (12) and $a \leq_1 b$, $a *_2 c \leq_1 b *_2 c$. ¿From (12) and $c \leq_1 d$, $c *_2 b \leq_1 d *_2 b$. Then $a *_2 c \leq_1 b *_2 c \leq_1 b *_2 d$. Conversely, the implication (14) $\Longrightarrow$ (12) is obvious. Hence (12) $\Longleftrightarrow$ (14). Similarly, (13) $\Longleftrightarrow$ (15).

Proposition 5 (12) $\Longrightarrow$ (16) *and* (13) $\Longrightarrow$ (17), *therefore* $[F] \Longrightarrow [C]$.

Proof We show that (12) $\Longrightarrow$ (16). ¿From (12) and $a \leq_1 a *_1 b$, $a *_2 (a *_1 b) \leq_1 (a *_1 b) *_2 (a *_1 b) = a *_1 b$. ¿From (12) and $b \leq_1 a *_1 b$, $b *_2 a \leq_1 (a *_1 b) *_2 a$. Then $b *_2 a \leq_1 (a *_1 b) *_2 a \leq_1 a *_1 b$.
Similarly, (13) $\Longrightarrow$ (17).

Proposition 6 (8) and (9) $\Longrightarrow$ (12) *and* (8) and (9) $\Longrightarrow$ (13), *therefore* $[D] \Longrightarrow [F]$.

Proof Suppose that (8) and (9). ¿From (9), if $a \leq_1 b$ then $b \leq_2 a$, then $b \leq_2 a \leq_2 a *_2 c$. Since $c \leq_2 a *_2 c$, $b *_2 c \leq_2 a *_2 c$. Hence $a *_2 c \leq_1 b *_2 c$, from (8). Therefore, (8) and (9) $\Longrightarrow$ (12).
Similarly, we can prove that (8) and (9) $\Longrightarrow$ (13).

This proposition shows "the absorption imply the [F]". If a bisemilattice is distributive, then it has the [F] even if it does not have the absorption. Hence, the condition [F] simutaneously generalize both the absorption and the distributive.

The quantitative law of the next [H] and [J] comes out from absorption law. Let $(A, *_1, *_2)$ be a bisemilattice. For $a, b \in A$, a pair of a and b is said to be **comparable** if any one of following four relations is satisfied.

$$a \leq_1 b, \quad a \leq_2 b, \quad b \leq_1 a, \quad b \leq_2 a \tag{18}$$

We denote $a \leftrightarrow b$ if a and b are comparable, and $a \perp b$ if a and b are not comparable.

The set $\Gamma(*_1, *_2, a, b)$ is the smallest set satisfying (i) and (ii):

(i) $a, b \in \Gamma(*_1, *_2, a, b)$,
(ii) If $x, y \in \Gamma(*_1, *_2, a, b)$, then $x *_1 y, \ x *_2 y \in \Gamma(*_1, *_2, a, b)$.

The $\gamma(a, b)$ is the number of $\Gamma(*_1, *_2, a, b)$, that is, $\gamma(a, b) = |\Gamma(*_1, *_2, a, b)|$.

$[H^+]$

$$\gamma(a,b) = \begin{cases} 1 & if \quad a = b \\ 2 & if \quad a \leftrightarrow b \ \text{ and } \ a \neq b \\ 4 & if \quad a \perp b \end{cases} \tag{19}$$

$[H]$

$$a \leftrightarrow b \implies \gamma(a,b) \leq 2 \tag{20}$$

The $\gamma(a,b) \leq 2$ is equivalent to "$a *_1 b = a$ or b *and* $a *_2 b = a$ or b."

$[H']$

$$\gamma(a,b) = \begin{cases} 1 & if \quad a = b \\ 2 & if \quad a \leftrightarrow b \text{ and } a \neq b \\ 3 \text{ or } 4 & if \quad a \perp b \end{cases} \tag{21}$$

$[H^-]$

$$\gamma(a,b) \leq 4 \tag{22}$$

Proposition 7 $[B] \implies [H^+] \implies [H] \iff [H'] \implies [H^-]$.

Proof The implication $[B] \implies [H^+] \implies [H'] \implies [H^-]$ and $[H'] \implies [H]$ are obvious. It remains to show that $[H] \implies [H']$.
Suppose that the bisemilattice A is satisfied with $[H]$ and $a \perp b$ $(a, b \in A)$.
(I) In the case of $a *_1 b = a *_2 b = c$. Immediately, $\gamma(a,b) = |\Gamma(*_1, *_2, a, b)| = |\{a,b,c\}| = 3$.
(II) In the case of $a *_1 b = c \neq d = a *_2 b$. Then, $a \leq_1 c$, $b \leq_1 c$, $a \leq_2 d$ and $b \leq_2 d$. ¿From $[H]$ and $a \leq_1 c$, $a \leq_2 c$ or $c \leq_2 a$. Similarly, $b \leq_2 c$ or $c \leq_2 b$, $a \leq_1 d$ or $d \leq_1 a$ and $b \leq_1 d$ or $d \leq_1 b$. As a and b are not comparable, there are the only four cases. (If $a \leq_2 c$ and $c \leq_2 b$, then $a \leq_2 b$. It is a contradiction.)
(i) $a \leq_2 c$, $b \leq_2 c$, $a \leq_1 d$ and $b \leq_1 d$.
(ii) $a \leq_2 c$, $b \leq_2 c$, $d \leq_1 a$ and $d \leq_1 b$.
(iii) $c \leq_2 a$, $c \leq_2 b$, $a \leq_1 d$ and $b \leq_1 d$.
(iv) $c \leq_2 a$, $c \leq_2 b$, $d \leq_1 a$ and $d \leq_1 b$.
Since $a \leftrightarrow c$, $b \leftrightarrow c$, $a \leftrightarrow d$ and $b \leftrightarrow d$, $\{a,c\}$, $\{b,c\}$, $\{a,d\}$ and $\{b,d\}$ can not make a new element by $*_1$ and $*_2$.
When (ii) or (iv), $d \leq_1 c$ from $a \leq_1 c$ and $d \leq_1 a$. When (iii), $c \leq_2 d$ from $c \leq_2 a$ and $a \leq_2 d$. When (i), $a \leq_2 c$ and $b \leq_2 c$. Since $d = a *_2 b$, $d \leq_2 c$. Hence, $c \leftrightarrow d$. Then, $\{c,d\}$ can not make a new element by $*_1$ and $*_2$.
Therefore, $\gamma(a,b) = |\Gamma(*_1, *_2, a, b)| = |\{a,b,c,d\}| = 4$.

Let $(A, *_1, *_2)$ be a bisemilattice, a, $b \in A$ and let n denote a non negative integer. Then, $d(a,b)$ difined by

(i) $d(a,b) = 0$ if and only if $a = b$,
(ii) $d(a,b) \leq 1$ if $a \leq_1 b$ or $a \leq_2 b$,
(iii) $d(a,b) \leq n+1$ if there exist $c_1, c_2, ..., c_n \in A$ and $e_1, e_2, ..., e_{n+1} \in \{1,2\}$ such that $a \leq_{e_1} c_1 \leq_{e_2} c_2 \ ... \ c_{n-1} \leq_{e_n} c_n \leq_{e_{n+1}} b$,
(iv) $d(a,b) = n+1$ if and only if $d(a,b) \leq n+1$ and $d(a,b) \not\leq n$,
(v) $d(a,b) = \infty$ if there is no n such that $d(a,b) \leq n$.

$[J^-]$

$$\min(d(a,b), d(b,a)) \leq 2. \tag{23}$$

$[J]$

$$\max(d(a,b), d(b,a)) \leq 2. \tag{24}$$

The implication $[J] \Longrightarrow [J^-]$ is obvious.

Proposition 8 (4) or (5) $\Longrightarrow$ (24), *therefore* $[B] \Longrightarrow [J]$.

Proof We show that (4) $\Longrightarrow$ (24). For any $a, b \in A$, $a \leq_2 a *_2 b$ and $a *_2 b \leq_1 (a *_2 b) *_1 b$. ¿From (4), $(a *_2 b) *_1 b = b$. Hence, $d(a,b) \leq 2$. Similarly, $d(b,a) \leq 2$.

1.2 Relations between laws on bisemilattice. Relations between each law are shown by the figure 1. The arrows in the figure are the symbols of implication. There is an example that a difference is shown in the place where there is no arrow. (See the examples of next subsection.)

$$\begin{tikzcd}
 & {[H^-]} & & \\
 & {[H] \Leftrightarrow [H']} \arrow[u] & & \\
 & {[H^+]} \arrow[u] & & \\
{[F] \Leftrightarrow [I]} \arrow[d] & {[B] \Leftrightarrow [D] \Leftrightarrow [G]} \arrow[l] \arrow[u] \arrow[d] \arrow[r] & {[J]} \arrow[r] & {[J^-]} \\
{[C]} & {[A] \Leftrightarrow [E]} & &
\end{tikzcd}$$

Figure 1: Relations between laws on bisemilattice

1.3 Examples of bisemilattice. Two left-right figures show the same set. A little circle expresses an element. An element in the same position in the left-right figures is the same. We depict two orders in the set by arrows of figures. We suppose that arrowhead is larger than the other end.

$\leq_1$ $\leq_2$

Example $b-01$

$\leq_1$ $\leq_2$

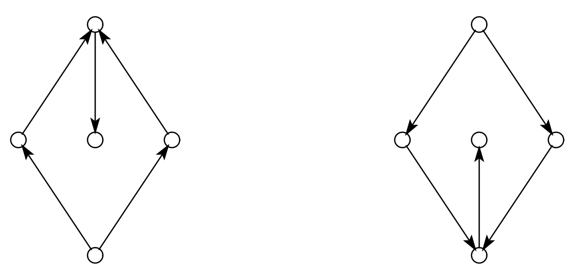
Example $b-02$

$\leq_1$ $\leq_2$

Example $b-03$

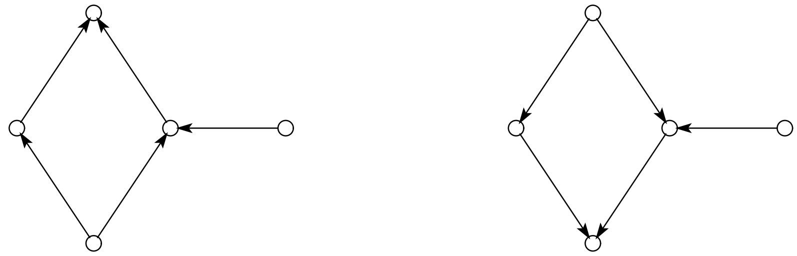
$\leq_1$ $\leq_2$

Example $b-04$



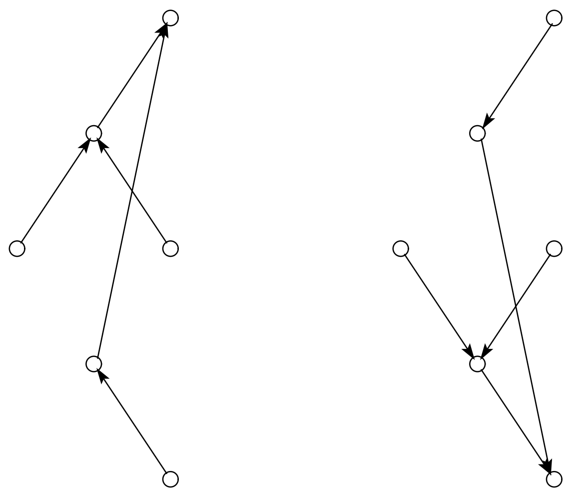
$\leq_1$ $\leq_2$

Example $b-05$



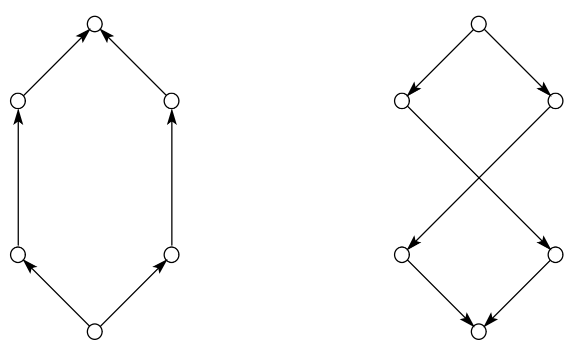
$\leq_1$ $\leq_2$

Example $b-06$



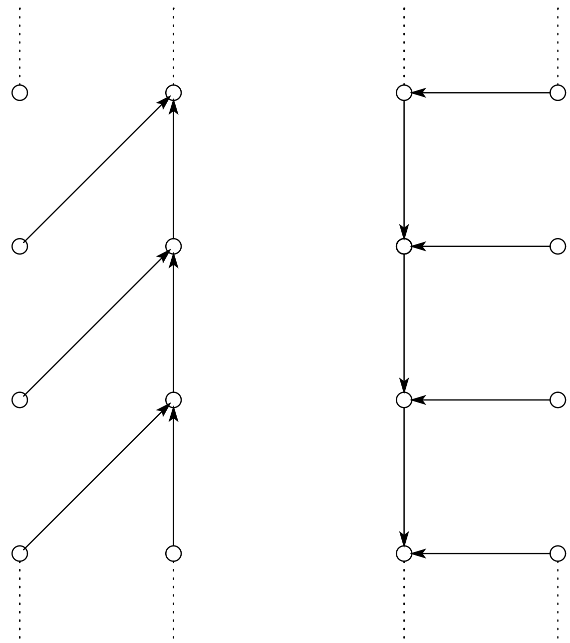
$\leq_1$ $\leq_2$

Example $b-07$



$\leq_1$ $\leq_2$

Example $b-08$



$\leq_1$ $\leq_2$

Example $b-09$

Example	$[B]$	$[A]$	$[C]$	$[F]$	$[H^-]$	$[H]$	$[H^+]$	$[J^-]$	$[J]$
$b-01$	×	×	◯	◯	◯	◯	×	×	×
$b-02$	×	◯	◯	◯	◯	×	×	×	×
$b-03$	×	×	×	×	◯	◯	◯	×	×
$b-04$	×	×	×	×	◯	×	×	◯	◯
$b-05$	×	×	◯	◯	◯	◯	◯	◯	×
$b-06$	×	×	◯	◯	×	×	×	◯	×
$b-07$	×	◯	×	×	×	×	×	×	×
$b-08$	×	◯	◯	×	◯	×	×	◯	◯
$b-09$	×	◯	◯	×	×	×	×	◯	◯

Table 1

1.4 Combination of weak laws on bisemilattice. We considered weaker law than the absorption law. It is shown here that the absorption law is led by that combination. !!

Proposition 9 $[A]$ and $[H]$ $\iff$ $[B]$.

Proof It is sufficient if $[A]$ and $[H] \Longrightarrow [B]$ to prove this proposition. Suppose that $[A]$ and $[H]$. Let $a \leq_1 b$ $(a \neq b)$. ¿From $[H]$, $a \leq_2 b$ or $b \leq_2 a$. If $a \leq_2 b$, then $a = b$ from $[A]$. It contradicts the assumption. Then, $b \leq_2 a$. The $[D]$ (that is, $[B]$) is satisfied.

Proposition 10 $[F]$ and $[J] \iff [B]$.

To prove this proposition, we will show the next lemma.

Lemma 1 $[F]$ and $[J] \Longrightarrow [A]$.

Proof Suppose that there is a bisemilattice A which is $[F]$ and $[J]$ but not $[A]$. Then, there exist $a, b \in A$ $(a \neq b)$ such that $a \leq_1 b$ and $a \leq_2 b$.
If $d(b,a) = 1$, then $b \leq_1 a$ or $b \leq_2 a$, hence $a = b$. It contradicts $a \neq b$.
If $d(b,a) = 2$, there exist c $(\neq a, \neq b)$ such that $b \leq_1 c \leq_2 a$ (case 1), or there exist c $(\neq a, \neq b)$ such that $b \leq_2 c \leq_1 a$ (case 2). In (case 1), $a \leq_1 b \leq_1 c$ and $c \leq_2 a \leq_2 b$. For $b \leq_1 c$, $b *_2 a = b \not\leq_1 a = c *_2 a$. It contradicts $[F]$. In (case 2), $a \leq_2 b \leq_2 c$ and $c \leq_1 a \leq_1 b$. For $b \leq_2 c$, $b *_1 a = b \not\leq_2 a = c *_1 a$. It contradicts $[F]$.
Hence, $d(b,a) \not\leq 2$. It contradicts $[J]$. Therefore, $[F]$ and $[J] \Longrightarrow [A]$.

Proof of Proposition 10 Suppose that there is a bisemilattice A which is $[F]$ and $[J]$ but not $[D]$ (that is, not $[B]$). Then, there exist $a, b \in A (a \neq b)$ such that $a \leq_1 b$ and $b \not\leq_2 a$.
If $a \leq_2 b$, then $a = b$ from the above lemma. It contradicts the supposition. Since $a \not\leq_2 b$ and $b \not\leq_2 a$, there is a d $(\neq a, \neq b) \in A$ such that $a *_2 b = d$. Then, $a \leq_2 d$. ¿From $a \leq_1 b$ and $[F]$, $a = a *_2 a \leq_1 b *_2 a = d$. Hence, $a \leq_1 d$. Since $[A]$, $a \leq_2 d$ and $a \leq_1 d$ indicates $a = d$. It contradicts the supposition.
Therefore, $[F]$ and $[J] \Longrightarrow [D]$ $(= [B])$. The converse is clear.

We notice following fact. Example $b - 08$ is $[A]$ and $[H^-]$ but not $[B]$. Example $b - 05$ is $[F]$ and $[J^-]$ (and $[H^+]$) but not $[B]$. Example $b - 02$ is $[A]$ and $[F]$ but not $[B]$. And Example $b - 09$ is $[C]$ and $[J]$ but not $[B]$.

2 On triple-semilattice

2.1 Some laws on triple-semilattice. A triple-semilattice $(T, *_1, *_2, *_3)$ is an algebra which has three semilattice operations. That is, $(T, *_1)$, $(T, *_2)$ and $(T, *_3)$ are semilattices, respectively！%We construct three ordered sets $(T, \leq_1)$, $(T, \leq_2)$ and $(T, \leq_3)$. In comparison with bisemilattice, there is much number of the equation. We have the way that we save paper (See [3]). However, we enumerate many equations to write relations with other law precisely. Next properties are adaptation from $[B]$, $[G]$ and $[D]$.

$[*B]$ if and only if the following six conditions hold:

(25) $$((a *_1 b) *_2 b) *_3 b = b$$
(26) $$((a *_1 b) *_3 b) *_2 b = b$$
(27) $$((a *_2 b) *_1 b) *_3 b = b$$
(28) $$((a *_2 b) *_3 b) *_1 b = b$$
(29) $$((a *_3 b) *_1 b) *_2 b = b$$
(30) $$((a *_3 b) *_2 b) *_1 b = b$$

$[*G]$ if and only if the following three conditions hold:

(31) $$a *_1 c \ \leq_2 \ a \ \text{ or } \ a *_1 c \ \leq_3 \ a$$

(32) $$a *_2 c \ \leq_3 \ a \ \text{ or } \ a *_2 c \ \leq_1 \ a$$

(33) $$a *_3 c \ \leq_1 \ a \ \text{ or } \ a *_3 c \ \leq_2 \ a$$

$[*D]$ if and only if the following three conditions hold:

(34) $$a \leq_1 b \implies b \leq_2 a \ \text{ or } \ b \leq_3 a$$

(35) $$a \leq_2 b \implies b \leq_3 a \ \text{ or } \ b \leq_1 a$$

(36) $$a \leq_3 b \implies b \leq_1 a \ \text{ or } \ b \leq_2 a$$

Proposition 11 (31) $\iff$ (34), (32) $\iff$ (35) *and* (33) $\iff$ (36), *therefore* $[*G] \iff [*D]$.

Proof We prove (31) $\Longrightarrow$ (34). Let $a \leq_1 b$. Then, $b = a *_1 b \leq_2 a$ or $b = a *_1 b \leq_3 a$ by (31). Conversely, we prove (34) $\Longrightarrow$ (31). ¿From $a \leq_1 a *_1 c$, we obtain $a *_1 c \leq_2 a$ or $a *_1 c \leq_3 a$ by (34). Hence, (31) $\iff$ (34). Similarly, (32) $\iff$ (35) and (33) $\iff$ (36).

However, $[*B] \neq [*D]$. (See Examaple $t-03$, $t-05$.) There are important simple examples, which are not $[*D]$ but $[*B]$. (See Examaple $t-12$, $t-13$.) As we consider that $[*B]$ is more important than $[*D]$, we named the laws $[*B]$ **roundabout-absorption laws** in [3].

In the same way as the case of bisemilattice, a pair of a and b is said to be **comparable** if any one of following six relations is satisfied.

(37) $$a \leq_1 b, \quad a \leq_2 b, \quad a \leq_3 b, \quad b \leq_1 a, \quad b \leq_2 a, \quad b \leq_3 a$$

We denote $a \leftrightarrow b$ if a and b are comparable, and $a \perp b$ if a and b are not comparable.

$[*D^+]$ if one of following six relation is satisfied for any comparable pair of a and b ($a \leftrightarrow b$):

(38) $$a \leq_1 b \ \text{ and } \ b \leq_2 a \ \text{ and } \ b \leq_3 a$$

(39) $$a \leq_2 b \ \text{ and } \ b \leq_3 a \ \text{ and } \ b \leq_1 a$$

(40) $$a \leq_3 b \ \text{ and } \ b \leq_1 a \ \text{ and } \ b \leq_2 a$$

(41) $$b \leq_1 a \ \text{ and } \ a \leq_2 b \ \text{ and } \ a \leq_3 b$$

(42) $$b \leq_2 a \ \text{ and } \ a \leq_3 b \ \text{ and } \ a \leq_1 b$$

(43) $$b \leq_3 a \ \text{ and } \ a \leq_1 b \ \text{ and } \ a \leq_2 b$$

$[*D^-]$ if and only if the following six conditions hold:

(44) $$a \leq_1 b \implies \exists c \in T \ s.t. \ b \leq_2 c \leq_3 a$$

(45) $$a \leq_1 b \implies \exists c \in T \ s.t. \ b \leq_3 c \leq_2 a$$

(46) $$a \leq_2 b \implies \exists c \in T \ s.t. \ b \leq_1 c \leq_3 a$$

(47) $$a \leq_2 b \implies \exists c \in T \ s.t. \ b \leq_3 c \leq_1 a$$

(48) $$a \leq_3 b \implies \exists c \in T \ s.t. \ b \leq_1 c \leq_2 a$$

(49) $$a \leq_3 b \implies \exists c \in T \ s.t. \ b \leq_2 c \leq_1 a$$

We named the $[*D^-]$ **return laws** (or **returnable**). Now, the implication $[*D^+] \implies [*D] \implies [*D^-]$ and $[*D^+] \implies [*B]$ are obvious.

Proposition 12 *(25) $\implies$ (44), (26) $\implies$ (45), (27) $\implies$ (46), (28) $\implies$ (47), (29) $\implies$ (48) and (30) $\implies$ (49), therefore $[*B] \implies [*D^-]$.*

Proof We prove (25) $\implies$ (44). Let T be a triple-semilattice with the property (25). Suppose that $a \leq_1 b$. Then, let $c = a *_2 b$. Clearly, $b \leq_2 c$. On the other hand,

$$\begin{aligned} c *_3 a &= (a *_2 b) *_3 a \\ &= (a *_2 (a *_1 b)) *_3 a \quad [a \leq_1 b \Leftrightarrow a *_1 b = b] \\ &= ((b *_1 a) *_2 a) *_3 a \\ &= a \end{aligned}$$

Hence, $c \leq_3 a$. Other proofs are the similar.

We adapted other laws on bisemilattice so that we can use them on triple-semilattice.

$[*A]$

$$a \leq_1 b \text{ and } a \leq_2 b \text{ and } a \leq_2 b \implies a = b \tag{50}$$

$[*E]$

$$a *_1 b = a *_2 b = a *_3 b \implies a = b \tag{51}$$

Proposition 13 $[*A] \iff [*E]$.

Proof This proof is the simulation of Proposition 2. To prove that $[*A] \implies [*E]$, let $a *_1 b = a *_2 b = a *_3 b = c$. ¿From $a \leq_1 c$, $a \leq_2 c$ and $a \leq_3 c$, we obtain $a = c$. ¿From $b \leq_1 c$, $b \leq_2 c$ and $b \leq_3 c$, we obtain $b = c$. Hence $a = c = b$. Conversely, let $a \leq_1 b$, $a \leq_2 b$ and $a \leq_3 b$. Then, $a *_1 b = a *_2 b = a *_3 b = b$. If $[*E]$ is satisfied, we obtain $a = b$.

Proposition 14 $[*B] \implies [*A]$.

Proof This proof is the simulation of Proposition 3. Let $a \leq_1 b$, $a \leq_2 b$ and $a \leq_3 b$. Then, $((a *_1 b) *_2 b) *_3 b = a$. ¿From (25), $((a *_1 b) *_2 b) *_3 b = b$. Hence $a = b$.

The implication $[*D] \implies [*A]$ is obvious.

$[*F]$ if and only if the following six conditions hold:

$$a \leq_1 b \implies a *_2 c \leq_1 b *_2 c \tag{52}$$
$$a \leq_1 b \implies a *_3 c \leq_1 b *_3 c \tag{53}$$
$$a \leq_2 b \implies a *_1 c \leq_2 b *_1 c \tag{54}$$
$$a \leq_2 b \implies a *_3 c \leq_2 b *_3 c \tag{55}$$
$$a \leq_3 b \implies a *_1 c \leq_3 b *_1 c \tag{56}$$
$$a \leq_3 b \implies a *_2 c \leq_3 b *_2 c \tag{57}$$

$[*I]$ if and only if the following six conditions hold:

$$a \leq_1 b \text{ and } c \leq_1 d \implies a *_2 c \leq_1 b *_2 d \tag{58}$$
$$a \leq_1 b \text{ and } c \leq_1 d \implies a *_3 c \leq_1 b *_3 d \tag{59}$$
$$a \leq_2 b \text{ and } c \leq_2 d \implies a *_1 c \leq_2 b *_1 d \tag{60}$$
$$a \leq_2 b \text{ and } c \leq_2 d \implies a *_3 c \leq_2 b *_3 d \tag{61}$$
$$a \leq_3 b \text{ and } c \leq_3 d \implies a *_1 c \leq_3 b *_1 d \tag{62}$$
$$a \leq_3 b \text{ and } c \leq_3 d \implies a *_2 c \leq_3 b *_2 d \tag{63}$$

$[*C]$ if and only if the following six conditions hold:

$$a *_2 b \leq_1 a *_1 b \tag{64}$$
$$a *_3 b \leq_1 a *_1 b \tag{65}$$
$$a *_1 b \leq_2 a *_2 b \tag{66}$$
$$a *_3 b \leq_2 a *_2 b \tag{67}$$
$$a *_1 b \leq_3 a *_3 b \tag{68}$$
$$a *_2 b \leq_3 a *_3 b \tag{69}$$

Proposition 15 (52) $\iff$ (58), (53) $\iff$ (59), (54) $\iff$ (60), (55) $\iff$ (61), (56) $\iff$ (62) *and* (57) $\iff$ (63), *therefore* $[*F] \iff [*I]$.

Proof This proof is the same way of Proposition 4.

Proposition 16 (52) $\implies$ (64), (53) $\implies$ (65), (54) $\implies$ (66), (55) $\implies$ (67), (56) $\implies$ (68) *and* (57) $\implies$ (69), *therefore* $[*F] \implies [*C]$.

Proof This proof is the same way of Proposition 5.

On bisemilattce, the absorption imply the $[F]$ form Proposition 6. However, $[*B] \not\Longrightarrow [*C]$. Moreover, $[*D^+] \not\Longrightarrow [*C]$. (See Examaple $t-06$.)

Let $(T, *_1, *_2, *_3)$ be a triple-semilattice, $a,\ b \in T$ and let n denote a non-negative integer. Then, $d(a,b)$ difined by

(i) $d(a,b) = 0$ if and only if $a = b$,

(ii) $d(a,b) \leq 1$ if $a \leq_1 b$ or $a \leq_2 b$ or $a \leq_3 b$,

(iii) $d(a,b) \leq n+1$ if there exist $c_1, c_2, ..., c_n \in T$ and $e_1, e_2, ..., e_{n+1} \in \{1,2,3\}$ such that $a \leq_{e_1} c_1 \leq_{e_2} c_2 \ ...\ c_{n-1} \leq_{e_n} c_n \leq_{e_{n+1}} b$,

(iv) $d(a,b) = n+1$ if and only if $d(a,b) \leq n+1$ and $d(a,b) \not\leq n$,

(v) $d(a,b) = \infty$ if there is no n such that $d(a,b) \leq n$.

$[*J^-]$ if and only if the following condition hold:

$$\min(d(a,b), d(b,a)) \leq 3. \tag{70}$$

$[*J]$ if and only if the following condition hold:

$$\max(d(a,b), d(b,a)) \leq 3. \tag{71}$$

$[*J^{+}]$ if and only if the following six conditions hold:

(72) $$\forall a,b \in T\ \exists c,d \in T\ s.t.\ a \leq_1 c \leq_2 d \leq_3 b$$

(73) $$\forall a,b \in T\ \exists c,d \in T\ s.t.\ a \leq_1 c \leq_3 d \leq_2 b$$

(74) $$\forall a,b \in T\ \exists c,d \in T\ s.t.\ a \leq_2 c \leq_1 d \leq_3 b$$

(75) $$\forall a,b \in T\ \exists c,d \in T\ s.t.\ a \leq_2 c \leq_3 d \leq_1 b$$

(76) $$\forall a,b \in T\ \exists c,d \in T\ s.t.\ a \leq_3 c \leq_1 d \leq_2 b$$

(77) $$\forall a,b \in T\ \exists c,d \in T\ s.t.\ a \leq_3 c \leq_2 d \leq_1 b$$

Then, $[*D^{-}] \implies [*J^{+}] \implies [*J] \implies [*J^{-}]$ is obvious. We call the $[*J^{+}]$ **attainment laws** (or **attainable**).

For a triple-semilattice $(T, *_1, *_2, *_3)$ and $a,\ b \in T$, the set $\Gamma(*_1, *_2, *_3, a, b)$ is the smallest set satisfying (i) and (ii):

(i) $a, b \in \Gamma(*_1, *_2, *_3, a, b)$,

(ii) If $x, y \in \Gamma(*_1, *_2, *_3, a, b)$, then $x *_1 y,\ x *_2 y,\ x *_3 y \in \Gamma(*_1, *_2, *_3, a, b)$.

The $\gamma(a, b)$ is the number of $\Gamma(*_1, *_2, *_3, a, b)$, that is, $\gamma(a, b) = |\Gamma(*_1, *_2, *_3, a, b)|$.

For $n \geq 2 (n \in \mathbf{N})$,

$[*H_n]$ if and only if the following condition hold:

(78) $$a \leftrightarrow b \implies \gamma(a, b) \leq n$$

$[*H_n^{-}]$ if and only if the following condition hold:

(79) $$\gamma(a, b) \leq n \quad (\forall a,\ b \in T)$$

Then, $[*H_n] \implies [*H_{n+1}]$ and $[*H_n^{-}] \implies [*H_{n+1}^{-}]$ are obvious. And it is obvious that $[*D^{+}] \implies [*H_2]$. But there is an example with $[*D]$ but not $[*H_n]$ for any $n \in \mathbf{N}$. (See Example $t-16$.) If T be a triple-semilattice with $[*B]$, the family of all binary trice polynomial functions $P^{(2)}(T)$ on T is a triple-semilattice with $[*B]$. If T is Example $t-02$, $P^{(2)}(T)$ is $[*B]$ and $[*H_{18}^{-}]$ (not $[*H_{17}^{-}]$). If T is Example $t-01$, $P^{(2)}(T)$ is $[*B]$ and $[*H_{729}^{-}]$ (not $[*H_{728}^{-}]$). (See [5].) At first sight, the $[*H_2]$ is to triple-semilattice as is the $[H]$ is to bisemilattice. And, the $[*H_4^{-}]$ is to triple-semilattice as is the $[H^{-}]$ is to bisemilattice. ¿From Proposition 7, if a bisemilattice has the property $[H]$ then it has the property $[H^{-}]$. However, there is a triple-semilattice which is not $[*H_4^{-}]$ but $[*H_2]$. (See Example $t-11$.)

Proposition 17 $[*H_2] \implies [*H_5^{-}]$.

Proof Suppose that the triple-semilattice T is satisfied with $[*H_2]$. Let $a \perp b$ $(a, b \in T)$. Let $a *_1 b = c$, $a *_2 b = d$, $a *_3 b = e$ and $c \neq d \neq e \neq c$. (As another cases are simpler, we omit the proof.)
¿From $a *_1 b = c$, $a \leq_1 c$. that is, $a \leftrightarrow c$. If $x \leftrightarrow y$ on the condition $[*H_2]$, then $x *_1 y = x$ or y, $x *_2 y = x$ or y and $x *_3 y = x$ or y. Hence, $\{a, c\}$ can not make a new element by $*_1$, $*_2$ and $*_3$. Similarly, $\{a, d\}$, $\{a, e\}$, $\{b, c\}$, $\{b, d\}$ and $\{b, e\}$ can not make a

new element by $*_1$, $*_2$ and $*_3$.
¿From $a *_2 b = d$, $a \leq_2 d$. ¿From $[*H_2]$, $a \leq_1 d$ or $d \leq_1 a$. Similarly, $b \leq_1 d$ or $d \leq_1 b$. (I) When $a \leq_1 d$, if $d \leq_1 b$ then $a *_1 b = b$, it contradicts $a \perp b$. Thus, $b \leq_1 d$. Hence, $c = a *_1 b \leq_1 d$. (II) When $d \leq_1 a$, from $a \leq_1 c$, $d \leq_1 c$. In either case (I) and (II), we obtain $c \leftrightarrow d$. Hence, $\{c, d\}$ can not make a new element by $*_1$, $*_2$ and $*_3$. Similarly, $\{c, d\}$ $\{d, e\}$ and $\{e, c\}$ can not make a new element by $*_1$, $*_2$ and $*_3$.
Therefore, $\{a, b, c, d, e\}$ can not make a new element by $*_1, *_2, *_3$.

2.2 Relations between laws on triple-semilattice. Relations between each law are shown by the figure 2. As we know $[*A] = [*E]$, $[*D] = [*G]$ and $[*F] = [*I]$, we omit $[*E]$, $[*G]$ and $[*I]$.

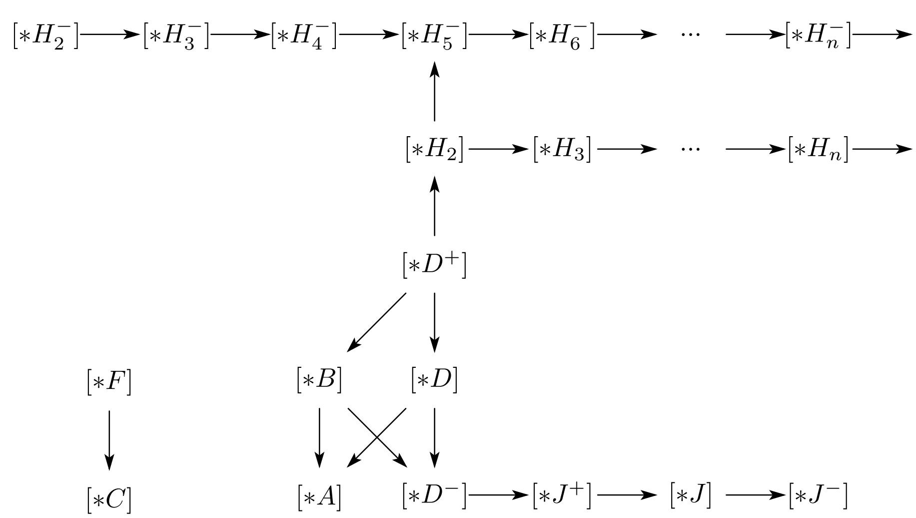
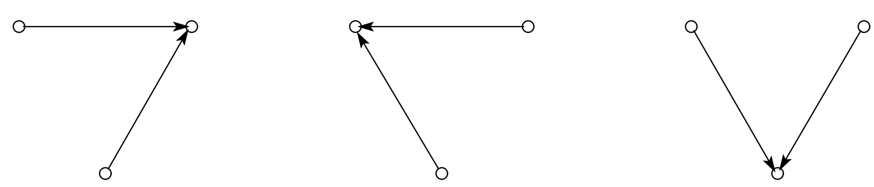


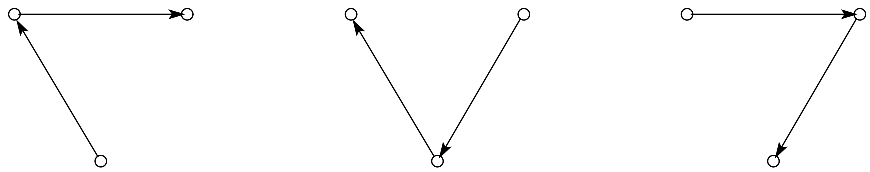
Figure 2: Relations between laws on triple-semilattice

2.3 Examples of triple-semilattice. Three figures show the same set. A little circle expresses an element. An element in the same position in figures is the same. We depict three orders in the set by arrows of figures. We suppose that arrowhead is larger than the other end.



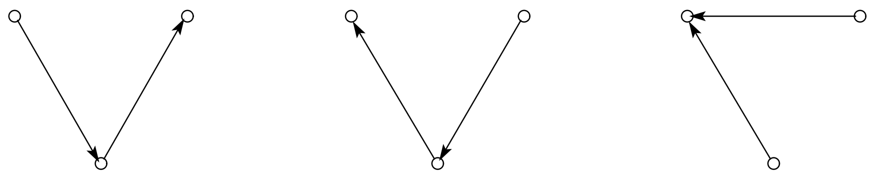
$\leq_1$ $\leq_2$ $\leq_3$

Example $t-01$ (trianglular situation)



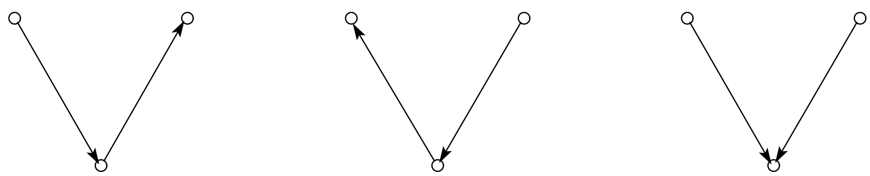
$\leq_1$ $\leq_2$ $\leq_3$

Example $t-02$ (trianglular circuit)



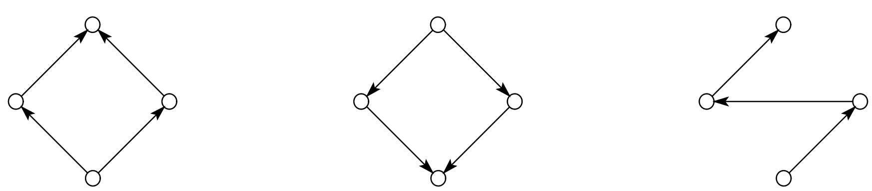
$\leq_1$ $\leq_2$ $\leq_3$

Example $t-03$



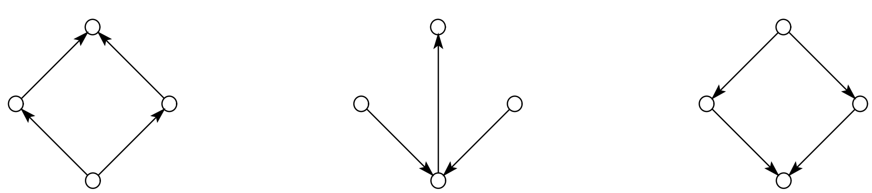
$\leq_1$ $\leq_2$ $\leq_3$

Example $t-04$



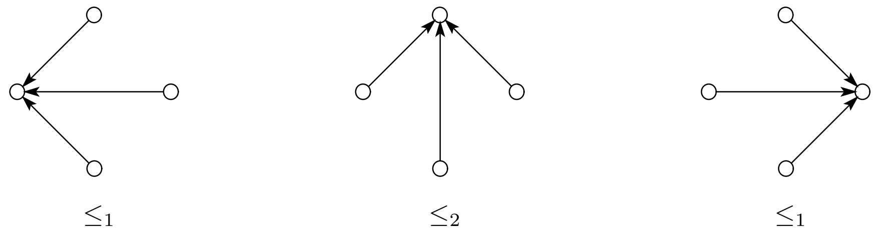
$\leq_1$ $\leq_2$ $\leq_3$

Example $t-05$

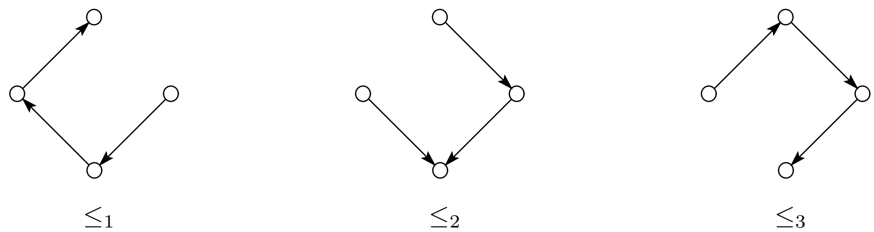


$\leq_1$ $\leq_2$ $\leq_1$

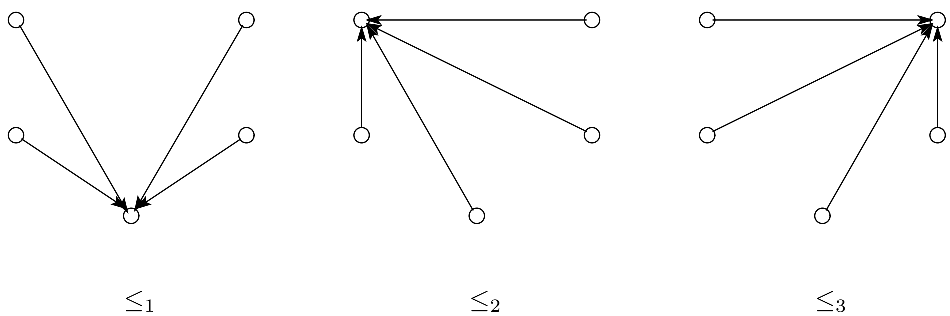
Example $t-06$



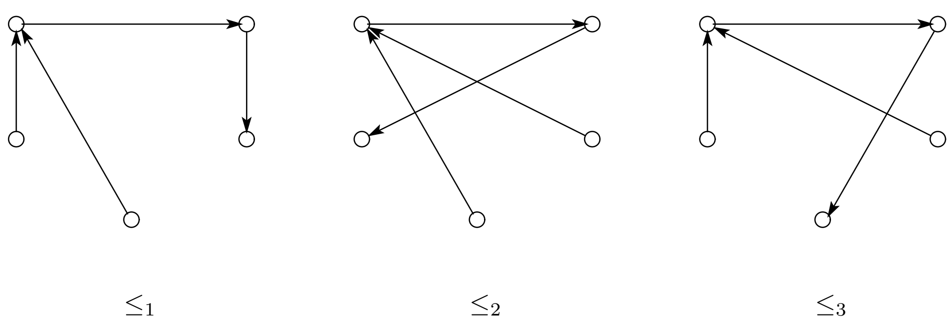
Example $t - 07$



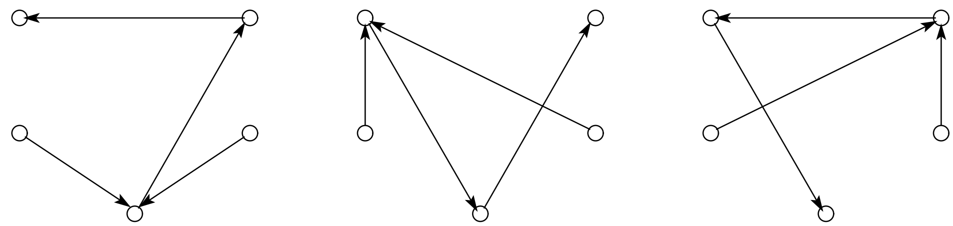
Example $t - 08$



Example $t - 09$

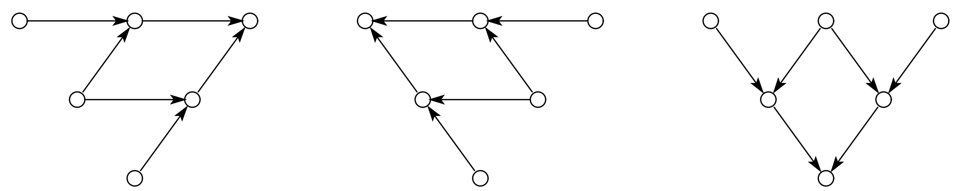


Example $t - 10$



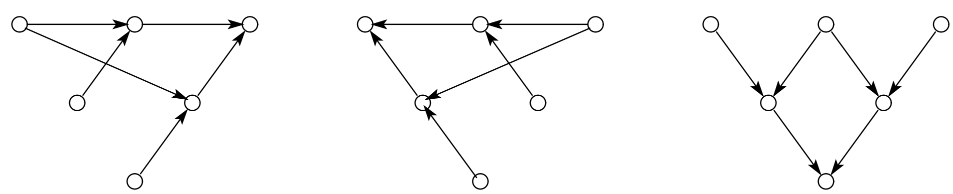
$\leq_1$ $\leq_2$ $\leq_3$

Example $t-11$



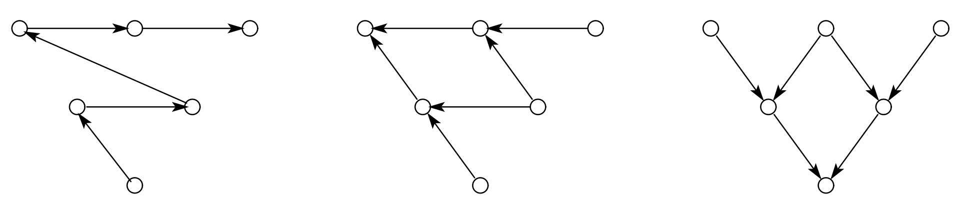
$\leq_1$ $\leq_2$ $\leq_3$

Example $t-12$



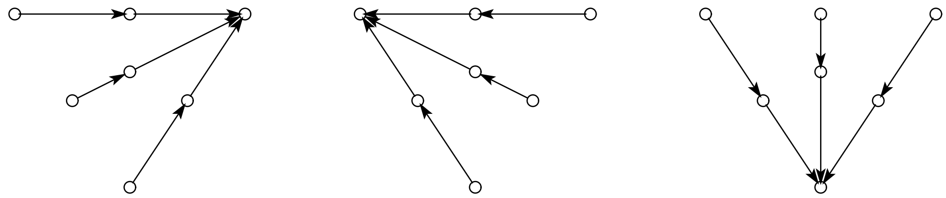
$\leq_1$ $\leq_2$ $\leq_3$

Example $t-13$



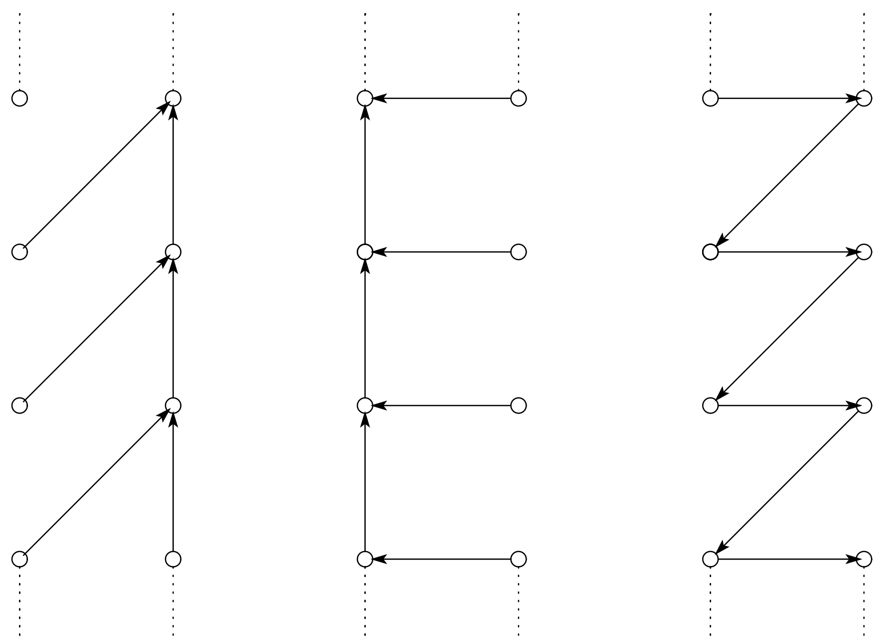
$\leq_1$ $\leq_2$ $\leq_3$

Example $t-14$



$\leq_1$ $\leq_2$ $\leq_3$

Example $t - 15$



$\leq_1$ $\leq_2$ $\leq_3$

Example $t - 16$

Example	$[*F]$	$[*C]$	$[*D^+]$	$[*B]$	$[*A]$	$[*D]$	$[*D^-]$	$[*J^+]$	$[*J]$	$[*J^-]$	$[*H_n]$	$[*H_n^-]$
$t-01$	×	◯	×	◯	◯	◯	◯	◯	◯	◯	n=2	n=3
$t-02$	×	◯	◯	◯	◯	◯	◯	◯	◯	◯	2	2
$t-03$	×	×	×	×	◯	◯	◯	◯	◯	◯	3	3
$t-04$	◯	◯	×	◯	◯	◯	◯	◯	◯	◯	3	3
$t-05$	×	×	×	◯	◯	×	◯	◯	◯	◯	4	4
$t-06$	×	×	◯	◯	◯	◯	◯	◯	◯	◯	2	4
$t-07$	×	◯	×	×	◯	×	×	×	×	◯	4	4
$t-08$	×	×	×	×	×	×	×	×	◯	◯	2	2
$t-09$	×	◯	×	×	◯	×	×	×	×	×	4	4
$t-10$	×	◯	×	×	×	×	◯	◯	◯	◯	3	3
$t-11$	×	×	×	×	×	×	×	×	×	◯	2	5
$t-12$	×	◯	×	◯	◯	×	◯	◯	◯	◯	6	6
$t-13$	×	◯	×	◯	◯	×	◯	◯	◯	◯	6	6
$t-14$	×	◯	×	×	◯	×	×	◯	◯	◯	4	4
$t-15$	×	×	×	×	◯	×	◯	◯	◯	◯	5	5
$t-16$	×	×	×	×	◯	◯	◯	◯	◯	◯	∞	∞

Table 2

2.4 Combination of weak laws on triple-semilattice. On bisemilattice, $[A] + [F] \not\Longrightarrow [B]$. (See Examaple $b-02$.) We can observe that $[*A] + [*F] \Longrightarrow [*B]$. But $[*A] + [*C] \not\Longrightarrow [*B]$. (See Examaple $t-07$ etc.)

Proposition 18 $[*A]$ and $[*F] \Longrightarrow [*B]$.

Proof It is clear that $b \leq_1 a *_1 b$. ¿From (52), $b \leq_1 (a *_1 b) *_2 b$. ¿From (53), $b \leq_1 ((a *_1 b) *_2 b) *_3 b$. It is clear that $b \leq_2 (a *_1 b) *_2 b$. ¿From (55), $b \leq_2 ((a *_1 b) *_2 b) *_3 b$. It is clear that $b \leq_3 ((a *_1 b) *_2 b) *_3 b$. Since $[*A]$, $b = ((a *_1 b) *_2 b) *_3 b$. This is, (25). Other formulae can be proved in the same way.

The next proposition can be cope with Proposition 9.

Proposition 19 $[*H_2]$ and $[*A] \Longleftrightarrow [*D^+]$.

The proof of this proposition is clear from the definition.

The $[*D^+]$ may be thought good property. But, triple-semilattices which actually satisfied $[*D^+]$ were limited. Example $t-01$, $t-04$ and $t-12$, which are basic and important, are not $[*D^+]$.

Lemma 2 *If a triple-semilattice $(T, *_1, *_2, *_3)$ has $[*F]$ and $[*D^+]$, there is not a trio $a, b, c \in T$ $(a \neq b \neq c \neq a)$ such that $a \leq_1 b \leq_1 c$ and $b \leq_2 a \leq_2 c$.*

Proof Suppose that there exists a trio $a, b, c \in T$ $(a \neq b \neq c \neq a)$ such that $a \leq_1 b \leq_1 c$ and $b \leq_2 a \leq_2 c$. ¿From $a \leq_1 c$ and $a \leq_2 c$, $c \leq_3 a$ by $[*D^+]$. ¿From $b \leq_1 c$ and $b \leq_2 c$, $c \leq_3 b$ by $[*D^+]$. Now, from $a \leftrightarrow b$, $a \leq_3 b$ or $b \leq_3 a$ by $[*D^+]$. (I) If $b \leq_3 a$, $a \leq_2 c$ and $a *_3 b = a \not\leq_2 b = c *_3 b$. (II) If $a \leq_3 b$, $b \leq_1 c$ and $b *_3 a = b \not\leq_1 a = c *_3 a$. In either case (I) and (II), it contradicts $[*F]$.

Lemma 3 *If a triple-semilattice $(T, *_1, *_2, *_3)$ has $[*F]$ and $[*D^+]$,*

$$\begin{aligned}
&(1) \quad a \leq_1 b \leq_1 c \text{ and } a \leq_2 b \implies b \leq_2 c \\
&(2) \quad a \perp b,\ a \leq_1 c,\ b \leq_1 c \text{ and } a \leq_2 c \implies b \leq_2 c \\
&(3) \quad a \leq_1 b \leq_1 c \text{ and } b \leq_2 c \implies a \leq_2 b \\
&(4) \quad a \leq_1 b \leq_1 c \text{ and } a \leq_2 c \implies b \leq_2 c.
\end{aligned}$$

Proof *(1)* Since $b \leq_1 c$ and $[*D^+]$, $b \leq_2 c$ or $c \leq_2 b$. Since $a \leq_1 c$ and $[*D^+]$, $a \leq_2 c$ or $c \leq_2 a$. The next three cases can be presumed from $a \leq_2 b$.

(i) $a \leq_2 b \leq_2 c$ (ii) $a \leq_2 c \leq_2 b$ (iii) $c \leq_2 a \leq_2 b$.

In the case (ii), $a \leq_1 b$ and $a *_2 c = c \not\leq_1 b = b *_2 c$. It contradicts $[*F]$. In the case (iii), $b \leq_1 c$ and $b *_2 a = b \not\leq_1 a = c *_2 a$. It contradicts $[*F]$. Hence, we obtain the case (i) $a \leq_2 b \leq_2 c$.

(2) Since $b \leq_1 c$ and $[*D^+]$, $b \leq_2 c$ or $c \leq_2 b$. If $c \leq_2 b$, $a \leq_2 c \leq_2 b$. That is, $a \leftrightarrow b$. It contradicts $a \perp b$. Hence, $b \leq_2 c$.

(3) Since $a \leq_1 b$ and $[*D^+]$, $a \leq_2 b$ or $b \leq_2 a$. Since $a \leq_1 c$ and $[*D^+]$, $a \leq_2 c$ or $c \leq_2 a$. The next three cases can be presumed from $b \leq_2 c$.

(i) $a \leq_2 b \leq_2 c$ (ii) $b \leq_2 c \leq_2 a$ (iii) $b \leq_2 a \leq_2 c$.

In the case (ii), $c \leq_2 a$ and $c *_1 b = c \not\leq_2 b = a *_1 b$. It contradicts $[*F]$. The case (iii) is impossible, because the Lemma 2. Hence, we obtain the case (i) $a \leq_2 b \leq_2 c$.

(4) Since $b \leq_1 c$ and $[*D^+]$, $b \leq_2 c$ or $c \leq_2 b$. If $c \leq_2 b$, $a \leq_2 c$ and $a *_1 b = b \not\leq_2 c = c *_1 b$. It contradicts $[*F]$. Hence, $b \leq_2 c$. Moreover, since the Lemma 2, we obtain $a \leq_2 b \leq_2 c$.

Proposition 20 *If a triple-semilattice $(T, *_1, *_2, *_3)$ has $[*F]$ and $[*D^+]$, two out of three operations $*_1, *_2, *_3$ are equal.*

Proof We can assume that there exists a pair a and b such that $a \leq_1 b$, $a \leq_2 b$ and $b \leq_3 a$. We show "$x \leq_1 y \iff x \leq_2 y$" on the assumption that $a \leq_1 b$ and $a \leq_2 b$. Let $a \leq_1 b$, $a \leq_2 b$ and $x \leq_1 y$.

(I) In the case of $b \perp y$, let $b *_1 y = z$. Then, $b \leq_1 z$. That is, $a \leq_1 b \leq_1 z$ and $a \leq_2 b$. Since lemma 3(*1*), $b \leq_2 z$. Now, $b \perp y$, $b \leq_1 z$, $b \leq_2 z$ and $y \leq_1 z$. Since lemma 3(*2*), $y \leq_2 z$. Then, $x \leq_1 y \leq_1 z$ and $y \leq_2 z$. Since lemma 3(*3*), we obtain $x \leq_2 y$.

(II) In the case of $b \leq_1 y$, $a \leq_1 b \leq_1 y$ and $a \leq_2 b$. Since lemma 3(*1*), $b \leq_2 y$.

(i) If $x \perp b$, then $b \leq_1 y$, $x \leq_1 y$ and $b \leq_2 y$. Since lemma 3(*2*), $x \leq_2 y$.

(ii) If $b \leq_1 x$, then $b \leq_1 x \leq_1 y$ and $b \leq_2 y$. Since lemma 3(*4*), $x \leq_2 y$.

(iii) If $x \leq_1 b$, then $x \leq_1 b \leq_1 y$ and $b \leq_2 y$. Since lemma 3(*3*), $x \leq_2 b \leq_2 y$.

Hence, in the case (II), we obtain $x \leq_2 y$.

(III) In the case of $y \leq_1 b$.

(i) If $a \perp y$, then $a \leq_1 b$, $y \leq_1 b$ and $a \leq_2 b$. Since lemma 3(*2*), $y \leq_2 b$. Then, $x \leq_1 y \leq_1 b$ and $y \leq_2 b$. Since lemma 3(*3*), $x \leq_2 y$.

(ii) If $a \leq_1 y$, then $a \leq_1 y \leq_1 b$ and $a \leq_2 b$. Since lemma 3(*4*), $y \leq_2 b$. Then $x \leq_1 y \leq_1 b$, $y \leq_2 b$. Since lemma 3(*3*), $x \leq_2 y$.

(iii) If $y \leq_1 a$, then $y \leq_1 a \leq_1 b$ and $a \leq_2 b$. Since lemma 3(*3*), $y \leq_2 a$. Then $x \leq_1 y \leq_1 a$, $y \leq_2 a$. Since lemma 3(*3*), $x \leq_2 y$.

Hence, in the case (III), we obtain $x \leq_2 y$.

Therefore, $x \leq_1 y \Longrightarrow x \leq_2 y$ on the assumption that $a \leq_1 b$ and $a \leq_2 b$.

Similarly, $x \leq_2 y \Longrightarrow x \leq_1 y$. The proof of Proposition 20 is completed.

References

[1] G. Birkhoff. *Lattice Theory (third ed.)* Amer. Math. Soc. Colloq. Publ., 1967.

[2] G. Grätzer. *General Lattice Theory (Second ed.)* Birkhäuser, 1998.

[3] K. Horiuchi. Trice and Two delegates operation. Scientiae Mathematicae, 2-3 (1999)373–384.

[4] K. Horiuchi, A. Tepavčević. On distributive trices. Discussiones Mathematicae General Algebra and Applications, 21(2001)21–29.

[5] K. Horiuchi. Some polynomials on trice. to appear.

Faculty of Science and Engineering, Konan University
Okamoto, Higashinada, Kobe 658-8501, Japan