SUM THEOREMS FOR C-SPACES

Chieko Komoda

Received October 30, 2002

Abstract.

Let X be a space with a hereditarily closure-preserving closed cover $\mathcal{F}$ consisting of C-spaces. In this paper we prove the following: (1) if X is paracompact, then X is a C-space, (2) if X is hereditarily collectionwise normal, then X is a C-space.

1 Introduction

In this paper we assume that all spaces are normal unless otherwise stated. We refer the readers to [3] for dimension theory.

If $\mathcal{A}$ and $\mathcal{B}$ are collections of subsets of a space X, then $\mathcal{A} < \mathcal{B}$ means that $\mathcal{A}$ is a refinement of $\mathcal{B}$, i.e. for every $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ such that $A \subset B$. Notice that $\mathcal{A}$ need not be a cover even if $\mathcal{B}$ is a cover.

Haver [7] introduced the notion of C-spaces for the class of metric spaces. Addis and Gresham [1] extended this notion to normal spaces. A space X is a *C-space* if for every countable collection $\{\mathcal{G}_i : i \in \mathbb{N}\}$ of open covers of X there exists a countable collection $\{\mathcal{H}_i : i \in \mathbb{N}\}$ of collections of pairwise disjoint open subsets of X such that $\mathcal{H}_i < \mathcal{G}_i$ for every $i \in \mathbb{N}$ and $\bigcup_{i=1}^{\infty} \mathcal{H}_i$ covers X. We call $\{\mathcal{H}_i : i \in \mathbb{N}\}$ a *C-refinement* of $\{\mathcal{G}_i : i \in \mathbb{N}\}$. In particular if all $\mathcal{H}_i$ are discrete, then we call $\{\mathcal{H}_i : i \in \mathbb{N}\}$ a *discrete C-refinement* of $\{\mathcal{G}_i : i \in \mathbb{N}\}$.

It is well-known that every hereditarily paracompact countable-dimensional space is a C-space and every C-space is A-weakly infinite-dimensional. Pol [9] constructed a compact metrizable C-space which is not countable-dimensional. However, it is still unknown whether every compact A-weakly infinite-dimensional metrizable space is a C-space.

Let X be a space with a closed cover $\mathcal{F}$ consisting of A-weakly infinite-dimensional subspaces. Hadziivanov [6] proved that X is A-weakly infinite-dimensional provided that X is countably paracompact and $\mathcal{F}$ is locally finite. Polkowski [10] proved that X is A-weakly infinite-dimensional provided that X is hereditarily normal and $\mathcal{F}$ is hereditarily closure-preserving. Here a collection $\{A_s : s \in S\}$ of subsets of a space X is *hereditarily closure-preserving* if for every collection $\{B_s : s \in S\}$, where $B_s \subset A_s$ for every $s \in S$, we have $\mathrm{Cl}(\bigcup_{s \in S} B_s) = \bigcup_{s \in S} \mathrm{Cl} B_s$. Let us note that every locally finite collection is hereditarily closure-preserving. By using the same method of Polkowski [10], we can easily show that Hadziivanov's result above remains true if '$\mathcal{F}$ is locally-finite' is weakened to '$\mathcal{F}$ is hereditarily closure-preserving'.

Let X be a space with a closed cover $\mathcal{F}$ consisting of C-spaces. Addis and Gresham [1] proved that X is a C-space provided that X is paracompact and hereditarily collectionwise normal and $\mathcal{F}$ is locally finite.

Hence it is natural to ask whether a hereditarily closure-preserving sum theorem for C-spaces holds. In this paper we shall prove the following theorem and some related theorems.

2000 *Mathematics Subject Classification.* Primary 54F45.

Key words and phrases. C-space, A-weakly infinite-dimensional, sum theorem, hereditarily closure-preserving.

1.1. Theorem. (i) *If a paracompact space X can be represented as the union of a hereditarily closure-preserving collection of closed C-spaces, then X is a C-space.*
(ii) *If a hereditarily collectionwise normal space X can be represented as the union of a hereditarily closure-preserving collection of closed C-spaces, then X is a C-space.*

1.2. Remark. In Theorem 1.1.(i), 'paracompact' can be replaced by 'countably paracompact' since a countably paracompact space with a hereditarily closure-preserving closed cover consisting of C-spaces is paracompact. Indeed, this is a consequence of the following two facts (1) and (2).
(1) Every countably paracompact C-space is paracompact (see a remark after the proof of Lemma 2.1 below).
(2) If a space X can be represented as the union of a hereditarily closure-preserving collection of closed paracompact spaces, then X is paracompact(cf. [4, 5.1.G]).

2 Hereditarily Closure-preserving Sum Theorems for C-spaces

We begin with basic symbols. For a collection $\mathcal{A}$ of subsets of a space X and for $Y \subset X$ we write $\mathcal{A}|Y$ for $\{A \cap Y : A \in \mathcal{A}\}$, $\mathrm{Cl}\mathcal{A}$ for $\{\mathrm{Cl}A : A \in \mathcal{A}\}$ and $\bigcup \mathcal{A}$ for $\bigcup\{A : A \in \mathcal{A}\}$.

2.1. Lemma. *Let X be a countably paracompact C-space. Then for every collection $\{\mathcal{G}_i : i \in \mathbb{N}\}$ of open covers of X there exists a discrete C-refinement of $\{\mathcal{G}_i : i \in \mathbb{N}\}$.*

Proof. Since X is a C-space, take a C-refinement $\{\mathcal{U}_i : i \in \mathbb{N}\}$ of $\{\mathcal{G}_i : i \in \mathbb{N}\}$. We set $U_i = \bigcup \mathcal{U}_i$ for every $i \in \mathbb{N}$. Obviously, $\{U_i : i \in \mathbb{N}\}$ is a countable open cover of X. Thus there exists an open cover $\{V_i : i \in \mathbb{N}\}$ of X such that $\mathrm{Cl}V_i \subset U_i$ for every $i \in \mathbb{N}$. Let us set $H(U) = U \cap V_i$ for every $U \in \mathcal{U}_i$. The collection $\mathcal{H}_i = \{H(U) : U \in \mathcal{U}_i\}$ is discrete. Obviously, we have $\mathcal{H}_i < \mathcal{G}_i$ for every $i \in \mathbb{N}$ and $X = \bigcup \bigcup_{i=1}^{\infty} \mathcal{H}_i$. This completes the proof of Lemma 2.1.

¿From the above lemma it follows that every countably paracompact C-space is paracompact.

The following lemma is easily checked.

2.2. Lemma. *Let E be a closed subset of a collectionwise normal space X. For every discrete collection $\mathcal{U}$ of open subsets of E there exists a discrete collection $\mathcal{V}$ of open subsets of X which satisfies $\mathcal{V}|E = \mathcal{U}$.*

2.3. Lemma. *Let E be a closed subset of a hereditarily collectionwise normal space X. For every collection $\mathcal{U}$ of pairwise disjoint open subsets of E there exists a collection $\mathcal{V}$ of pairwise disjoint open subsets of X which satisfies $\mathcal{V}|E = \mathcal{U}$.*

Proof. Let $\mathcal{U} = \{U_s : s \in S\}$ and $Y = (X - E) \cup \bigcup\{U_s : s \in S\}$. Since $\mathrm{Cl}_Y\mathcal{U} = \{\mathrm{Cl}_Y U_s : s \in S\}$ is a discrete collection of closed subsets of Y, by collectionwise normarity of Y, there exists a discrete collection $\mathcal{V} = \{V_s : s \in S\}$ of open subsets of Y such that $\mathrm{Cl}_Y U_s \subset V_s$ for every $s \in S$. As Y is an open subset of X, V_s is an open subset of X for every $s \in S$. We get the required collection $\mathcal{V} = \{V_s : s \in S\}$.

2.4. Lemma. *Let X be a countably paracompact and collectionwise normal space which is the union of a closed subspace E and a C-space F. If $\{\mathcal{G}_i : i \in \mathbb{N}\}$, where $\mathcal{G}_i = \{G_\lambda : \lambda \in \Lambda_i\}$, is a collection of open covers of X and $\{\mathcal{H}_i : i \in \mathbb{N}\}$, where $\mathcal{H}_i = \{H_\lambda : \lambda \in \Lambda_i\}$, is a discrete C-refinement of $\{\mathcal{G}_i|E : i \in \mathbb{N}\}$ in E such that $H_\lambda \subset G_\lambda$ for every $\lambda \in \Lambda_i$, then there exists a discrete C-refinement $\{\tilde{\mathcal{H}}_i : i \in \mathbb{N}\}$, where $\tilde{\mathcal{H}}_i = \{\tilde{H}_\lambda : \lambda \in \Lambda_i\}$, of $\{\mathcal{G}_i : i \in \mathbb{N}\}$ in X such that $\tilde{H}_\lambda \subset G_\lambda$ and $\tilde{H}_\lambda \cap E = H_\lambda$ for every $\lambda \in \Lambda_i$.*

Proof. By Lemma 2.2, we can take a discrete collection $\mathcal{H}'_i = \{H'_\lambda : \lambda \in \Lambda_i\}$ of open subsets of X for every $i \in \mathbb{N}$ such that $H'_\lambda \subset G_\lambda$ and $H'_\lambda \cap E = H_\lambda$ for every $\lambda \in \Lambda_i$. Take a closed G_δ-set Z in X such that $E \subset Z \subset \bigcup\bigcup_{i=1}^\infty \mathcal{H}'_i$. Let $X - Z = \bigcup_{n=1}^\infty K_n$, where K_n is a closed subset of X. For every $n \in \mathbb{N}$ take two open subsets U_n and V_n of X such that $Z \subset U_n, K_n \subset V_n$ and $\mathrm{Cl}U_n \cap \mathrm{Cl}V_n = \emptyset$.

Represent the set $\mathbb{N}$ of all positive integers as the union $\bigcup_{n=1}^\infty N_n$ of infinite sets, where $N_n \cap N_m = \emptyset$ whenever $n \neq m$. Since K_n is a C-space, by Lemma 2.1, for every $i \in N_n$ there exists a discrete C-refinement $\{\mathcal{O}_i : i \in N_n\}$, where $\mathcal{O}_i = \{O_\lambda : \lambda \in \Lambda_i\}$, of $\{\mathcal{G}_i|K_n : i \in N_n\}$ in K_n such that $O_\lambda \subset G_\lambda$ for every $\lambda \in \Lambda_i$. By Lemma 2.2, for every $i \in N_n$there exists a discrete collection $\mathcal{O}'_i = \{O'_\lambda : \lambda \in \Lambda_i\}$ of open subsets of X such that $O'_\lambda \cap K_n = O_\lambda$, $O'_\lambda \subset G_\lambda$ and $O'_\lambda \subset V_n$ for every $\lambda \in \Lambda_i$. For every $i \in \mathbb{N}$ and every $\lambda \in \Lambda_i$, choose $n \in \mathbb{N}$ with $i \in N_n$ and let $\tilde{H}_\lambda = (H'_\lambda \cap U_n) \cup O'_\lambda$. Let us set $\tilde{\mathcal{H}}_i = \{\tilde{H}_\lambda : \lambda \in \Lambda_i\}$ for every $i \in \mathbb{N}$. Obviously, $\tilde{H}_\lambda \subset G_\lambda$ and $\tilde{H}_\lambda \cap E = H_\lambda$ for every $i \in \mathbb{N}$ and every $\lambda \in \Lambda_i$, and $\bigcup\bigcup_{i=1}^\infty \tilde{\mathcal{H}}_i = X$ because $X - Z = \bigcup_{n=1}^\infty K_n$. It remains to prove that $\tilde{\mathcal{H}}_i$ is discrete for every $i \in \mathbb{N}$. For every $i \in \mathbb{N}$ take $n \in \mathbb{N}$ such that $i \in N_n$. Since the collection $\mathcal{H}''_i = \{H'_\lambda \cap U_n : \lambda \in \Lambda_i\}$ and $\mathcal{O}'_i$ are discrete, and since we have $\bigcup \mathcal{H}''_i \subset U_n$, $\bigcup \mathcal{O}'_i \subset V_n$ and $\mathrm{Cl}U_n \cap \mathrm{Cl}V_n = \emptyset$, $\tilde{\mathcal{H}}_i$ is discrete. This completes the proof of Lemma 2.4.

Similarly to the proof of Lemma 2.4, we obtain the next lemma by using Lemma 2.3 instead of Lemma 2.2 and by replacing 'discrete C-refinement' by 'C-refinement'.

2.5. Lemma. *Let X be a hereditarily collectionwise normal space which is the union of a closed subspace E and a C-space F. If $\{\mathcal{G}_i : i \in \mathbb{N}\}$, where $\mathcal{G}_i = \{G_\lambda : \lambda \in \Lambda_i\}$, is a collection of open covers of X and $\{\mathcal{H}_i : i \in \mathbb{N}\}$, where $\mathcal{H}_i = \{H_\lambda : \lambda \in \Lambda_i\}$, is a C-refinement of $\{\mathcal{G}_i|E : i \in \mathbb{N}\}$ in E such that $H_\lambda \subset G_\lambda$ for every $\lambda \in \Lambda_i$, then there exists a C-refinement $\{\tilde{\mathcal{H}}_i : i \in \mathbb{N}\}$, where $\tilde{\mathcal{H}}_i = \{\tilde{H}_\lambda : \lambda \in \Lambda_i\}$, of $\{\mathcal{G}_i : i \in \mathbb{N}\}$ in X such that $\tilde{H}_\lambda \subset G_\lambda$ and $\tilde{H}_\lambda \cap E = H_\lambda$ for every $\lambda \in \Lambda_i$.*

2.6. Proof of Theorem 1.1.(i) Suppose that X is paracompact and is the union of a hereditarily closure-preserving cover $\{F_\alpha : \alpha < \xi\}$ such that F_α is a closed C-space. Let $\{\mathcal{G}_i : i \in \mathbb{N}\}$, where $\mathcal{G}_i = \{G_\lambda : \lambda \in \Lambda_i\}$, be a collection of open covers of X for every $i \in \mathbb{N}$. Let $E_\alpha = \bigcup_{\beta<\alpha} F_\beta$ for every $\alpha \leq \xi$. For every $\alpha \leq \xi$, inductively, we shall construct a collection $\mathcal{H}^\alpha_i = \{H^\alpha_\lambda : \lambda \in \Lambda_i\}$ of open subsets of E_α for every $i \in \mathbb{N}$ satisfying the following conditions:

> $\{\mathcal{H}^\alpha_i : i \in \mathbb{N}\}$ is a discrete C-refinement of $\{\mathcal{G}_i|E_\alpha : i \in \mathbb{N}\}$ such that, for every $\lambda \in \Lambda_i$ and every $i \in \mathbb{N}$, $H^\alpha_\lambda \subset G_\lambda$ and if $\beta < \alpha$, then $H^\alpha_\lambda \cap E_\beta = H^\beta_\lambda$.

The space X will be a C-space if we complete the induction, since $\{\mathcal{H}^\xi_i : i \in \mathbb{N}\}$ is then a C-refinement of $\{\mathcal{G}_i : i \in \mathbb{N}\}$.

First we set $\mathcal{H}^0_i = \{\emptyset\}$ for every $i \in \mathbb{N}$. Assume that a sequence $\{\mathcal{H}^\beta_i : i \in \mathbb{N}\}$ has been constructed for every $\beta < \alpha$. We shall construct a sequence $\{\mathcal{H}^\alpha_i : i \in \mathbb{N}\}$. If $\alpha = \beta + 1$, then applying Lemma 2.4 to the case where $X = E_\alpha, E = E_\beta, F = F_\alpha, \mathcal{G}_i = \mathcal{G}_i|E_\alpha$ and $\mathcal{H}_i = \mathcal{H}^\beta_i$, we obtain the required sequence $\{\mathcal{H}^\alpha_i : i \in \mathbb{N}\}$.

In the case when α is a limit number, let us set

$$H^\alpha_\lambda = \bigcup_{\beta<\alpha} H^\beta_\lambda \text{ for every } \lambda \in \Lambda_i \text{ and } \mathcal{H}^\alpha_i = \{H^\alpha_\lambda : \lambda \in \Lambda_i\}.$$

Since $\{F_\beta : \beta < \alpha\}$ is hereditarily closure-preserving, $H_\lambda^\alpha = E_\alpha - \bigcup_{\beta<\alpha}(F_\beta - H_\lambda^{\beta+1})$ is open in E_α. Thus $\mathcal{H}_i^\alpha$ is a collection of open subsets of E_α.

As $H_\lambda^\beta \subset G_\lambda$ for every $\beta < \alpha$, we have $H_\lambda^\alpha = \bigcup_{\beta<\alpha} H_\lambda^\beta \subset G_\lambda$ for every $\lambda \in \Lambda_i$.

Clearly $E_\alpha \supset \bigcup\bigcup_{i=1}^\infty \mathcal{H}_i^\alpha$. Consider an arbitrary point $x \in E_\alpha$. Take $\beta_0 < \alpha$ such that $x \in E_{\beta_0}$. Since $\bigcup_{i=1}^\infty \mathcal{H}_i^{\beta_0}$ is a cover of E_{β_0}, there are $i \in \mathbb{N}$ and $\lambda \in \Lambda_i$ such that $x \in H_\lambda^{\beta_0}$ Since $x \in H_\lambda^{\beta_0} \subset H_\lambda^\alpha \in \mathcal{H}_i^\alpha$, we have $E_\alpha \subset \bigcup\bigcup_{i=1}^\infty \mathcal{H}_i^\alpha$. Thus the equality $E_\alpha = \bigcup\bigcup_{i=1}^\infty \mathcal{H}_i^\alpha$ holds.

It is easy to see that $H_\lambda^\alpha \cap E_\beta = H_\lambda^\beta$ for every $\beta < \alpha$.

It remains to prove that $\mathcal{H}_i^\alpha$ is discrete in E_α for every $i \in \mathbb{N}$. First, we show that $\mathrm{Cl}\mathcal{H}_i^\alpha$ is pairwise disjoint. Assume on the contrary that there exist $x \in X$ and $\lambda, \lambda' \in \Lambda_i$ such that $\lambda' \neq \lambda$ and $x \in \mathrm{Cl}H_\lambda^\alpha \cap \mathrm{Cl}H_{\lambda'}^\alpha$. We have $\mathrm{Cl}H_\lambda^\alpha = \mathrm{Cl}(\bigcup_{\beta<\alpha}(H_\lambda^\alpha \cap F_\beta)) = \bigcup_{\beta<\alpha} \mathrm{Cl}(H_\lambda^\alpha \cap F_\beta)$, because $\{F_\beta : \beta < \alpha\}$ is hereditarily closure-preserving. Take $\beta < \alpha$ such that $x \in \mathrm{Cl}(H_\lambda^\alpha \cap F_\beta) \subset \mathrm{Cl}(H_\lambda^\alpha \cap E_{\beta+1}) = \mathrm{Cl}H_\lambda^{\beta+1}$. Similarly, we can take $\beta' < \alpha$ such that $x \in \mathrm{Cl}H_{\lambda'}^{\beta'+1}$. Thus $x \in \mathrm{Cl}H_\lambda^{\beta''} \cap \mathrm{Cl}H_{\lambda'}^{\beta''}$, where $\beta'' = \max\{\beta, \beta'\} + 1$, which contradicts the discreteness of $\mathcal{H}_i^{\beta''}$. Next, we show that $\mathrm{Cl}\mathcal{H}_i^\alpha$ is closure-preserving. It suffices to show that the set

$$F = \bigcup\{\mathrm{Cl}H_\lambda^\alpha : \lambda \in M\}$$

is closed in E_α for every $M \subset \Lambda_i$. We have

$$\begin{aligned} F &= \bigcup\{\bigcup_{\beta<\alpha} \mathrm{Cl}(H_\lambda^\alpha \cap F_\beta) : \lambda \in M\} \\ &= \bigcup_{\beta<\alpha} \bigcup\{\mathrm{Cl}(H_\lambda^\alpha \cap F_\beta) : \lambda \in M\} \\ &= \bigcup_{\beta<\alpha} \bigcup\{\mathrm{Cl}(H_\lambda^{\beta+1} \cap F_\beta) : \lambda \in M\} \end{aligned}$$

because the equalities $H_\lambda^\alpha \cap F_\beta = H_\lambda^\alpha \cap (E_{\beta+1} \cap F_\beta) = (H_\lambda^\alpha \cap E_{\beta+1}) \cap F_\beta = H_\lambda^{\beta+1} \cap F_\beta$ hold. $\mathcal{H}_\lambda^{\beta+1}$ being discrete in $E_{\beta+1}$, $\bigcup\{\mathrm{Cl}(H_\lambda^{\beta+1} \cap F_\beta) : \lambda \in M\}$ is closed in F_β. Since $\{F_\beta : \beta < \alpha\}$ is hereditarily closure-preserving, F is closed in E_α. Since a closure-preserving, pairwise disjoint collection of closed sets is discrete, $\mathcal{H}_i^\alpha$ is discrete in E_α. This completes the proof of Theorem 1.1.(i).

Similarly, we can prove (ii) by using Lemma 2.5 instead of Lemma 2.4 and by omitting the discussion on discreteness.

3 Countable Sum Theorems for C-spaces

Let X be a space with a countable closed cover $\mathcal{F}$. Levšenko [8] proved that X is A-weakly infinite-dimensional provided that X is countably paracompact (or hereditarily normal) and each $F \in \mathcal{F}$ is A-weakly infinite-dimensional. On the other hand, Addis and Gresham [1] proved that X is a C-space provided that X is hereditarily collectionwise normal and each $F \in \mathcal{F}$ is a C-space. This result is a counterpart for C-spaces of Levšenko's result in the case when X is hereditarily normal.

As an application of a selection theorem to sum theorems, Gutev and Valov [5] obtained the following countable sum theorem for C-spaces.

3.1. Theorem (Gutev and Valov [5]). *If a paracompact space X can be represented as the union of a countable collection of closed C-spaces, then X is a C-space.*

As stated in Remark 1.2, every countably paracompact C-space is paracompact. It is also easily proved that a collectionwise normal space is paracompact if it is the union of countably many paracompact closed subspaces. Thus, Theorem 3.1 can be restated as the following corollary; here, we give a direct proof as an application of our arguments.

3.2. Corollary. *If a countably paracompact and collectionwise normal space X can be represented as the union of a countable collection of closed C-spaces, then X is a C-space.*

Proof. Suppose that X is the union of a countable cover $\{F_i : i \in \mathbb{N}\}$ such that F_i is a closed C-space. Let $\{\mathcal{G}_i : i \in \mathbb{N}\}$ be a collection of open covers of X. Represent the set $\mathbb{N}$ of all positive integers as the union $\bigcup_{j=1}^{\infty} N_j$ of infinite sets, where $N_j \cap N_k = \emptyset$ whenever $j \neq k$. As F_j is a C-space, by Lemma 2.1, there exists a discrete C-refinement $\{\mathcal{H}_i : i \in N_j\}$ of $\{\mathcal{G}_i|F_j : i \in N_j\}$. By Lemma 2.2, there exists a discrete collection $\tilde{\mathcal{H}}_i$ of open subsets of X for every $i \in N_j$ such that $\tilde{\mathcal{H}}_i < \mathcal{G}_i$ and $\tilde{\mathcal{H}}_i|F_j = \mathcal{H}_i$. Then $F_j \subset \bigcup\bigcup_{i\in N_j} \tilde{\mathcal{H}}_i$, and thus $\{\tilde{\mathcal{H}}_i : i \in \mathbb{N}\}$ is a C-refinement of $\{\mathcal{G}_i : i \in \mathbb{N}\}$. Corollary 3.2 has been proved.

Corollary 3.2 is a counterpart for C-spaces of Levšenko's result in the case when X is countably paracompact.

In Corollary 3.2 the assumption of collectionwise normality of X can not be dropped.

3.3. Example. Bing [2] constructed a perfectly normal space X which is not collectionwise normal. What we need is the fact that X has an uncountable discrete closed subset F_0 such that $X - F_0$ is discrete. Since X is perfectly normal, there are countably many closed subsets F_n, $n \in \mathbb{N}$, such that $X - F_0 = \bigcup_{n=1}^{\infty} F_n$. As F_n is discrete for every $n < \omega$, F_n is a C-space for every $n < \omega$. As $X = \bigcup_{n<\omega} F_n$, X is the union of countably many closed C-spaces. On the other hand, since X is perfectly normal, X is countably paracompact. Since X is not collectionwise normal, X is not a C-space (see Remark 1.2.(1)).

4 Sum Theorems for C_f-spaces

Not all finite-dimensional spaces are C-spaces(See [1]). We introduce the notion of the class of C_f-spaces which contains all finite-dimensional spaces. A space X is a *C_f-space* if for every countable collection $\{\mathcal{G}_i : i \in \mathbb{N}\}$ of finite open covers of X there exists a countable collection $\{\mathcal{H}_i : i \in \mathbb{N}\}$ of pairwise disjoint collections of open subsets of X such that $\mathcal{H}_i < \mathcal{G}_i$ for every $i \in \mathbb{N}$ and $\bigcup_{i=1}^{\infty} \mathcal{H}_i$ covers X. We call $\{\mathcal{H}_i : i \in \mathbb{N}\}$ a *C_f-refinement* of $\{\mathcal{G}_i : i \in \mathbb{N}\}$. In particular if all $\mathcal{H}_i$ are discrete, then we call $\{\mathcal{H}_i : i \in \mathbb{N}\}$ a *discrete C_f-refinement* of $\{\mathcal{G}_i : i \in \mathbb{N}\}$.

It is easily seen that every C-space is a C_f-space and every C_f-space is A-weakly infinite-dimensional. Addis and Gresham [1] proved that all finite-dimensional, paracompact spaces are C-spaces. By the same proof, we can show that all finite-dimensional spaces are C_f-spaces.

4.1. Proposition. *All finite-dimensional spaces are C_f-spaces.*

Proof. Let X be a space with $\dim X = n$. Consider a collection $\{\mathcal{G}_i : i \in \mathbb{N}\}$ of finite open covers of X. Applying Ostrand's theorem (see [4, p.184]) to the finite open cover $\mathcal{G} = \{G_1 \cap G_2 \cap ... \cap G_{n+1} : G_i \in \mathcal{G}_i \text{ for } i = 1, 2, ..., n+1\}$ of X we obtain a refinement $\mathcal{U}$ of the cover $\mathcal{G}$ which can be represented as the union of $n+1$ collections $\mathcal{U}_1, \mathcal{U}_2, ..., \mathcal{U}_{n+1}$ of pairwise disjoint open sets. Letting

$$\mathcal{H}_i = \mathcal{U}_i \text{ for } i \leq n+1 \text{ and } \mathcal{H}_i = \{\emptyset\} \text{ for } i > n+1$$

we get the required collection $\{\mathcal{H}_i : i \in \mathbb{N}\}$.

The following Lemmas 4.2, 4.3 and 4.4 are parallel to Lemmas 2.1, 2.2 and 2.3 respectively, and Lemma 4.5 is parallel to Lemmas 2.4 and 2.5. The proofs are left to the reader since they are similar.

4.2. Lemma. *Let X be a countably paracompact C_f-space. Then for every collection $\{\mathcal{G}_i : i \in \mathbb{N}\}$ of finite open covers of X there exists a discrete C_f-refinement of $\{\mathcal{G}_i : i \in \mathbb{N}\}$.*

4.3. Lemma. *Let E be a closed subset of a space X. For every finite discrete collection $\mathcal{U}$ of open subsets of E there exists a discrete collection $\mathcal{V}$ of open subsets of X which satisfies $\mathcal{V}|E = \mathcal{U}$.*

4.4. Lemma. *Let E be a closed subset of a hereditarily normal space X. For every finite collection $\mathcal{U}$ of pairwise disjoint open subsets of E there exists a finite collection $\mathcal{V}$ of pairwise disjoint open subsets of X which satisfies $\mathcal{V}|E = \mathcal{U}$.*

4.5. Lemma. *Let X be a countably paracompact* (*resp. hereditarily normal*) *space which is the union of a closed subspace E and a C_f-space F. If $\{\mathcal{G}_i : i \in \mathbb{N}\}$, where $\mathcal{G}_i = \{G_\lambda : \lambda \in \Lambda_i\}$, is a collection of finite open covers of X and $\{\mathcal{H}_i : i \in \mathbb{N}\}$, where $\mathcal{H}_i = \{H_\lambda : \lambda \in \Lambda_i\}$, is a discrete C_f-refinement* (*resp. C_f-refinement*) *of $\{\mathcal{G}_i|E : i \in \mathbb{N}\}$ such that $H_\lambda \subset G_\lambda$ for every $\lambda \in \Lambda_i$, then there exists a discrete C_f-refinement* (*resp. C_f-refinement*) *$\{\tilde{\mathcal{H}}_i : i \in \mathbb{N}\}$ of $\{\mathcal{G}_i : i \in \mathbb{N}\}$, where $\tilde{\mathcal{H}}_i = \{\tilde{H}_\lambda : \lambda \in \Lambda_i\}$, such that $\tilde{H}_\lambda \subset G_\lambda$ and $\tilde{H}_\lambda \cap E = H_\lambda$ for every $\lambda \in \Lambda_i$.*

By using the same methods as the proofs of Theorem 1.1 and Corollary 3.2, we obtain the hereditarily closure-preserving sum theorem and the countable sum theorem for C_f-spaces.

4.6. Theorem. (i) *If a space X is either countably paracompact or hereditarily normal, and can be represented as the union of a hereditarily closure-preserving collection of closed C_f-spaces, then X is a C_f-space.*
(ii) *If a space X is either countably paracompact or hereditarily normal, and can be represented as the union of a countable collection of closed C_f-spaces, then X is a C_f-space.*

4.7. Remark. It is unknown whether hereditarily normal case of Theorem 4.6.(i) remains true if 'C_f' is replaced by 'C', while Theorem 4.6.(ii) is not true for C-spaces as we showed in Example 3.4 above. Both Theorem 4.6.(i) and (ii) are parallel to the results for A-weakly infinite dimensional spaces stated in the introduction.

References

[1] D. F. Addis and J. H. Gresham, *A class of infinite-dimensional spaces, Part I: Dimension theory and Alexandroff's Problem*, Fund. Math. 101(1978), 195-205.

[2] R. H. Bing, *Metrization of topological spaces*, Canad. J. Math. 3(1951), 175-186.

[3] R. Engelking, General Topology, Heldermann Verlag, Lemgo, 1989.

[4] R. Engelking, Theory of Dimensions, Finite and Infinite, Heldermann Verlag, Lemgo, 1995.

[5] V. Gutev and V. Valov, *Continuous selections and C-spaces*, Proc. Amer. Math. Soc. 130 (2002), 233-242.

[6] N. G. Hadziivanov, *On infinite-dimensional spaces*, Bull. Acad. Pol. Sci. Sér. Math. 19(1971), 491-500.

[7] W. E. Haver, *A covering property for metric spaces*, Topology Conference at Virginia Polytechnic Institute 1973, Lecture Notes in Math. 375(1974), 108-113.

[8] B. T. Levšenko, *On strongly infinite-dimensional spaces*, Vestnik Moskov. Univ. 5(1959), 219-228.

[9] R. Pol, *A-weakly infinite-dimensional compactum which is not countable-dimensional*, Proc. Amer. Math. Soc. 82 (1981), 634-636.

[10] L. Polkowski, *A sum theorem for A-weakly infinite-dimensional spaces*, Fund. Math. 119 (1983), 7-10.

Chieko Komoda
Department of Mathematics, Faculty of Science, Shizuoka University
Ohya, Shizuoka, 422-8529, Japan