SPAN MATES AND SPAN

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Abstract. We show that the span mate width of a simple closed curve dose not exceed the span of that simple closed curve.

1. INTRODUCTION

We shall start by recalling the definitions introduced by A. Lelek in [4] and [5]. Let X be a nonempty connected metric space. The span $\sigma(X)$ in the least upper bound of the set of real numbers $r, r \geq 0$, that satisfy the following condition:

There exists a connected space Y and a pair of continuous functions $f, g : Y \longrightarrow X$ such that

$$f(Y) = g(Y) \tag{1}$$

and dist $[f(Y), g(Y)] \geq r$, for every $y \in Y$.

Relaxing (1) to the inclusion $g(Y) \supset f(Y)$ one obtains the definition of the semispan $\sigma_0(X)$ of X. We omit here the surjective varieties of span and semispan since they do not present different concepts for a simple closed curve.

Clearly, $0 \leq \sigma(X) \leq \sigma_0(X) \leq \text{diam}(X)$. It was proven by Lelek in [5, p.39] that when X is a continuum $\sigma_0(X) \leq \varepsilon(X)$, where $\varepsilon(X)$ is the infimum of the set of meshes of the chains that cover X. A different, direct proof of this inequality can be found in [1]. There are two other estimates of span involving mesh. In [1], it was proven, by this author, that the dual effectively monotone span of a simple closed curve X does not exceed $\varepsilon(X)$. In [2], the dual monotone span of a starlike curve X was shown to be not smaller than $\varepsilon(X)$.

The problem of calculating the span of a simple closed curve is a difficult one. It has been solved only in the cases when additional assumptions have been imposed on the curve. In [6], this author solved it for the "convex case", that is the case when X is a boundary of a convex region. It has been determined that both span and semispan of X are equal to the infimum of the set of directional diameters of X, which in convex case coincides with $\varepsilon(X)$. In [3], a class of starlike curves was identified for which $\sigma(X) = \sigma_0(X) = \varepsilon(X)$. The latter introduced the concept of span mates. We now review the main definitions.

A simple closed polygonal path is a simple closed curve in the Cartesian plane **C** which consists of finitely many line segments. Let X be a simple closed polygonal path. A vertex

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$W \in X$ is outer if and only if the angle at W in the bounded component of $\mathbf{C} \setminus X$ is less than π. A connected subset of X between two consecutive outer vertices is called a segment. Each segment inherits the positive orietation from X and hence has a uniquely determined beginning and end. A segment with the beginning A and end B will be represented by AB. The distance $\text{dist}[A, Y]$ from a point A to a set Y in the plane is defined, as usual, by letting $\text{dist}[A, Y] = \inf_{P \in Y} \text{dist}[A, P]$, where P is a point in Y.

Definition 1.1. Let AB and CD be two segments of a polygonal path X. The span distance $s(AB, CD)$ between AB and CD is defined as

$$s(AB, CD) = \max\{\min(\text{dist}[A, CD], \text{dist}[D, AB]), \min(\text{dist}[B, CD], \text{dist}[C, AB])\}.$$

Definition 1.2. Let AB and CD be two different segments of a polygonal path X. We say that AB is first with respect to CD if $s(AB, CD) = \min(\text{dist}[B, CD], \text{dist}[C, AB])$. We say that AB is second with respect to CD if $s(AB, CD) = \min(\text{dist}[A, CD], \text{dist}[D, AB])$.

Definition 1.3. Let V_{i-1}, V_i, V_{i+1} be three consecutive, in the positive direction, outer vertices on X, and let AB be a segment on $X, AB \neq V_{i-1}V_i, AB \neq V_iV_{i+1}$. We say that V_i is significant with respect to AB if $V_{i-1}V_i$ is first with respect to AB, V_iV_{i+1} is second with respect to AB, and V_iV_{i+1} is not first with respect to AB.

Definition 1.4. Let AB be a segment on X, let V_i be a significant vertex with respect to AB, and let V_{i+1} be the next, in the positive direction, outer vertex on X. The segment V_iV_{i+1} is called a span mate of AB.

1. SPAN MATES AND SPAN

We shall use the concept of span mates to obtain an estimate from below for the span of any simple closed curve. First, we need the following definition and a lemma for a simple closed polygonal path. We shall denote the outer vertices of a simple closed polygonal path by $V_0, V_1, \ldots, V_N$, with the understanding that whenever an arbitrary segment V_jV_{j+1} is considered, $j = 0, \ldots, N, j+1$ is taken modulo $N+1$.

Definition 2.1. Let X be simple closed polygonal path, and let $V_0, V_1, \ldots, V_N$ be the outer vertices of X, in their consecutive positive order. The span mate width $\text{smw}(X)$ of X is defined as

$$\text{smw}(X) = \min \begin{cases} \min\limits_{0 \le j \le N} s(V_jV_{j+1}, V_mV_{m+1}) \\ m, \text{such that } V_mV_{m+1} \text{ is a span mate of } V_jV_{j+1} \\ \\ \min\limits_{0 \le j \le N} s(V_jV_{j+1}, V_mV_{m+1}) \\ m, \text{such that } V_mV_{m+1} \text{ is first with respect to } V_jV_{j+1} \\ \quad \text{and } V_jV_{j+1} \text{ is first with respect to } V_mV_{m+1}. \end{cases}$$

Lemma 2.2. If X is a simple closed polygonal path then $\sigma(X) \ge \text{smw}(X)$.

Proof. Let V_jV_{j+1} be a fixed, but arbitrary, segment on X. Note, that V_jV_{j+1} has at least one span mate. Ineed, the segment $V_{j+1}V_{j+2}$ is first with respect to V_jV_{j+1} and the segment $V_{j-1}V_j$ is second, and not first, with respect to V_jV_{j+1}. Therefore, there exists at least one outer vertex on X that is significant with respect to V_jV_{j+1}. Consequently, there exists at least one span mate of V_jV_{j+1}. Suppose V_mV_{m+1} is a span mate of V_jV_{j+1}. We next claim that there exists a span mate of V_mV_{m+1} on the positive arc $V_{j+1}V_m$. Indeed, since V_mV_{m+1} is second, and not first, with respect to V_jV_{j+1} it follows that V_jV_{j+1} is first, and not second, with respect to V_mV_{m+1}. This, together with the fact that $V_{m-1}V_m$ is second, and not first with respect to V_mV_{m+1} implies the existence of an outer vertex significant with respect to V_mV_{m+1} on the positive arc $V_{j+1}V_m$. Hence, the claim.

Partition the unit interval $[0,1]$ into $2(N+1)$ line segments by letting $P_0 = 0, P_1 = 1/2(N+1), \dots, P_n = n/2(N+1), \dots, P_{2(N+1)} = 1$. For a line segment P_iP_{i+1} on $[0,1]$, $0 \le i \le 2N+1$, and a segment V_jV_{j+1} on X, $0 \le j \le N$, $L[P_iP_{i+1} \longrightarrow V_jV_{j+1}]$ shall represent an affine transformation of P_iP_{i+1} onto V_jV_{j+1}, with P_i and P_{i+1} corresponding to V_j and V_{j+1}, respectively.

We shall define two mappings $f, g : [0,1] \longrightarrow X$ as follows. Suppose that V_jV_{j+1} is a span mate of V_0V_1, and set $f(t) = L[P_0P_1 \longrightarrow V_0V_1]$ and $g(t) = V_j$, for $t \in [P_0, P_1]$. Note that, as argued above, a span mate of V_jV_{j+1} must exist on the positive arc V_1V_j. Let V_mV_{m+1} be the first, in the positive direction on V_1V_j, span mate of V_jV_{j+1}. There are two possible cases. Either $m = 1$ or $m > 1$. If $m = 1$ then set $f(t) = V_1$ and $g(t) = L[P_1P_2 \longrightarrow V_jV_{j+1}]$, for $t \in [P_1, P_2]$. If $m > 1$ then set $f(t) = L[P_kP_{k+1} \longrightarrow V_kV_{k+1}]$, for $t \in [P_k, P_{k+1}]$, $k = 0, \ \dots, \ m-1$, and $g(t) = V_j$, for $t \in [P_1, P_m]$, and after that set $f(t) = V_m$ and $g(t) = L[P_mP_{m+1} \longrightarrow V_jV_{j+1}]$, for $t \in [P_m, P_{m+1}]$.

Again, by the argument presented at the beginning of the proof, a span mate of V_mV_{m+1} must exist on the positive arc $V_{j+1}V_m$. We take the first, in the positive direction on $V_{j+1}V_m$, span mate of V_mV_{m+1} and continue defining f and g in the manner described above. Note, that whenever f covers an arbitrary segment V_iV_{i+1} while $g(t) = V_k$, for some k, at least one of the following conditions holds:

1) V_kV_{k+1} is a span mate of V_iV_{i+1}

2) V_kV_{k+1} is first with respect to of V_iV_{i+1} and V_iV_{i+1} is first with respectt to V_kV_{k+1}.

The same is true whenever g covers an arbitrary segment V_iV_{i+1} while $f(t) = V_k$, for some k. Therefore, it follows that $\text{dist}[f(t), g(t)] \ge \text{smw}(X)$ for all $t \in [0,1]$.

Since, as t covers $[0,1]$, f and g cover, between them, $2(N+1)$ segments on X in the positive direction, there are two cases to consider. Either both f and g are onto or one of them is not onto. Suppose the latter holds, and assume, without loss of generality, then g is not onto. Then continue the definition of f and g, in the manner described above, beyond the interval $[0,1]$, on $[1, 1+1/2(N+1)]$ and as many additional intervals of length $1/2(N+1)$ as needed until g covers X. Let n be a natural number such that $g([0, 1+n/2(N+1)]) = X$ and put $a = 1 + n/2(N+1)$.

We have defined two mappings $f, g : [0,a] \longrightarrow X$ for some $a \ge 1$, $f([0,a]) = g([0,a]) = X$ and $\text{dist}[(f(t), g(t)] \ge \text{smw}(X)$ for all $t \in [0,a]$. Hence, $\sigma(X) \ge \text{smw}(X)$.

The, above result can be extended, via approximation, to a simple closed curve. Let X be a simple closed curve, let L be a simple closed polygonal path with all of its vertices on X, and let $\varepsilon > 0$. For each line segment AB on L, we shall denote by $AB(X)$ the arc on X that corresponds to AB, i.e. $AB(X)$ has the same endpoints as AB and $AB(X) \cap (L \setminus AB) = \emptyset$. We consider L to be an ε-approximation of X if and only if

$$\forall AB \text{ on } L \ \forall P \in AB(X) \quad \text{dist}[P, AB] < \varepsilon. \tag{2}$$

For each natural n, we shall denote a $1/n$-approximation of X by L_n.

Definition 2.3. Let X be a simple closed curve. The span mate width smw(X) of X is defined as

$$\text{smw}(X) = \lim_n \sup \text{smw}(L_n),$$

where L_n is a 1/n-approximation of x.

Theorem 2.4. If X is a simple closed curve then $\sigma(X) \geq \text{smw}(X)$.

Proof. Let ε be an arbitrary positive number, and let L_n be a simple closed polygonal path and a 1/n-approximation of X, for each natural n. We choose a natural number $k, 1/k < \varepsilon/4$, such that

$$\text{smw}(L_k) > \text{smw}(X) - \varepsilon/4. \tag{3}$$

Since, by virtue of Lemma 2.2, $\sigma(L_k) \geq \text{smw}(L_k)$, there exist two mappings $f, g : [0,1] \longrightarrow L_k$ such that

$$\forall t \in [0,1] \quad \text{dist}[f(t), g(t)] \geq \text{smw}(L_k) - \varepsilon/4. \tag{4}$$

Let $h : L_k \longrightarrow X$ be a homeomorphism such that, for each line segment AB on L_k $h(AB) = AB(X)$, $h(A) = A, h(B) = B$, and for each x on L_k $\text{dist}[x, h(x)] < 1/k$. We define two mappings $F, G : [0,1] \longrightarrow X$ by putting $F(t) = h(f(t))$ and $G(t) = h(g(t))$ for all $t \in [0,1]$. Notice that

$$\forall t \in [0,1] \quad \text{dist}[f(t), F(t)] < 1/k < \varepsilon/4, \tag{5}$$

$$\forall t \in [0,1] \quad \text{dist}[g(t), G(t)] < 1/k < \varepsilon/4. \tag{6}$$

Therefore, in view of (3) - (6), we have

$$\text{dist}[F(t), G(t)] \geq \text{dist}[f(t), g(t)] - \varepsilon/2 \geq \text{smw}(L_k) - 3\varepsilon/4 > \text{smw}(X) - \varepsilon, \quad \text{for all } t \in [0,1].$$

Hence, since ε was arbitrary, it follows that $\sigma(X) \geq \text{smw}(X)$.

Finally, we point out that the equality holds in a known case. Let $d(X)$ denote the infimum of the set of the directional diameters of X, as defined in [6] for a simple closed curve X. The following theorem follows from the proof of Theorem 3 in [6] and the definition of the span mate width.

Theorem 2.5. Let X be the boundary of a bounded convex region in the plane. Then $\sigma(X) = \sigma_0(X) = \text{d}(X) = \text{smw}(X)$.

References

[1] K.T. Hallenbeck, Estimates of span of a simple closed curve involving mesh, Houston J. Math. 26 (2000), 741–745.

[2] K.T. Hallenbeck, On the span of starlike curves, Scientiae Mathematicae Japonicae 55, No. 3 (2002), 547–554.

[3] K.T. Hallenbeck, Span mates and mesh, Glasnik Matematicki Vol. 37 (57) (2002), 175–186.

[4] A. Lelek, Disjoint mappings and the span of spaces, Fund. Math. 55 (1964), 199–214.

[5] A. Lelek, On the surjective span and semispan of connected metric spaces, Colloq. Math. 37 (1977), 35–45.

[6] K. Tkaczynska, The span and semispan of some simple closed curves, Proc. Amer. Math. Soc., Vol. 111, No. 1 (1991), 247–253.

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