INEQUALITIES ON DERIVATIVES OF HARMONIC BERGMAN FUNCTIONS

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Abstract. We give a necessary and sufficient condition for positive measures which satisfy a Carleson type inequality for the harmonic Bergman space on the upper half-space of Euclidean spaces.

1. Introduction

Let H be the upper half-space of the n-dimensional Euclidean space $\mathbb{R}^n$ $(n \geq 2)$, that is, $H = \{z = (x, y) \in \mathbb{R}^n \ ; \ y > 0\}$, where we have written a point $z \in \mathbb{R}^n$ as $z = (x, y)$ with $x = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}$. For $0 < p < \infty$, let $b^p = b^p(H, dV)$ be the class of all harmonic functions u on H such that

$$\| u \|_p = \left(\int_H |u|^p dV \right)^{1/p} < \infty$$

where dV denotes the Lebesgue volume measure on H. The class b^p is called the harmonic Bergman space. Properties of functions in the harmonic Bergman space on the upper half-space were studied by Ramey and Yi [11] when $1 \leq p < \infty$, and by the author [13] when $0 < p \leq 1$.

Let μ and ν be σ-finite positive Borel measures on H. We consider conditions on μ and ν for which there exists a constant $C > 0$ such that $\int_H |u| d\mu \leq C \int_H |D_y u| d\nu$ for all u in b^1, where D_y denotes the differetiation operator with respect to y. More generally, we have the problem of determining conditions on μ and ν such that $\int_H |D^\alpha u|^p d\mu \leq C \int_H |D_y^m u|^p d\nu$, where α is a multi-index and D^α is the corresponding the partial differentiation operator. Such inequalities on the unit disk Δ in the complex plane were studied by Stegenga, and multipliers of the Dirichlet space were characterized in [12]. When $d\nu = (1 - |\zeta|)^r dA$ and $r \geq 1$, Stegenga proved that finite positive Borel measures μ and ν on the unit disk satisfy the inequality $\int_\Delta |f|^2 d\mu \leq C \int_\Delta |f'|^2 d\nu$ for all holomorphic functions f, $f(0) = 0$ if and only if there is a constant K such that $\mu(S_I) \leq K|I|^r$ for any interval I in the unit circle, where dA denotes the Lebesgue area measure, $|I|$ denotes the normalized arc length of I, and S_I is the corresponding Carleson square over I. It was also proved that when $0 \leq r < 1$ such measures are those satisfying $\mu(\cup S_{I_j}) \leq K\mathrm{Cap}(\cup I_j)$ for all finite disjoint collections of intervals $\{I_j\}$, where Cap is an appropriate Bessel capacity (if $r < 0$ any finite Borel measure satisfies this inequality). It is known that these characterizations can be generalized to the case of $p > 1$ (see also [12]). When $0 < p \leq 1$, $d\nu = (1 - |\zeta|)^r dA$, and $-1 < r \leq p - 1$, Ahern and Jevtić [1] proved that there is a constant $C > 0$ such that $\int_\Delta |f|^p d\mu \leq C \int_\Delta |f'|^p d\nu$ if and only if $\mu(S_I) \leq K|I|^{2-p+r}$. Using this result, Ahern and Jevtić characterized inner multipliers of the Besov space in case $0 < p \leq 1$. Such investigations on the unit ball of $\mathbb{C}^n$ are in [3]. In these investigations, when $p > 1$ necessary and sufficient conditions were

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not obtained completely. It was also shown that, in general, the above condition is not necessary. When $0 < p \le 1$ and $d\nu = y^r dV$, such a inequality on the upper half-space H of $\mathbb{R}^n$ was studied by author [13]. For the inequality $\int_\Delta |f|^p d\mu \le C \int_\Delta |f|^p d\nu$ on the unit disk, the properties of measures satisfying the inequality were studied in [6], [7], and [10], and partial results were obtained for more general measures μ and ν.

If $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index of nonnegative integers with order ℓ, then D^α denotes the partial differentiation operator $\partial^\ell / \partial x_1^{\alpha_1} \cdots \partial x_{n-1}^{\alpha_{n-1}} \partial y^{\alpha_n}$. We also use the absolute value symbol $|\cdot|$ to denote the Euclidean norm in $\mathbb{R}^n$. For $z = (x, y) \in H$, let $\bar{z} = (x, -y)$. The pseudohyperbolic metric ρ in H is defined by $\rho(z, w) = |w - z|/|\bar{w} - z|$. It is clear that ρ is invariant under horizontal translations. Let $D_\varepsilon(w) = \{z \in H \ ; \ \rho(z, w) < \varepsilon\}$ when $0 < \varepsilon < 1$. For $w = (s, t) \in H$, $D_\varepsilon(w)$ is a Euclidean ball whose center and radius are $\left(s, \dfrac{1+\varepsilon^2}{1-\varepsilon^2} t\right)$ and $\dfrac{2\varepsilon t}{1-\varepsilon^2}$ respectively. It follows that there is a constant $C = C_\varepsilon > 0$ such that $C^{-1} t^n \le V(D_\varepsilon(w)) \le C t^n$ for all $w \in H$. Let $S(w) = \{z = (x, y) \in H \ ; \ |x - s| < t, y < 2t\}$. $S(w)$ is called a Carleson box. We now state our main result in this paper.

THEOREM 1. *Let $0 < p \le 1$ and ℓ, m be nonnegative integers. Suppose that μ is a σ-finite positive Borel measure on H, $d\nu = \omega dV$ and ω satisfies the $(A_q)_\partial$-condition for some $1 < q < \infty$. Then, the following (1) $\sim$ (3) are equivalent.*

(1) *There is a constant $C > 0$ such that*

$$\int_H |D^\alpha u|^p d\mu \le C \int_H |D_y^m u|^p d\nu$$

for all $u \in b^p$ and multi-indices α of order ℓ.

(2) *There is a constant $C > 0$ such that*

$$\int_H |D_y^\ell u|^p d\mu \le C \int_H |D_y^m u|^p d\nu$$

for all $u \in b^p$.

(3) *There are constants $K > 0$ and $0 < \varepsilon < 1$ such that $\mu(S(w)) \le K t^{(\ell - m)p} \nu(D_\varepsilon(w))$ for all $w = (s, t) \in H$.*

In §2, we give the notation and some preliminary results. In Theorem 1, we assume that $d\nu = \omega dV$ and ω satisfies $(A_p)_\partial$-condition. We define and discuss these conditions. The $(A_p)_\partial$-condition on the unit disk of the complex plane is defined in [10]. In the definition of the $(A_p)_\partial$-condition on the unit disk, the normalized reproducing kernel in the holomorphic Bergman space is used. However, on the upper half-space of $\mathbb{R}^n$, we cannot use arguments in the complex plane. Therefore, we will extend the notion of the $(A_p)_\partial$-condition to H of $\mathbb{R}^n$ using another function. In §3, we give a sufficient condition for measures μ and ν which satisfy the inequality in (1) of Theorem 1. A necessary condition for the inequality in (2) of Theorem 1 is shown in §4. In §3 and §4, we will not assume that ω satisfies the $(A_p)_\partial$-condition. In §5, assuming that ω satisfies the $(A_p)_\partial$-condition, we give the proof of Theorem 1.

Throughout this paper, C will denote a positive constant whose value is not necessary the same at each occurrence; it may vary even within a line.

2. Preliminaries

In this section, we state some preliminary results for our investigations. The following lemma is in [13].

LEMMA 1. *Let $0 < \varepsilon < 1$. Then, the following are true.*

(1) *If z, w, ζ are in H and $\rho(z,w) < \varepsilon$, then $C^{-1}|\bar{\zeta} - z| \le |\bar{\zeta} - w| \le C|\bar{\zeta} - z|$ with a positive constant C depending only on ε.*

(2) *If $z = (x,y), w = (s,t)$ are in H and $\rho(z,w) < \varepsilon$, then $C^{-1}y \le t \le Cy$ with a positive constant C depending only on ε.*

(3) *If $0 < \varepsilon < 1/2$ then there exist a positive integer N and a sequence $\{\zeta_j\}$ in H satisfying the following conditions :* (a) *$H = \cup D_\varepsilon(\zeta_j)$,* (b) *any point in H belongs to at most N of the sets $D_{2\varepsilon}(\zeta_j)$.*

For a function u on H and $\delta > 0$, let $\tau_\delta u$ denote a function on H defined by $\tau_\delta u(x,y) = u(x, y+\delta)$, and let $\mathcal{T}^p = \{\tau_\delta u \ ; \ u \in b^p, \delta > 0\}$. The following lemma is stated in [13].

LEMMA 2. *Let $0 < p \le 1$. Then, the following are true.*

(1) *For any $u \in b^p$, there is a constant $C > 0$ such that $|D^\alpha u(s,t)| \le C/t^{n/p+|\alpha|}$ for all $(s,t) \in H$.*

(2) *For any $u \in b^p$, there is a constant $C > 0$ such that $|(D^\alpha \tau_\delta u)(s,t)| \le C/(t+\delta)^{n/p+|\alpha|}$ for all $(s,t) \in H$.*

The following lemma is useful and stated in [11, Lemma 3.1]

LEMMA 3. *Let $0 < c < 1$. Then, there is a constant $C > 0$ depending on c and n such that*

$$\int_H \frac{y^{-c}}{|\bar{w} - z|^n} dV(z) \le Ct^{-c}$$

for all $w = (s,t) \in H$.

For $w = (s,t) \in H$, let P_w be the Poisson kernel on the upper half-space H, that is, $P_w(x) = P(s-x, t) = \gamma_n t/(|s-x|^2 + t^2)^{n/2}$ $\quad (x \in \partial H)$ (where $\gamma_n = 2/(nV(\mathbb{B}_n))$, and $\mathbb{B}_n$ denotes the unit ball in $\mathbb{R}^n$). The harmonic extension of this function to H is $P(s-x, t+y)$. If $z = (x,y) \in H$, then we may write $P_w(z)$. We note that $P_w(z) = \gamma_n(t+y)/|\bar{w} - z|^n$, $|D_z^\alpha P_w(z)| \le C/|\bar{w} - z|^{n+|\alpha|-1}$, and $D_z^\alpha P_w(z) = (-1)^{\alpha_1 + \cdots + \alpha_{n-1}} D_w^\alpha P_w(z)$. Let m be a nonnegative integer and let $c_m = (-2)^m/m!$. The following Lemma 4 is given in [13].

LEMMA 4. *Let $0 < p \le 1$. If $u \in \mathcal{T}^p$, then*

$$u(w) = -2c_{m+k} \int_H y^{m+k} (D_y^m u)(z) D_y^{k+1} P_w(z) dV(z)$$

for all $m, k \ge 0$ and $w \in H$.

We show that Lemma 4 is also valid for $u \in b^p$ when the integer k is sufficiently large.

LEMMA 5. *Let $0 < p \le 1$ and k be a nonnegative integer such that $k > n/p$. If $u \in b^p$, then*

$$u(w) = -2c_{m+k} \int_H y^{m+k} (D_y^m u)(z) D_y^{k+1} P_w(z) dV(z)$$

for all $m \geq 0$ and $w \in H$.

PROOF. Let $u \in b^p$ and $k > n/p$. Then, Lemma 4 implies that

$$\tau_\delta u(w) = -2c_{m+k}\int_H y^{m+k}(D_y^m \tau_\delta u)(z) D_y^{k+1} P_w(z) dV(z)$$

for all $m \geq 0$ and $w \in H$. We show that the integrand is dominated by a integrable function $y^{-c}/|\bar{w}-z|^n$ $(0 < c < 1)$ for all $\delta > 0$. In fact, (2) of Lemma 2 implies that there is a constant $C > 0$ such that $|y^{m+k}(D_y^m \tau_\delta u)(z) D_y^{k+1} P_w(z)| \leq C y^{m+k}/\{(y+\delta)^{n/p+m}|\bar{w}-z|^{n+k}\} \leq C y^{k-n/p}/|\bar{w}-z|^{n+k}$. Since $k > n/p$, we have there is a constant $0 < c < 1$ such that $y^{k-n/p}/|\bar{w}-z|^{n+k} \leq y^{k-n/p}/\{|\bar{w}-z|^n (y+t)^k\} \leq y^{k-n/p}/\{|\bar{w}-z|^n y^{k-n/p+c} t^{n/p-c}\} \leq t^{c-n/p} y^{-c}/|\bar{w}-z|^n$. Thus, Lemma 3 implies that $t^{c-n/p}y^{-c}/|\bar{w}-z|^n$ is integrable. If $\delta \to 0$, then Lebesgue's dominated convergence theorem implies that

$$u(w) = -2c_{m+k}\int_H y^{m+k}(D_y^m u)(z) D_y^{k+1} P_w(z) dV(z).$$

For a nonnegative integrable function ω on the unit circle $\partial\Delta$ in the complex plane, the function ω satisfies the Muckenhoupt's A_2-condition if there is a constant $\gamma > 0$ such that $1/|I|\int_I \omega d\theta (1/|I|\int_I \omega^{-1} d\theta)^{-1} \leq \gamma$ for all intervals $I \subset \partial\Delta$. For $w \in \Delta$, let $p_w(\zeta)$ $(\zeta \in \partial\Delta)$ be the Poisson kernel on the unit disk. It is well known that ω satisfies the A_2-condition if and only if there is a constant $\gamma > 0$ such that $\int_{\partial\Delta} p_w \omega d\theta (\int_{\partial\Delta} p_w \omega^{-1} d\theta)^{-1} \leq \gamma$ for all $w \in \Delta$ (see [5]). In [10], for a nonnegative integrable function ω on Δ, using the function $p_w(z)$ $(z \in \Delta)$, $(A_2)_\partial-$condition is defined, that is, a function ω on Δ satisfies the $(A_2)_\partial-$condition if there is a constant $\gamma > 0$ such that $\int_\Delta p_w^2 \omega dA (\int_\Delta p_w^2 \omega^{-1} dA)^{-1} \leq \gamma$ for all $w \in \Delta$. We will consider the condition for a function ω on H. When $n = 2$, for $w = (s,t) \in H$ the Poisson kernel $P_w(x)$ is given by $P_w(x) = \gamma_2 t/|\overline{w}-(x,0)|^2$ $(x \in \partial H)$. Using a function $t/|\overline{w}-z|^2$ $(z \in H)$, we will define a $(A_p)_\partial-$condition on H.

Let $1 < p < \infty$, and ω be a non-negative L^1_{loc} function on H of $\mathbb{R}^n$. We say that the function ω satisfies the $(A_p)_\partial-$condition on H if there is a constant $\gamma > 0$ such that

$$\int_H \left(\frac{t}{|\overline{w}-z|^2}\right)^n \omega dV(z) \left(\int_H \left(\frac{t}{|\overline{w}-z|^2}\right)^n \omega^{\frac{-1}{p-1}} dV(z)\right)^{p-1} \leq \gamma$$

for all $w = (s,t) \in H$.

Since an elementary calculation shows that $\int_H \frac{1}{|\overline{w}-z|^{2n}} dV(z) = (n2^{n-1}t^n)^{-1}$, ω is bounded and bounded below then ω satisfies the $(A_p)_\partial-$condition. Moreover, since $|\overline{w}-z| \leq \sqrt{10}\, t$ for $z \in S(w)$ and there is a constant $0 < \varepsilon < 1$ such that $D_\varepsilon(w) \subset S(w)$ for all $w \in H$, there are constants $C, C' > 0$ such that

$$\frac{1}{V(D_\varepsilon(w))}\int_{D_\varepsilon(w)} \omega dV \leq C \frac{1}{V(S(w))}\int_{S(w)} \omega dV \leq C' \int_H \left(\frac{t}{|\overline{w}-z|^2}\right)^n \omega dV(z)$$

for all $w \in H$. Therefore, the $(A_p)_\partial$-condition implies the C_p-condition which is defined in [8]. Since ω satisfies the C_p-condition, ω satisfies the doubling condition. Hence, for $0 < \varepsilon, \delta < 1$ there is a constant $C > 0$ such that $\int_{D_\varepsilon(w)} \omega dV \leq C \int_{D_\varepsilon(\zeta)} \omega dV$ whenever $\rho(w,\zeta) < \delta$ (see Corollary 3.8 in [8]).

3. Sufficient condition for the inequality

We give a sufficient condition for measures μ and ν which satisfy the inequality in (1) of Theorem 1, when $d\nu = \omega dV$.

PROPOSITION 2. *Let $0 < p \le 1$, $1 < q < \infty$, and $k > n/p$. Suppose that ℓ, m be nonnegative integers. Assume that μ is a σ-finite positive Borel measure on H and $d\nu = \omega dV$ such that $\omega \in L^1_{loc}(H, dV)$. If there are constants $K > 0$ and $0 < \varepsilon < 1$ such that*

$$\int_H \frac{t^{p(m+k+n)-nq}}{|\overline{z} - w|^{p(n+\ell+k)}} d\mu(z) \le K \left(\int_{D_\varepsilon(w)} \omega^{\frac{-1}{q-1}} dV \right)^{-(q-1)}$$

for all $w = (s,t) \in H$, then there is a constant $C > 0$ such that

$$\int_H |D^\alpha u|^p d\mu \le C \int_H |D^m_y u|^p d\nu$$

for all $u \in b^p$ and multi-indices α of order ℓ.

PROOF. Let $u \in b^p$. By Lemma 5 and the remark above Lemma 4, we have

$$\begin{aligned} |D^\alpha u(w)| &\le C \int_H |y^{m+k} (D^m_y u)(z) D^\alpha_w D^{k+1}_y P_w(z)| dV(z) \\ &\le C \int_H \frac{y^{m+k}}{|\overline{w} - z|^{n+\ell+k}} |D^m_y u(z)| dV(z) \end{aligned}$$

For $0 < \varepsilon < 1/2$, by (3) of Lemma 1, we can choose a integer N and a sequence $\{\zeta_j\}$ in H satisfying the conditions : (a) $H = \cup D_\varepsilon(\zeta_j)$, (b) any point in H belongs to at most N of the sets $D_{2\varepsilon}(\zeta_j)$. We will write $\zeta_j = (\xi_j, \eta_j)$. Since $D^m_y u$ is harmonic, Lemma 2 in [4, §9] implies that $|D^m_y u(z)|^{p/q} \le C/y^n \int_{D_\varepsilon(z)} |D^m_y u|^{p/q} dV$. Therefore, (1) and (2) of Lemma 1 show that

$$\begin{aligned} |D^\alpha u(w)| &\le C \sum_j \int_{D_\varepsilon(\zeta_j)} \frac{y^{m+k}}{|\overline{w} - z|^{n+\ell+k}} |D^m_y u(z)| dV(z) \\ &\le C \sum_j \frac{\eta_j^{m+k}}{|\overline{w} - \zeta_j|^{n+\ell+k}} \int_{D_\varepsilon(\zeta_j)} \left(\frac{1}{y^n} \int_{D_\varepsilon(z)} |D^m_y u|^{p/q} dV \right)^{q/p} dV(z) \\ &\le C \sum_j \frac{\eta_j^{m+k}}{|\overline{w} - \zeta_j|^{n+\ell+k}} \int_{D_\varepsilon(\zeta_j)} \left(\frac{1}{\eta_j^n} \int_{D_{2\varepsilon}(\zeta_j)} |D^m_y u|^{p/q} \omega^{1/q} \omega^{-1/q} dV \right)^{q/p} dV(z) \\ &\le C \sum_j \frac{\eta_j^{m+k+n-nq/p}}{|\overline{w} - \zeta_j|^{n+\ell+k}} \left(\int_{D_{2\varepsilon}(\zeta_j)} |D^m_y u|^p \omega dV \right)^{1/p} \left(\int_{D_{2\varepsilon}(\zeta_j)} \omega^{\frac{-1}{q-1}} dV \right)^{(q-1)/p} \\ &\le C \sum_j \left(\int_{D_{2\varepsilon}(\zeta_j)} \left[\frac{y^{p(m+k+n)-nq}}{|\overline{w} - z|^{p(n+\ell+k)}} \left(\int_{D_{4\varepsilon}(z)} \omega^{\frac{-1}{q-1}} dV \right)^{q-1} \right] |D^m_y u|^p \omega dV(z) \right)^{1/p} \\ &\le C \left(\sum_j \int_{D_{2\varepsilon}(\zeta_j)} \left[\frac{y^{p(m+k+n)-nq}}{|\overline{w} - z|^{p(n+\ell+k)}} \left(\int_{D_{4\varepsilon}(z)} \omega^{\frac{-1}{q-1}} dV \right)^{q-1} \right] |D^m_y u|^p \omega dV(z) \right)^{1/p} \\ &\le C \left(N \int_H \left[\frac{y^{p(m+k+n)-nq}}{|\overline{w} - z|^{p(n+\ell+k)}} \left(\int_{D_{4\varepsilon}(z)} \omega^{\frac{-1}{q-1}} dV \right)^{q-1} \right] |D^m_y u|^p \omega dV(z) \right)^{1/p}. \end{aligned}$$

Thus, integrating p-th power of the inequality with respect to μ, Fubini's theorem implies that

$$\int_H |D^\alpha u(w)|^p d\mu(w) \le C \int_H \left[\int_H \frac{y^{p(m+k+n)-nq}}{|\overline{w}-z|^{p(n+\ell+k)}} d\mu(w) \left(\int_{D_{4\varepsilon}(z)} \omega^{\frac{-1}{q-1}} dV \right)^{q-1} \right] |D_y^m u|^p \omega dV(z).$$

This completes the proof.

4. Necessary condition for the inequality

We give a necessary condition for measures μ and ν which satisfy the inequality in (2) of Theorem 1. When $w = (s,t)$ in H, we may write a Carleson box $S(w) = S(s,t)$. We need the following lemma, and Lemma 6 is stated in [13].

LEMMA 6. *Let k be a nonnegative integer. Then, there exist constants $0 < \sigma \le 1$ and $C > 0$ such that $|D_y^k P_w(z)| \ge C/t^{n+k-1}$ for all $w = (s,t) \in H$ and $z \in S(s, \sigma t)$.*

In Lemma 6, we do not know that the constant σ can be taken $\sigma = 1$. We give a necessary condition for the inequality.

PROPOSITION 3. *Let $0 < p \le 1$, and k be a nonnegative integer which is sufficiently large. Suppose that ℓ, m be nonnegative integers. Assume that μ and ν are σ-finite positive Borel measures on H. If there is a constant $C > 0$ such that*

$$\int_H |D_y^\ell u|^p d\mu \le C \int_H |D_y^m u|^p d\nu$$

for all $u \in b^p$, then there are constants $0 < \sigma \le 1$ and $K = K_\sigma > 0$ such that

$$\mu(S(s,\sigma t)) \le K t^{p(\ell+n+k)} \int_H \frac{1}{|\overline{w}-z|^{p(n+m+k)}} d\nu$$

for all $w = (s,t) \in H$.

PROOF. Suppose that the inequality in (2) of Theorem 1 is satisfied. We can choose a nonnegative integer k such that $u(z) = (D_y^{k+1} P_w)(z)$ is in b^p. Then, we have

$$\int_H |D_y^m u|^p d\nu = \int_H |D_y^{m+k+1} P_w|^p d\nu \le C \int_H \frac{1}{|\overline{w}-z|^{p(n+m+k)}} d\nu.$$

Moreover, Lemma 6 implies that

$$\begin{aligned} \int_H |D_y^\ell u|^p d\mu &= \int_H |D_y^{\ell+k+1} P_w|^p d\mu \ge \int_{S(s,\sigma t)} |D_y^{\ell+k+1} P_w|^p d\mu \\ &\ge \frac{C_\sigma}{t^{p(\ell+n+k)}} \int_{S(s,\sigma t)} d\mu = \frac{C_\sigma}{t^{p(\ell+n+k)}} \mu(S(s,\sigma t)). \end{aligned}$$

Therefore, it follows that

$$\frac{C_\sigma}{t^{p(\ell+n+k)}} \mu(S(s,\sigma t)) \le C \int_H \frac{1}{|\overline{w}-z|^{p(n+m+k)}} d\nu.$$

5. Proof of Theorem 1

We give a proof of Theorem 1. The implication (1) $\Rightarrow$ (2) is trivial. Therefore, we show that (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1).

(2) $\Rightarrow$ (3). We suppose that the inequality in (2) of Theorem 1 is hold. Then, Proposition 3 implies that there are constants $0 < \sigma \le 1$ and $K = K_\sigma > 0$ such that $\mu(S(s,\sigma t)) \le Kt^{p(\ell+n+k-1)} \int_H 1/|\overline{w}-z|^{p(m+k+n-1)} d\nu$ for all $w=(s,t) \in H$. Since $|\overline{w}-z| \ge t$, We have $\mu(S(s,\sigma t)) \le Kt^{p(\ell-m)+n} \int_H t^n/|\overline{w}-z|^{2n} d\nu$. Moreover, since ω satisfies the $(A_q)_\partial$-condition, we obtain $\mu(S(s,\sigma t)) \le Kt^{p(\ell-m)}\nu(D_\varepsilon(s,\sigma t))$. Since s and t are arbitrary, we can replace t by t/σ. This implies that $\mu(S(w)) \le Ct^{p(\ell-m)}\nu(D_\varepsilon(w))$.

(3) $\Rightarrow$ (1). Let $c = p(\ell-m)$ and suppose that $\mu(S(\zeta)) \le K\eta^c\nu(D_\varepsilon(\zeta))$ for all $\zeta = (\xi,\eta) \in H$. Since ω satisfies the $(A_q)_\partial$-condition, the sufficient condition in Proposition 2 is equivalent to a condition $\int_H t^{p(n+m+k)}/|\overline{w}-z|^{p(n+\ell+k)} d\mu(z) \le K\nu(D_\varepsilon(w))$. Therefore, it is enough to prove that $\int_H 1/|\overline{w}-z|^\gamma d\mu(z) \le Ct^{c-\gamma}\nu(D_\varepsilon(w))$ for all $w=(s,t)\in H$, where $\gamma = p(n+\ell+k)$ and k is sufficiently large. Let $w \in H$. Clearly, if $z \notin S(s,2^{j-1}t)$, then $|w-\bar{z}| \ge 2^{j-1}t$ $(j \ge 1)$. Therefore, the hypothesis implies that

$$\begin{aligned}
\int_H \frac{1}{|w-\bar{z}|^\gamma} d\mu(z) &\le t^{-\gamma}\int_{S(s,t)} d\mu + t^{-\gamma}\sum_{j=1}^\infty \frac{1}{2^{\gamma(j-1)}}\int_{S(s,2^j t)\setminus S(s,2^{j-1}t)} d\mu \\
&\le t^{-\gamma}\mu(S(s,t)) + t^{-\gamma}\sum_{j=1}^\infty \frac{1}{2^{\gamma(j-1)}}\mu(S(s,2^j t)) \\
&\le Kt^{c-\gamma}\nu(D_\varepsilon(s,t)) + Kt^{-\gamma}\sum_{j=1}^\infty \frac{1}{2^{\gamma(j-1)}}(2^j t)^c \nu(D_\varepsilon(s,2^j t)) \\
&= Kt^{c-\gamma}\left(\nu(D_\varepsilon(s,t)) + 2^\gamma \sum_{j=1}^\infty \frac{1}{2^{(\gamma-c)j}}\nu(D_\varepsilon(s,2^j t))\right)
\end{aligned}$$

Since ω satisfies the $(A_q)_\partial$-condition, ω satisfies the C_q-condition. Therefore, Corollary 3.8 in [8] implies that there is a constant $\lambda > 0$ such that $\nu(D_\varepsilon(s,2t)) \le 2^\lambda \nu(D_\varepsilon(s,t))$. Hence, we have

$$\begin{aligned}
\int_H \frac{1}{|w-\bar{z}|^\gamma} d\mu(z) &\le Kt^{c-\gamma}\left(\nu(D_\varepsilon(w)) + 2^\gamma \sum_{j=1}^\infty \frac{1}{2^{(\gamma-c)j}} 2^{\lambda j}\nu(D_\varepsilon(w))\right) \\
&= Kt^{c-\gamma}\left(1 + 2^\gamma \sum_{j=1}^\infty \frac{1}{2^{(\gamma-c-\lambda)j}}\right)\nu(D_\varepsilon(w)).
\end{aligned}$$

If we choose an integer k such that $\gamma - c - \lambda = p(n+m+k) - \lambda > 0$, then we obtain $\int_H 1/|\overline{w}-z|^\gamma d\mu(z) \le Ct^{c-\gamma}\nu(D_\varepsilon(w))$.

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