CUSTOMS vs. SMUGGLER GAME WITH A RANDOM AMOUNT OF CARGO

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Abstract. Customs vs. Smuggler (player I and II, respectively) game where the amount of cargo is a random variable is discussed. II wants to cross the strait a motorboat carrying illegal cargo during one of n nights. I wants to stop it, and can patrol at most k nights. The amount X_i of cargo in the i-th night is supposed to be $U_{[0,1]}$-distributed random variable. We suppose that the realized value of X_i in each night is, by some information agent, communicated to I. Payoff to I is $X_i(-X_i)$, if patrol-go (no=patrol-go) are chosen. I (II) wants to maximize (minimize) the expected payoff to I. This game G_k^n is formulated and solved by deriving the triangular recursion of values $V_k^n = \mathrm{Val}(G_k^n), 1 \le k \le n, n = 1, 2, \cdots$. It is shown that $V_k^n \downarrow -1$ as $n \to \infty$, for every fixed k.

1 Customs vs. Smuggler Game with a Random Amount of Cargo. Let $X_i, i = 1, 2, \cdots, n$, be *i.i.d.* random variables each with uniform distribution on $[0, 1]$. As each X_i comes up, each player I and II must choose simultaneously and independently of other player's choice, either to accept (A) or to reject (R) it. If the choices are R-A (A-A), then player I gets the amount zero (X_i), and the game terminates. If the choices are R-R or A-R, then the X_i is rejected, the next X_{i+1} is presented and the game continues.

(*) Player $\left\{\begin{array}{c}\text{II}\\ \text{I}\end{array}\right\}$ must choose A $\left\{\begin{array}{c}\text{just once}\\ \text{at most } k \text{ times}\end{array}\right\}$ during the n stages, where $1 \le k \le n$. Player I (II) aims to maximize (minimize) the expected payoff to I.

Let v_k^n be the value of the n-stage game (Γ_k^n, say). Then the Optimality Equation is

$$v_k^n = \mathrm{E}\left[\mathrm{val}\left\{\begin{array}{c} \\ \text{R} \\ \text{A}\end{array} \begin{array}{cc} \text{R} & \text{A} \\ \multicolumn{2}{c}{\left(\begin{array}{cc} v_k^{n-1} & 0 \\ v_{k-1}^{n-1} & X \end{array}\right)}\end{array}\right\}\right] \tag{1.1}$$

with the boundary conditions

$$v_0^n = 0, \qquad \forall n \ge 0, \tag{1.2}$$

$$v_n^n = \nu_n \qquad (n \ge 1, \nu_1 = 1/2)\,, \tag{1.3}$$

where the sequence $\{\nu_n\}$ is determined by the recursion

$$\nu_n = \nu_{n-1} - \frac{1}{2}\nu_{n-1}^2 \qquad (n \ge 1, \nu_1 = 1/2)\,. \tag{1.4}$$

There is another closely related game, G_k^n, say. The only one difference from the game Γ_k^n is : If the choices are R-A, then player I pays player II the amount X_i (insted of zero).

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So, denoting by V_k^n the value of the game G_k^n, the Optimality Equation is

$$V_k^n = \mathrm{E}\left[\mathrm{val}\left\{ \begin{array}{c} \\ \mathrm{R} \\ \mathrm{A} \end{array} \begin{array}{c} \begin{array}{cc} \mathrm{R} & \mathrm{A} \end{array} \\ \begin{pmatrix} v_k^{n-1} & -X \\ v_{k-1}^{n-1} & X \end{pmatrix} \end{array} \right\}\right] \tag{1.5}$$

with the boundary conditions

$$V_0^n = -\mu_n \qquad (n \geq 1, \mu_1 = 1/2) \tag{1.6}$$

$$V_n^n = \nu_n \qquad (n \geq 1, \nu_1 = 1/2) \tag{1.7}$$

where $\{\mu_n\}$ is so-called Moser's sequence determined by the recursion

$$\mu_n = \frac{1}{2}\left(1 + \mu_{n-1}^2\right) \qquad (n \geq 1, \mu_0 = 0). \tag{1.8}$$

The solutions to the games Γ_k^n and G_k^n are given in Sections 2 and 3, respectively, together with the proofs of the boundary conditions (1.2)-(1.3) and (1.6)-(1.7) for these games.

These games correspond to the scene found in the Customs vs. Smuggler game, where
Player I (Customs) : A=patrol, R=no patrol.
Player II (Smuggler) : A=go, R=don't go.

II wants to cross the strait by a motorboat carrying illegal cargo during one of n nights. I wants to stop it, and can patrol at most k nights. The amount of II's cargo is supposed to be $U_{[0,1]}$-distributed random variable. We suppose that its realized value in each night, by some information agent, communicated to I.

Or, another interpretation for Γ_k^n is ; X_i is the probability that I catches II, if I patrols and II goes in the same i-th night. It randomly changes depending of the weather, *etc.*

2 Solution to the Game Γ_k^n. We define state $(n, k|x)$ to mean that (1) the game still continues, n random values remain to be observed, and player I can choose A at most k times, and (2) the first random variable has just been observed with value x.

lemma 1.1 *Game Γ_k^n has the boundary conditions* (1.2) *and* (1.3).

Proof. If I cannot choose A during the whole stages, (1.1) becomes

$$v_0^n = v_0^{n-1} \wedge 0 = 0, \qquad \forall n \geq 1,$$

since we have evidently $v_0^n \geq 0, \forall n$. Player II chooses A in the first stage, for any $x \in [0,1]$, terminating the game. This proves (1.2).

If $k = n$, I doesn't choose R in the first stage, since the second row in the payoff matrix in (1.1) dominates the first row. It then follows that choosing A is optimal for I. Hence I chooses A during the whole stages. Therefore $\nu_n \equiv v_n^n$ satisfies

$$\nu_n = \mathrm{E}(\nu_{n-1} \wedge X), \qquad (\ \forall n \geq 2, \nu_1 = \mathrm{EX} = 1/2\ )$$

which gives the recursion (1.4). Note that $\nu_1 = 1/2$ is derived from the condition (*) of the game. Player II chooses A as soon as the stage $(n, n|x)$ satisfying $x < \nu_{n-1}$ appears. This proves that (1.3) is true. □

We find that $\nu_n \downarrow 0$, as $n \to \infty$, since (1.4) is rewritten as

$$\nu_n = T(\nu_{n-1}), \text{ with } T(y) = y - \frac{1}{2}y^2,$$

and $T(y), 0 \leq y \leq 1$, is concave and increasing with $T(0) = 0$ and $T(1) = 1/2$.

Theorem 1 *The value v_k^n of the game Γ_k^n is given by the triangular recursion*

$$v_k^n = \frac{1}{2}b^2 + \bar{b}a + a(a-b)\log\frac{a}{a-b+1} \qquad (1 \le k \le n, v_0^n = 0, v_n^n = \nu_n) \tag{2.1}$$

where $a = v_k^{n-1}, b = v_{k-1}^{n-1}$ and $0 < b < a < 1$.
As $n \to \infty, v_k^n \downarrow 0$, for every fixed k.
The optimal play in state $(n, k|x)$ is :
Both players choose A, if $x < b$;
I *and* II *employ the mixed strategies $\langle \bar{\alpha}(x), \alpha(x) \rangle$ and $\langle \bar{\beta}(x), \beta(x) \rangle$, resp., where $\alpha(x) = \frac{a}{x+a-b}$ and $\beta(x) = \frac{a-b}{x+a-b}$, if $x > b$.*

Proof. By induction on n, it is clear that

$$0 < b \equiv v_{k-1}^{n-1} \le v_k^{n-1} \equiv a < 1, \qquad \forall n \ge 1.$$

Hence in state $(n, k|x)$, we have

$$\text{val}\begin{pmatrix} a & 0 \\ b & x \end{pmatrix} = \begin{cases} x, & \text{if } x \le b \text{ (A-A is optimal)} \\ \dfrac{ax}{x+a-b}, & \text{if } x \ge b. \end{cases} \tag{2.2}$$

Therefore, from (1.1),

$$v_k^n = \int_0^b x\,dx + \int_b^1 \frac{ax}{x+a-b}dx = \frac{1}{2}b^2 + a\left\{\bar{b} + (a-b)\log\frac{a}{a-b+1}\right\}$$

which is (2.1). Also we have

$$v_k^n - v_k^{n-1} = v_k^n - a = \frac{1}{2}(a-b)^2 - \frac{1}{2}a^2 + a(a-b)\log\frac{a}{a-b+1} < 0,$$

since $0 < b < a < 1$.

$\{v_k^n\}_n$ converges. The limit α_k satisfies the recursion

$$0 = \frac{1}{2}(\alpha_k - \alpha_{k-1})^2 - \frac{1}{2}\alpha_k^2 + \alpha_k(\alpha_k - \alpha_{k-1})\log\frac{\alpha_k}{\alpha_k - \alpha_{k-1} + 1} \qquad (k = 1, 2, \cdots; \alpha_0 = 0)$$

and induction on k gives that $\alpha_k \equiv 0, \ \forall k$.
The rest part stated in the theorem is evident from (2.2). □

Table 1. $\{v_k^n\}$, for $k, n = 1(1)10$.

$\downarrow k$ \ $\to n$	1	2	3	4	5	6	7	8	9	10
1	500	225	139	99	75	60	50	42	36	32
2		375	253	185	158	124	101	85	73	64
3			305	240	193	164	139	120	104	103
4				258	219	187	162	142	126	113
5					225	198	175	156	140	126
6						200	178	161	146	134
7							180	163	148	137
8								164	149	138
9									150	139
10										139

In Table 1 we show $\{v_k^n\}$ values computed from (2.1). The first row is derived from the recursion

$$v_1^n = a - a^2 \log(1 + a^{-1}), \quad (a = v_1^{n-1} \text{ and } b = 0 \text{ in } (2.1))$$

and the unit of the figures is 0.001. They are under rounding errors.

For example, $v_3^{10} = 0.103$, and the optimal play in stay $(10, 3|x)$ is ;
Both players choose A, if $x < v_2^9 = 0.073$;
I and II employ the mixed strategies $\left\langle \frac{x - 0.073}{x + 0.031}, \frac{0.104}{x + 0.031} \right\rangle$ and $\left\langle \frac{x}{x + 0.031}, \frac{0.031}{x + 0.031} \right\rangle$, resp., if $x > 0.073$ (since $a - b = v_3^9 - v_2^9 = 0.031$).

The result in Theorem 1 is well compared with the case where the cargo has the fixed amount EX $= \frac{1}{2}$. Denote this game by $\widetilde{\Gamma}_k^n$. The equation corresponding to (2.2) is

$$w_k^n = \text{val} \begin{pmatrix} w_k^{n-1} & 0 \\ w_{k-1}^{n-1} & 1/2 \end{pmatrix} = \frac{\frac{1}{2} w_k^{n-1}}{w_k^{n-1} - w_{k-1}^{n-1} + 1/2},$$

with $w_0^n = 0$, and $w_n^n = 1/2$. This gives the very simple solution

$$w_k^n = k/(2n)$$

and the optimal strategy-pair in state (n, k)

$$\langle 1 - k/n, k/n \rangle \text{ for I, and } \left\langle 1 - n^{-1}, n^{-1} \right\rangle \text{ for II.}$$

We find, for example, $\text{Val}\left(\widetilde{\Gamma}_3^{10}\right) = \dfrac{3}{20} = 0.15$, whereas $v_3^{10} = 0.103$.

3 Solution to the Game $\mathrm{G}_\mathrm{k}^\mathrm{n}$.

Lemma 2.1 *Game G_k^n has the boundary condition* (1.6) *and* (1.7).

Proof. If I cannot choose A during the whole stages, (1.5) becomes

$$V_0^n = \mathrm{E}\left[V_0^{n-1} \wedge (-X)\right],$$

and hence $W_0^n = -V_0^n$ satisfies $W_0^n = \mathrm{E}\left(W_0^{n-1} \vee X\right)$, $(n \geq 1, W_0^0 = 0)$, implying that $\{W_0^n\}$ is identical to the Moser's sequence (1.8). Note that $V_0^1 = -\text{EX} = -\frac{1}{2}$ is derived from (*). Player II chooses A as soon as the state $(n, 0|x)$ with $x > \mu_{n-1}$ appears. This proves (1.6).

The proof of (1.7) is the same as in the proof of (1.3) made in Lemma 1.1. □

Theorem 2 (i). *The value V_k^n of the game G_k^n is given by* (3.4)-(3.5)*, where* $-1 < b \equiv V_{k-1}^{n-1} < a \equiv V_k^{n-1} < 1$.
(ii). *As $n \to \infty, V_k^n \downarrow -1$, for every fixed k.*
(iii). *The optimal play in state $(n, k|x)$ is ; In Case* 1 *(i.e., $a + b < 0$),*
Both players choose R*, if $x < -a$;*
I *and* II *employ the mixed strategies $\langle \bar{\alpha}(x), \alpha(x) \rangle$ and $\left\langle \bar{\beta}(x), \beta(x) \right\rangle$, resp., where* $\alpha(x) = \dfrac{x + a}{2x + a - b}$ *and* $\beta(x) = \dfrac{a - b}{2x + a - b}$*, if $x > -a$,*
and in Case 2 *(i.e., $a + b > 0$),*
Both players choose A, if $x < b$;
Players employ the same mixed strategies as in Case 1*, if $x > b$ (See Figure* 1*).*

Proof. (i) : Induction on n gives

$$-1 \le V_{k-1}^n \le V_k^n \le 1 \qquad \forall n \ge 1.$$

So, from (1.5), we have

$$V_k^n = \mathrm{E}\left[\mathrm{val}\begin{pmatrix} V_k^{n-1} & -X \\ V_{k-1}^{n-1} & X \end{pmatrix}\right] = -\frac{1}{2} + 2\mathrm{E}\left[\mathrm{val}\begin{pmatrix} \frac{1}{2}(a+X) & 0 \\ \frac{1}{2}(b+X) & X \end{pmatrix}\right], \tag{3.1}$$

where $a = V_k^{n-1}, b = V_{k-1}^{n-1}$ and $-1 \le b \le a \le 1$.

Let $M(x) = \begin{pmatrix} \frac{1}{2}(a+x) & 0 \\ \frac{1}{2}(b+x) & x \end{pmatrix}$, then

$$\mathrm{val}\ M(x) = \begin{cases} \frac{1}{2}(a+x), & \text{if } x < -a, \\ x, & \text{if } x < b, \\ \dfrac{x(a+b)}{2x+a-b} (\equiv g(x), \text{say}), & \text{if otherwise.} \end{cases}$$

We consider the two cases ; Case 1. $a+b<0$, and Case 2. $a+b>0$. (see Figure 1)

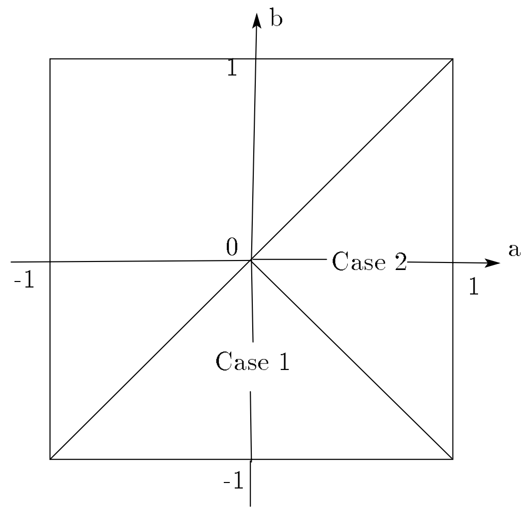

Figure 1. Domain of two cases.

Then we find that

$$\mathrm{val}\ M(x) = \begin{cases} \frac{1}{2}(a+x), & \text{if } x < -a \ \text{(R-R is optimal)} \\ g(x), & \text{if } x > -a, \end{cases} \tag{3.2}$$

in Case 1 (since A-A is not optimal), and

$$= \begin{cases} x, & \text{if } x < b \ \text{(A-A is optimal)} \\ g(x), & \text{if } x > b. \end{cases} \tag{3.3}$$

in Case 2 (since R-R is not optimal).

By performing the integration and considering (3.1), we find that ; In Case 1,

$$\mathrm{E\ val}\ M(x) = \begin{cases} \displaystyle\int_0^{-a} \frac{1}{2}(a+x)dx + \int_{-a}^1 g(x)dx, & \text{if } a<0 \ (i.e., 0<-a<1) \\ \displaystyle\int_0^1 g(x)dx, & \text{if } a>0. \end{cases}$$

and hence

$$(3.4)\qquad V_k^n = \begin{cases} -a^2 + \frac{1}{2}(a+b)(1+a) - \frac{1}{4}(a^2-b^2)\log\frac{2+a-b}{-(a+b)}, & \text{if } a \le 0, \\ \frac{1}{2}b + \frac{1}{4}b^2 \log\frac{2-b}{-b}, & \text{if } a = 0, \\ \frac{1}{2}(a+b) - \frac{1}{4}(a^2-b^2)\log\frac{2+a-b}{a-b}, & \text{if } a \ge 0. \end{cases}$$

In case 2, the analogous computation gives the result ;

$$\text{E val } M(x) = \begin{cases} \int_0^1 g(x)dx, & \text{if } b < 0, \\ \int_0^b xdx + \int_b^1 g(x)dx, & \text{if } b > 0 \end{cases}$$

and hence

$$(3.5)\qquad V_k^n = \begin{cases} \frac{1}{2}(a+b) - \frac{1}{4}(a^2-b^2)\log\frac{2+a-b}{a-b}, & \text{if } b \le 0, \\ \frac{1}{2}a - \frac{1}{4}a^2 \log\frac{2+a}{a}, & \text{if } b = 0, \\ \frac{1}{2}(a+b-ab) - \frac{1}{4}(a^2-b^2)\log\frac{2+a-b}{a+b}, & \text{if } b \ge 0. \end{cases}$$

For the bordering case of Cases 1 and 2 *i.e.*, $a+b=0$, both of (3.4) and (3.5) give the same value 0.

(ii) : We want to prove that V_k^n is decreasing in n for fixed k.

In Case 1, we have, from (3.4),

$$(3.6)\qquad V_k^n - V_k^{n-1} = V_k^n - a = \begin{cases} \frac{1}{2}(1+a)(b-a) + \frac{1}{4}(a^2-b^2)\log\frac{-(a+b)}{2+a-b}, & \text{if } a \le 0, \\ \frac{1}{2}(b-a) + \frac{1}{4}(a^2-b^2)\log\frac{a-b}{2+a-b}, & \text{if } a \ge 0. \end{cases}$$

Using the universal inequality $\alpha \log(\alpha/\beta) \ge \alpha - \beta$, we have

$$\frac{1}{4}(b-a)(-(a+b))\log\frac{-(a+b)}{2+a-b} \le \frac{1}{4}(b-a)(-2)(1+a) = \frac{1}{2}(a-b)(1+a),$$

$$\frac{1}{4}(a+b)(a-b)\log\frac{a-b}{2+a-b} \le \frac{1}{4}(a+b)(-2) = -\frac{1}{2}(a+b).$$

Thus $V_k^n - V_k^{n-1} < 0$, for $\forall a \in (-1,1)$.

In Case 2, we have, from (3.5),

$$(3.7)\qquad V_k^n - V_k^{n-1} = V_k^n - a = \begin{cases} \frac{1}{2}(b-a) + \frac{1}{4}(a^2-b^2)\log\frac{a-b}{2+a-b}, & \text{if } b \le 0, \\ \frac{1}{2}(b-a-ab) + \frac{1}{4}(a^2-b^2)\log\frac{a+b}{2+a-b}, & \text{if } b \ge 0. \end{cases}$$

Here, $a^2 - b^2 = (a+b)(a-b) > 0$, and

$$b - a - ab < 0, \ \text{ if } (-a) \vee 0 < b < a.$$

Therefore all four terms are negative. So, $V_k^n < V_k^{n-1}$ follows. $\{V_k^n\}_n$ converges. The limit α_k satisfies the recursion $D_k = 0, (k \geq 0, \alpha_0 = -1)$, where D_k is given by the r.h.s. of (3.6)-(3.7), with a, b replaced by α_k, α_{k-1}. Suppose that $\alpha_{k-1} = -1$. Then $\alpha_k + \alpha_{k-1} = \alpha_k - 1 < 0$. So, Case 1 applies, and

$$D_k = \begin{cases} -(1+\alpha_k)\left\{\dfrac{1}{2}(1+\alpha_k) + \dfrac{1}{4}(1-\alpha_k)\log\dfrac{1-\alpha_k}{3+\alpha_k}\right\}, & \text{if } \alpha_k \leq 0, \\ -(1+\alpha_k)\left\{\dfrac{1}{2} + \dfrac{1}{4}(1-\alpha_k)\log\dfrac{1+\alpha_k}{3+\alpha_k}\right\}, & \text{if } \alpha_k \geq 0. \end{cases}$$

Then $D_k = 0$ gives a unique root $\alpha_k = -1$, since the equation $(1-t)\log\dfrac{1+t}{3+t} = -2$ has no root in $t \geq 0$. It follows by induction arguments, that $\alpha_k = -1, \forall k \geq 0$.

(iii) : Evident from (3.1)∼(3.3) in the proof of part (i).

Thus we have completed the proof of Theorem 2. □

Table 2. $\{V_k^n\}$, for $k, n = 1(1)10$.

k \ n →	1	2	3	4	5	6	7	8	9	10
0	-500	-625	-695	-742	-775	-800	-820	-836	-850	-861
1	500	0	-172	-268	-334	-385	-427	-460	-487	-513
2		375	123	-17	-103	-166	-215	-256	-291	-321
3			305	161	56	-43	-87	-127	-162	-193
4				258	172	94	21	-28	-66	-99
5					225	171	114	58	13	-23
6						200	164	123	81	42
7							180	155	125	93
8								164	145	123
9									150	136
10										

In Table 2, we show the values of $\{V_k^n\}$, computed from (3.4)-(3.5). They are decreasing in n, for every fixed k, and increasing in k for every fixed n. The figures are in 0.001 unit, and subject to rounding errors. The upper(lower) part of the bold line in the table corresponds to Case 1(Case 2). $V_1^2 = 0$ is on the bordering case.

We see that, for example, $V_3^{10} = -0.193$, and the optimal play in state $(10, 3|x)$ is : Both players choose R, if $x < -V_3^9 = 0.162$;

$$\left\{\begin{array}{c} \text{I} \\ \text{II} \end{array}\right\} \text{employs the mixed strategy} \left\{\begin{array}{ll} \langle\bar{\alpha}(x), \alpha(x)\rangle, & \alpha(x) = \frac{x-0.162}{2x+0.129} \\ \langle\bar{\beta}(x), \beta(x)\rangle, & \beta(x) = \frac{0.129}{2x+0.129} \end{array}\right\}, \text{ if } x > 0.162,$$

(since $a - b = V_3^9 - V_2^9 = 0.129$)

The result in Theorem 2 is well compared with the case where the cargo carries the fixed amount 1. Then the equation corresponding to (1.5) is

$$V_k^n = \text{val}\begin{pmatrix} V_k^{n-1} & -1 \\ V_{k-1}^{n-1} & 1 \end{pmatrix} \tag{3.8}$$

with $V_0^n = -1$, and $V_n^n = 0$. Let $W_k^n = \frac{1}{2}(1 - V_k^n)$. Then (3.8) becomes

$$W_k^n = \text{val} \begin{pmatrix} W_k^{n-1} & W_{k-1}^{n-1} \\ 1 & 0 \end{pmatrix} \tag{3.9}$$

with $W_0^n = 1$ and $W_n^n = \frac{1}{2}$. Baston and Bostock [1], Garnaev [3] and Sakaguchi [5] suggest that

$$W_k^n = s_k^{n-1}/s_k^n, \quad \text{where } s_k^n = \sum_{j=0}^{k} \binom{n}{j} \tag{3.10}$$

is the solution of (3.9) which satisfies the two boundary conditions. Proof is as follows : Equation (3.9) gives

$$W_k^n = \frac{W_{k-1}^{n-1}}{1 - W_k^{n-1} + W_{k-1}^{n-1}},$$

which is rewritten as

$$\frac{1}{W_k^{n-1}} \left(\frac{1}{W_k^n} - 1 \right) = \frac{1}{W_{k-1}^{n-1}} \left(\frac{1}{W_k^{n-1}} - 1 \right). \tag{3.11}$$

By using the identity $s_k^n = s_k^{n-1} + s_{k-1}^{n-1}$, we find that both sides of (3.11) substituted by (3.10) are equal to the same s_{k-1}^{n-1}/s_k^{n-2}.

Summarizing the above we arrive at : The solution of the equation (3.6) is

$$V_k^n = 1 - 2W_k^n = 1 - 2s_k^{n-1}/s_k^n = -\binom{n-1}{k} \Big/ \sum_{j=0}^{k} \binom{n}{j}.$$

Denote by $\widetilde{G}_k^n$ the game G_k^n with X_i replaced by a fixed constant $\frac{1}{2}$. Then

$$\text{Val } \widetilde{G}_3^{10} = -\frac{1}{2} \binom{9}{3} \Big/ \sum_{j=0}^{3} \binom{10}{j} \cong -0.2386,$$

whereas $\text{Val}G_3^{10} = V_3^{10} \cong -0.193$.

4 Final Remarks.

1. As $n \to \infty$, Val $\Gamma_k^n \downarrow 0$, and Val $G_k^n \downarrow -1$ for every fixed k. The same is true for non-random version, *i.e.*, Val $\widetilde{\Gamma}_k^n \downarrow 0$ and Val $\widetilde{G}_k^n \downarrow -\frac{1}{2}$.

2. Multistage games discussed in the present paper has some variants. One of the open problem is the case where Smuggler must cross the strait twice (or more generally $m \geq 2$ times). Let $(1 \leq)\tau_1 \leq \tau_2 (\leq n)$ be II's "go" stages. Payoff to I is $\sum_{i=1,2} X_{\tau_i}$. The games $\widetilde{\Gamma}_k^n$ along this line of extension are investigated in [2, 4, 5].

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