ON BH-RELATIONS IN BH-ALGEBRAS

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ABSTRACT. As a generalization of a BH-homomorphism, the notion of a relation on BH-algebras, called a *BH-relation*, is introduced. Some fundamental properties related to BH-subalgebras are discussed.

1. INTRODUCTION

It is well-known that the class of BCH-algebras is a generalization of the class of BCK/BCI-algebras. It is important for us to generalize some algebraic structures. Jun, Roh and Kim [2] introduced a new notion, called a *BH-algebra*, which is a generalization of $BCK/BCI/BCH$-algebras. In this paper, we introduce the notion of a relation on BH-algebras, called a *BH-relation*, which is a generalization of a BH-homomorphism, and then we discuss the fundamental properties related to BH-subalgebras.

2. PRELIMINARIES

A *BH-algebra* is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following conditions:

(I) $x * x = 0$,
(II) $x * y = 0$ and $y * x = 0$ imply $x = y$
(III) $x * 0 = x$

for all $x, y \in X$. A nonempty subset S of a BH-algebra X is called a *BH-subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A nonempty subset J of a BH-algebra X is called a *BH-ideal* of X if it satisfies

- $0 \in J$.
- $\forall x, y \in X$, $x * y \in J$, $y \in J \Rightarrow x \in J$.

A mapping $f : X \to Y$ of BH-algebras is called a *BH-homomorphism* if $f(x*y) = f(x)*f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a BH-homomorphism, then $f(0_X) = 0_Y$, where 0_X and 0_Y are constants of X and Y, respectively.

3. BH-RELATIONS

Definition 3.1. Let X and Y be BH-algebras. A nonempty relation $\mathcal{H} \subseteq X \times Y$ is called a *BH-relation* if

(R1) for every $x \in X$ there exists $y \in Y$ such that $x\mathcal{H}y$,
(R2) $x\mathcal{H}a$ and $y\mathcal{H}b$ imply $(x * y)\mathcal{H}(a * b)$.

We usually denote such relation by $\mathcal{H} : X \to Y$. It is clear from (R1) and (R2) that $0_X\mathcal{H}0_Y$.

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Example 3.2. Consider a proper BH-algebra $X = \{0, a, b\}$ having the following Cayley table (see [2]):

$*$	0	a	b
0	0	a	b
a	a	0	a
b	b	a	0

Define a relation $\mathcal{H} : X \to X$ as follows: $0\mathcal{H}0$, $a\mathcal{H}a$, $b\mathcal{H}b$. It is easy to verify that $\mathcal{H}$ is a BH-relation. A relation $\mathcal{D} : X \to X$ given by $0\mathcal{D}0$, $0\mathcal{D}a$, $a\mathcal{D}0$, $a\mathcal{D}a$, $b\mathcal{D}0$, and $b\mathcal{D}a$ is a BH-relation.

Theorem 3.3. *Every BH-homomorphism is a BH-relation.*

Proof. Let $\mathcal{H} : X \to Y$ be a BH-homomorphism. Clearly, $\mathcal{H}$ satisfies conditions (R1) and (R2). □

Note that every diagonal BH-relation on a BH-algebra X (i.e., a BH-relation satisfying $x\mathcal{H}x$ for all $x \in X$ in which $x\mathcal{H}y$ is false whenever $x \neq y$) is clearly a BH-homomorphism. But, in general, the converse of Theorem 3.3 need not be true as seen in the following example.

Example 3.4. The BH-relation $\mathcal{D}$ in Example 3.2 is not a BH-homomorphism.

Let $\mathcal{H} : X \to Y$ be a BH-relation. For any $x \in X$ and $y \in Y$, let

$$\mathcal{H}[x] := \{y \in H \mid x\mathcal{H}y\} \quad \text{and} \quad \mathcal{H}^{-1}[y] := \{x \in X \mid x\mathcal{H}y\}.$$

Note that $\mathcal{H}[x]$ and $\mathcal{H}^{-1}[y]$ are not BH-subalgebras of Y and X, respectively, as seen in the following example:

Example 3.5. Let $\mathcal{H}$ be a BH-relation in Example 3.2(1). Then $\mathcal{H}^{-1}[b] = \{b\}$ (resp. $\mathcal{H}[a] = \{a\}$) is not a BH-subalgebra of X (resp. Y).

Theorem 3.6. *For any BH-relation $\mathcal{H} : X \to Y$, we have*

(i) *$\mathcal{H}[0_X]$, called the zero image of $\mathcal{H}$, is a BH-subalgebra of Y.*
(ii) *$\mathcal{H}^{-1}[0_Y]$, called the kernel of $\mathcal{H}$ and denoted by* Ker$\mathcal{H}$, *is a BH-subalgebra of X.*

Proof. (i) Let $y_1, y_2 \in \mathcal{H}[0_X]$. Then $0_X\mathcal{H}y_1$ and $0_X\mathcal{H}y_2$. It follows from (R2) and (I) that $0_X\mathcal{H}(y_1 * y_2)$, that is, $y_1 * y_2 \in \mathcal{H}[0_X]$.

(ii) Let $x_1, x_2 \in \text{Ker}\mathcal{H}$. Then $x_1\mathcal{H}0_Y$ and $x_2\mathcal{H}0_Y$. By using (R2) and (I), we get $(x_1 * x_2)\mathcal{H}0_Y$ and so $x_1 * x_2 \in \text{Ker}\mathcal{H}$. This completes the proof. □

Proposition 3.7. *Let $\mathcal{H} : X \to Y$ be a BH-relation.*

(i) *If $\mathcal{H}[a] \cap \mathcal{H}[b] \neq \emptyset$ where $a, b \in X$, then $a * b \in \text{Ker}\mathcal{H}$.*
(ii) *If $\mathcal{H}^{-1}[u] \cap \mathcal{H}^{-1}[v] \neq \emptyset$ where $u, v \in Y$, then $u * v \in \mathcal{H}[0_X]$.*

Proof. (i) Let $a, b \in X$ be such that $\mathcal{H}[a] \cap \mathcal{H}[b] \neq \emptyset$. Taking $y \in \mathcal{H}[a] \cap \mathcal{H}[b]$, we have $a\mathcal{H}y$ and $b\mathcal{H}y$. It follows from (R2) and (I) that $(a*b)\mathcal{H}(y*y) = (a*b)\mathcal{H}0_Y$ so that $a*b \in \text{Ker}\mathcal{H}$.

(ii) Let $x \in \mathcal{H}^{-1}[u] \cap \mathcal{H}^{-1}[v]$. Then $x\mathcal{H}u$ and $x\mathcal{H}v$. Using (R2) and (I), we obtain $(x * x)\mathcal{H}(u * v) = 0_X\mathcal{H}(u * v)$, i.e., $u * v \in \mathcal{H}[0_X]$. This completes the proof. □

Theorem 3.8. *Let $\mathcal{H} : X \to Y$ be a BH-relation and let S be a BH-subalgebra of X. Then*

$$\mathcal{H}[S] := \{y \in H \mid x\mathcal{H}y \text{ for some } x \in S\}$$

is a BH-subalgebra of Y.

Proof. Clearly, $\mathcal{H}[S] \neq \emptyset$ since $0_X\mathcal{H}0_Y$. Let $y_1, y_2 \in \mathcal{H}[S]$. Then $x_1\mathcal{H}y_1$ and $x_2\mathcal{H}y_2$ for some $x_1, x_2 \in S$. Using (R2), we obtain $(x_1 * x_2)\mathcal{H}(y_1 * y_2)$ which implies that $y_1 * y_2 \in \mathcal{H}[S]$ since $x_1 * x_2 \in S$. Therefore $\mathcal{H}[S]$ is a BH-subalgebra of Y. □

Corollary 3.9. *Let $\mathcal{H} : X \to Y$ be a BH-relation. Then*

(i) $\mathcal{H}[X]$ *is a BH-subalgebra of Y.*
(ii) $\mathcal{H}[X] = \bigcup_{x \in X} \mathcal{H}[x]$.
(iii) *The zero image of $\mathcal{H}$ is a BH-subalgebra of $\mathcal{H}[X]$.*

Proof. (i) and (ii) are straightforward.

(iii) Let $a, b \in \mathcal{H}[0_X]$. Then $0_X\mathcal{H}a$ and $0_X\mathcal{H}b$, and hence $0_X\mathcal{H}(a * b)$, i.e., $a * b \in \mathcal{H}[0_X]$. Therefore $\mathcal{H}[0_X]$ is a BH-subalgebra of $\mathcal{H}[X]$. □

For any BH-relation $\mathcal{H} : X \to Y$, we know that there is a BH-ideal J of X in which $\mathcal{H}[J]$ is not a BH-ideal of Y. Indeed, consider the BH-relation $\mathcal{D}$ in Example 3.2. Note that $J := \{0, 2\}$ is a BH-ideal of X, but $\mathcal{H}[J] = \{0, 1\}$ is not a BH-ideal of X.

Theorem 3.10. *Let $\mathcal{H} : X \to Y$ be a BH-relation and let T be a BH-subalgebra of Y. Then*

$$\mathcal{H}^{-1}[T] := \{x \in X \mid x\mathcal{H}y \text{ for some } y \in T\}$$

is a BH-subalgebra of X.

Proof. Obviously, $\mathcal{H}^{-1}[T] \neq \emptyset$ since $0_X\mathcal{H}0_Y$. Let $x_1, x_2 \in \mathcal{H}^{-1}[T]$. Then there exist $y_1, y_2 \in T$ such that $x_1\mathcal{H}y_1$ and $x_2\mathcal{H}y_2$. Note that $y_1 * y_2 \in T$ since T is a subalgebra of Y. It follows from (R2) that $(x_1 * x_2)\mathcal{H}(y_1 * y_2)$ so that $x_1 * x_2 \in \mathcal{H}^{-1}[T]$. Hence $\mathcal{H}^{-1}[T]$ is a BH-subalgebra of X. □

Corollary 3.11. *Let $\mathcal{H} : X \to Y$ be a BH-relation. Then*

(i) $\mathcal{H}^{-1}[Y]$ *is a BH-subalgebra of X.*
(ii) $\mathcal{H}^{-1}[Y] = \bigcup_{y \in Y} \mathcal{H}^{-1}[y]$.
(iii) *The kernel of $\mathcal{H}$ is a BH-subalgebra of $\mathcal{H}^{-1}[Y]$.*

Proof. (i) and (ii) are straightforward.

(iii) Let $x, y \in \mathrm{Ker}\mathcal{H}$. Then $x\mathcal{H}0_Y$ and $y\mathcal{H}0_Y$. It follows from (R2) and (I) that

$$(x * y)\mathcal{H}(0_Y * 0_Y) = (x * y)\mathcal{H}0_Y$$

so that $x * y \in \mathrm{Ker}\mathcal{H}$. Hence $\mathrm{Ker}\mathcal{H}$ is a BH-subalgebra of $\mathcal{H}^{-1}[Y]$. This completes the proof. □

Open Problem 3.12. In Theorem 3.10, if T is a BH-ideal of Y, then is $\mathcal{H}^{-1}[T]$ a BH-ideal of X?

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