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EXISTENCE AND BOUNDEDNESS OF g_λ^*-FUNCTION AND MARCINKIEWICZ FUNCTIONS ON CAMPANATO SPACES

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Abstract. Let $g(f)$, $S(f)$, $g_\lambda^*(f)$ be the Littlewood-Paley g function, Lusin area function, and Littlewood-Paley g_λ^* function of f, respectively. In 1990 Chen Jiecheng and Wang Silei showed that if, for a BMO function f, one of the above functions is finite for a single point in $\mathbb{R}^n$, then it is finite a.e. on $\mathbb{R}^n$, and BMO boundedness holds. Recently, Sun Yongzhong extended this result to the case of Campanato spaces (i.e. Morrey spaces, BMO, and Lipschitz spaces). We improve his g_λ^* result further. His assumption is $\lambda > 3+2/n$. We show this is relaxed to $\lambda > \max(1, 2/p)$ $(-n/p \le \alpha < 0)$, $\lambda > 1$ $(0 \le \alpha < 1/2)$, and $\lambda > 1 + 2\alpha/n$ $(1/2 \le \alpha < 1)$. We also treat generalized Marcinkiewicz functions $\mu^\rho(f)$, $\mu_S^\rho(f)$ and $\mu_\lambda^{*,\rho}(f)$.

1. Introduction

In this note we study the existence and boundedness property of square function operators, such as Littlewood-Paley's g_λ^*-function and Marcinkiewicz functions, on Campanato spaces. First, we recall the definition of Littlewood-Paley's functions (generalized ones) in the n-dimensional Euclidean space $\mathbb{R}^n$.

Definition 1. A continuous function ψ on $\mathbb{R}^n$ is called an LP function, if there exist positive constants C_0, C_1, δ, η and γ such that

(i) $\psi \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \psi(x)\,dx = 0$;
(ii) $|\psi(x)| \le C_0(1+|x|)^{-n-\delta}$;
(iii) $|\psi(x+h) - \psi(x)| \le C_1|h|^\gamma(1+|x|)^{-n-\eta}$ for $|h| \le |x|/2$.

¿From (ii) and (iii) it follows

(iii') $\int_{\mathbb{R}^n} |\psi(x+h) - \psi(x)|\,dx \le C|h|^\gamma$ for $h \in \mathbb{R}^n$.

In fact, we have $\int_{\mathbb{R}^n} |\psi(x+h)-\psi(x)|\,dx \le \int_{|x|\ge 2|h|} |\psi(x+h)-\psi(x)|\,dx + \int_{|x|<2|h|}(|\psi(x+h)| + |\psi(x)|)\,dx \le C|h|^\gamma \int_{\mathbb{R}^n}(1+|x|)^{-n-\eta}\,dx + C\min(|h|^n, \int_{\mathbb{R}^n}(1+|x|)^{-n-\delta}\,dx) \le C|h|^\gamma$.

For an LP function we define Littlewood-Paley's g and Lusin's area functions as follows. Here and hereafter, $f_t(x)$ denotes $t^{-n}f(x/t)$.

$$g(f)(x) = \left(\int_0^\infty \frac{|\psi_t * f(x)|^2}{t}\,dt\right)^{\frac{1}{2}},$$

$$S(f)(x) = \left(\int_{\Gamma(x)} |\psi_t * f(y)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}},$$

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where $\Gamma(x) = \{(t,y) \in \mathbb{R}^{n+1}_+; |x-y| < t\}$.

$$g^*_\lambda(f)(x) = \left(\int_{\mathbb{R}^{n+1}_+} \Big(\frac{t}{t+|x-y|}\Big)^{\lambda n} |\psi_t * f(y)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}}$$

where $\lambda > 1$. L^p boundedness of these operators are known like as the classical Littlewood-Paley's g-functions. That is, g and S are L^p bounded for $1 < p < \infty$, and g^*_λ is L^p bounded for $1 < p < \infty$ if $\lambda > \max(1, \frac{2}{p})$ (see for example Torchinsky [12, pp. 309–318]). Here and hereafter, the letter C denotes a constant depending on main parameters and may vary at each occurrence.

Stein's generalization of the Marcinkiewicz function is as follows [8]: Let $\Omega(x)$ be a function on $\mathbb{R}^n$ which satisfies the following two conditions:

(i) $\Omega(x)$ is homogeneous of degree 0 and continuous on the unit sphere S^{n-1}, and satisfies for some $0 < \beta \le 1$

$$|\Omega(x') - \Omega(y')| \le C|x'-y'|^\beta, \qquad x', y' \in S^{n-1}.$$

(ii) $\int_{S^{n-1}} \Omega(x')\,d\sigma(x') = 0$, where $d\sigma$ is the surface Lebesgue measure on S^{n-1}.

Define $\mu(f)(x)$ by

$$\mu(f)(x) = \left(\int_0^\infty \frac{|\psi_t * f(x)|^2}{t}\,dt\right)^{\frac{1}{2}},$$

where $\psi(x) = \dfrac{\Omega(x)}{|x|^{n-1}} \chi_{\{|x|\le 1\}}$.

In their work on Marcinkiewicz integral, A. Torchinsky and S. Wang [13] introduced the Marcinkiewicz functions $\mu_S(f)$ and $\mu^*_\lambda(f)$ corresponding to the S function and g^*_λ function. On the other hand, in the connection of $\mu(f)$ a parametrized Marcinkiewicz function $\mu^\rho(f)$ was considered by L. Hörmander [3]. It corresponds to the case $\psi(x) = \Omega(x)|x|^{\rho-n} \times \chi_{\{|x|\le 1\}}$. Thus, we have considered in [15] parametrized $\mu^\rho_S(f)$ and $\mu^{*,\rho}_\lambda(f)$, where $\psi(x) = \Omega(x)|x|^{\rho-n}\chi_{\{|x|\le 1\}}$, for $\rho \in \mathbb{C}$ with $\operatorname{Re}\rho > 0$. L^p boundedness for these operators are well discussed in [7, 15], and will be used in this paper. We recall also the definition of Campanato spaces [5].

Definition 2. For $1 \le p < \infty$ and $-n/p \le \alpha \le 1$, the Campanato space $\mathcal{E}^{\alpha,p}$ is defined by the set of functions for which

$$\|f\|_{\mathcal{E}^{\alpha,p}} = \sup_{x_0 \in \mathbb{R}^n} \sup_B \frac{1}{|B|^{\alpha/n}} \left(\frac{1}{|B|}\int_B |f(x) - f_B|^p\,dx\right)^{1/p} < \infty,$$

where B moves over all balls centered at x_0, and f_B is the average of f over B, $(1/|B|)\int_B f(t)\,dt$.

It is known that for $0 < \alpha \le 1$, $\mathcal{E}^{\alpha,p} = \mathrm{Lip}_\alpha$: the Banach space of Lipschitz continuous functions of exponent α, and the norms are equivalent. If $\alpha = 0$, $\mathcal{E}^{\alpha,p}$ coincides with BMO: the space of functions of bounded mean oscillation. And if $\alpha < 0$, $\mathcal{E}^{\alpha,p}$ is equivalent to the Morrey space $L^{p,n+p\alpha}$. It is also easily checked that $\|f\|_{\alpha,p} \le C \sup_B \inf_{a\in\mathbb{C}} |B|^{-\alpha/n}\big(|B|^{-1}\int_B |f(x)-a|^p\,dx\big)^{1/p}$ $(-n/p \le \alpha \le 1)$, and hence these norms are equivalent. We note that balls can be replaced by cubes with sides parallel to the coordinate axes and the norms are equivalent. In [7, 14] we have deduced the boundedness from the existence of Marcinkiewicz functions on a set of positive measure. Recently, Sun Yongzhong [11] gives the following results, extending the BMO results by Wang and Chen [18].

Theorem 1. *Let $1 < p < \infty$ and $-n/p \le \alpha < \min(1, \delta, \gamma, \eta)$. If $f \in \mathcal{E}^{\alpha,p}$ and $g(f)(x_0)$ is finite for a point $x_0 \in \mathbb{R}^n$, then $g(f)(x) < \infty$ a.e. on $\mathbb{R}^n$, and there is a constant C independent of f, such that*

$$\|g(f)\|_{\mathcal{E}^{\alpha,p}} \le C\|f\|_{\mathcal{E}^{\alpha,p}}.$$

Theorem 2. *Let $1 < p < \infty$ and $-n/p \le \alpha < \max\big(\min(\frac{1}{2}, \delta), \min(\delta, \gamma, \eta)\big)$. If $f \in \mathcal{E}^{\alpha,p}$ and $S(f)(x_0)$ is finite for a point $x_0 \in \mathbb{R}^n$, then $S(f)(x) < \infty$ a.e. on $\mathbb{R}^n$, and there is a constant C independent of f, such that*

$$\|S(f)\|_{\mathcal{E}^{\alpha,p}} \le C\|f\|_{\mathcal{E}^{\alpha,p}}.$$

He shows the above results in the case $|\psi(x)|$, $(1+|x|)|\nabla\psi(x)| \le C(1+|x|)^{-n-1}$ ($\delta = 1$, $\eta = 2$ and $\gamma = 1$), but it is easily seen that his results hold in the above cases. He also gives the corresponding result for g_λ^* function. In this paper, we further improve his result on g_λ^* as follows.

Theorem 3. *Let $1 < p < \infty$, $-n/p \le \alpha < \min(1, \delta)$ and $\lambda > \lambda_0$, where $\lambda_0 = \max(1, 2/p)$ $(-n/p \le \alpha < 0)$, $\lambda_0 = 1$ $(0 \le \alpha < 1/2)$, and $\lambda_0 = 1 + 2\alpha/n$ $(1/2 \le \alpha < \min(1, \delta, \gamma, \eta)$. If $f \in \mathcal{E}^{\alpha,p}$ and $g_\lambda^*(f)(x_0)$ is finite for a point $x_0 \in \mathbb{R}^n$, then $g_\lambda^*(f)(x) < \infty$ a.e. on $\mathbb{R}^n$, and there is a constant C independent of f, such that*

$$\|g_\lambda^*(f)\|_{\mathcal{E}^{\alpha,p}} \le C\|f\|_{\mathcal{E}^{\alpha,p}}.$$

Sun's assumption is $\lambda > 3 + \frac{2}{n}$ (see also Wang and Chen [18] in the case $\alpha = 0$). Our result also improves the author's one in [14], the assumption was $\lambda > 1 + \frac{2}{n}$ in the case $\frac{1}{2} \le \alpha < 1$. As for Marcinkiewicz functions, we can improve our results in [15] as follows.

Theorem 4. *Let $\sigma > 0$, $1 < p < \infty$ and $-n/p \le \alpha < \beta \le 1$. Then, if $f \in \mathcal{E}^{\alpha,p}$ and $\mu^\rho(f)(x_0)$ is finite for a point $x_0 \in \mathbb{R}^n$, then $\mu^\rho(f)(x) < \infty$ a.e. on $\mathbb{R}^n$, and there is a constant C independent of f, such that*

$$\|\mu^\rho(f)\|_{\mathcal{E}^{\alpha,p}} \le C\|f\|_{\mathcal{E}^{\alpha,p}}.$$

Theorem 5. *Let $\sigma > 0$, $\max(1, \frac{2n}{n+2\sigma}) < p < \infty$, and $-n/p \le \alpha < \max\big(\frac{1}{2}, \min(\beta, \sigma)\big)$. Then, if $f \in \mathcal{E}^{\alpha,p}$ and $\mu_S^\rho(f)(x_0)$ is finite for a point $x_0 \in \mathbb{R}^n$, then $\mu_S^\rho(f)(x) < \infty$ a.e. on $\mathbb{R}^n$, and there is a constant C independent of f, such that*

$$\|\mu_S^\rho(f)(x)\|_{\mathcal{E}^{\alpha,p}} \le C\|f\|_{\mathcal{E}^{\alpha,p}}.$$

Theorem 6. *Let $\sigma > 0$, $\max(1, \frac{2n}{n+2\sigma}) < p < \infty$, $\lambda > \lambda_0$, and $-n/p \le \alpha < \max\big(\frac{1}{2}, \min(\beta, \sigma)\big)$. Then, if $f \in \mathcal{E}^{\alpha,p}$ and $\mu_\lambda^{*,\rho}(f)(x_0)$ is finite for a point $x_0 \in \mathbb{R}^n$, then $\mu_\lambda^{*,\rho}(f)(x) < \infty$ a.e. on $\mathbb{R}^n$, and there is a constant C independent of f, such that*

$$\|\mu_\lambda^{*,\rho}(f)\|_{\mathcal{E}^{\alpha,p}} \le C\|f\|_{\mathcal{E}^{\alpha,p}},$$

where $\lambda_0 = \max(1, 2/p)$ $(-n/p \le \alpha < 0)$, $\lambda_0 = 1$ $(0 \le \alpha < 1/2)$, and $\lambda_0 = 1 + 2\alpha/n$ $(1/2 \le \alpha < 1)$.

To prove the above theorems we use the following two key lemmas.

Lemma 1. *Let $1 \le p < \infty$. If $\delta > 0$ and $-n/p \le \alpha < \min(1, \delta/p)$, then there exists $C > 0$ such that for any ball $B = B(x, r)$ and any $f \in \mathcal{E}^{\alpha,p}$*

$$\left(\int_{\mathbb{R}^n} \frac{|f(y) - f_B|^p}{(r + |y-x|)^{n+\delta}}\, dy\right)^{\frac{1}{p}} \le C r^{\alpha - \frac{\delta}{p}} \|f\|_{\mathcal{E}^{\alpha,p}}.$$

This can be proved easily by modifying the proof of Lemma 2.3 in [1].

Lemma 2. *Let α, $\beta > 0$. Suppose $|\varphi(x)| \le C_1(1+|x|)^{-(n+\alpha)}$ and $|\psi(x)| \le C_2(1+|x|)^{-(n+\beta)}$. Then,*

$$|\varphi_t * \psi_t(x)| \le C_3 t^{-n}\Big(1+\Big|\frac{x}{t}\Big|\Big)^{-(n+\min(\alpha,\beta))}$$

Proof. Since $\varphi_t * \psi_t(x) = (\varphi * \psi)_t(x)$, we may assume $t = 1$. Note that if $|y-x| \le |x|/2$, then $|y| \ge |x|/2$. So, we have

$$\begin{aligned}\int_{|y-x|\le|x|/2} |\varphi(y)\psi(x-y)|\,dy &\le C_1\int_{|y-x|\le|x|/2}(1+|y|)^{-(n+\alpha)}|\psi(x-y)|\,dy\\ &\le C_1(1+|x|/2)^{-(n+\alpha)}\int_{\mathbb{R}^n}|\psi(x-y)|\,dy \le C_1\|\psi\|_1(1+|x|/2)^{-(n+\alpha)}.\end{aligned}$$

And,

$$\begin{aligned}\int_{|y-x|\ge|x|/2} |\varphi(y)\psi(x-y)|\,dy &\le C_2\int_{|y-x|\ge|x|/2}|\varphi(y)|(1+|x-y|)^{-(n+\beta)}\,dy\\ &\le C_2(1+|x|/2)^{-(n+\beta)}\int_{\mathbb{R}^n}|\varphi(y)|\,dy \le C_2\|\varphi\|_1(1+|x|/2)^{-(n+\beta)}.\end{aligned}$$

Hence, we obtain the conclusion. □

Finally in this section, we mention some examples of LP functions. Let $P(t,x) = c_n t(t^2+|x|^2)^{-\frac{n+1}{2}}$ and $Q(x) = \frac{\partial}{\partial t}P(t,x)\big|_{t=1}$. Then, $Q(x)$ is an LP function satisfying the conditions in Definition 1 with $\delta = 1$, $\eta = 2$, $\gamma = 1$. Let $R(x) = Q(x)\cos\sqrt{1+|x|^2}$. Then, $R(x)$ is an LP function satisfying the conditions in Definition 1 with $\delta = 1$, $\eta = 1$, $\gamma = 1$.

2. Proof of Theorem 3 for g_λ^*-functions

Let $\psi(x)$ be an LP function. Following the procedure of the proof by Sun, we use first the following:

Lemma 3. *Let $\lambda > 1$, $1 \le p < \infty$ and $-n/p \le \alpha < 1$. Then there exists $C > 0$ such that for any ball $B = B(x_0, r)$, any $x \in B$ and any $f \in \mathcal{E}^{\alpha,p}$*

$$g^*_{\lambda,\infty}(f_2)(x) + g^*_{\lambda,0,\infty}(f_2)(x) \le C r^\alpha \|f\|_{\mathcal{E}^{\alpha,p}},$$

where $f_2(x) = (f(x) - f_{4B})\chi_{4B}$ and

$$\begin{aligned} g^*_{\lambda,\infty}(f_2)(x) &:= \bigg(\int_r^\infty\int_{\mathbb{R}^n}\bigg|\int_{\mathbb{R}^n}\psi_t(x-u-y)f_2(y)\,dy\bigg|^2\Big(1+\frac{|u|}{t}\Big)^{-\lambda n}\frac{dudt}{t^{n+1}}\bigg)^{\frac12},\\ g^*_{\lambda,0,\infty}(f_2)(x) &:= \bigg(\int_0^r\int_{|u|\ge 8r}\bigg|\int_{\mathbb{R}^n}\psi_t(x-u-y)f_2(y)\,dy\bigg|^2\Big(1+\frac{|u|}{t}\Big)^{-\lambda n}\frac{dudt}{t^{n+1}}\bigg)^{\frac12}\end{aligned}$$

Proof. Since ψ is bounded, we have

$$\begin{aligned} g^*_{\lambda,\infty}(f_2)(x) &\le \bigg(\int_r^\infty\int_{\mathbb{R}^n}\bigg(\int_{\mathbb{R}^n}\frac{\|\psi\|_\infty}{t^n}|f_2(y)|\,dy\bigg)^2\Big(1+\frac{|u|}{t}\Big)^{-\lambda n}\frac{dudt}{t^{n+1}}\bigg)^{\frac12}\\ &\le \|\psi\|_\infty\int_{4B}|f(y)-f_{4B}|\,dy\bigg(\int_r^\infty\int_{\mathbb{R}^n}\Big(1+\frac{|u|}{t}\Big)^{-\lambda n}\frac{du}{t^n}\frac{dt}{t^{2n+1}}\bigg)^{\frac12}\\ &\le C r^{n+\alpha}\|f\|_{\mathcal{E}^{\alpha,1}}\bigg(\int_r^\infty\frac{dt}{t^{2n+1}}\bigg)^{\frac12} \le C r^\alpha\|f\|_{\mathcal{E}^{\alpha,p}}.\end{aligned}$$

Now for $x \in B$, $y \in 4B$ and $|u| \geq 8r$, we have $|x-u-y| \geq |u| - |x-x_0| - |x_0-y| \geq \frac{3}{8}|u|$, and hence

$$\int_0^r \int_{|u|\geq 8r} \left| \int_{\mathbb{R}^n} \psi_t(x-u-y) f_2(y)\,dy \right|^2 \left(1+\frac{|u|}{t}\right)^{-\lambda n} \frac{dudt}{t^{n+1}}$$
$$\leq C \int_0^r \int_{|u|\geq 8r} \left(\int_{4B} \frac{|f(y)-f_{4B}|\,dy}{t^n \left(1+\frac{|x-u-y|}{t}\right)^{n+\delta}} \right)^2 \left(1+\frac{|u|}{t}\right)^{-\lambda n} \frac{dudt}{t^{n+1}}$$
$$\leq C \int_0^r \int_{|u|\geq 8r} \left(\int_{4B} \frac{t^\delta |f(y)-f_{4B}|\,dy}{|u|^{n+\delta}} \right)^2 \left(1+\frac{|u|}{t}\right)^{-\lambda n} \frac{dudt}{t^{n+1}}$$
$$\leq C \int_0^r t^{2\delta+\lambda n-n-1}\,dt \int_{|u|\geq 8r} |u|^{-2n-2\delta-\lambda n}\,du \left(\int_{4B} |f(y)-f_{4B}|\,dy \right)^2$$
$$\leq C r^{2\delta+\lambda n-n} r^{-2n-2\delta-\lambda n+n} r^{2n+2\alpha} \|f\|^2_{\mathcal{E}^{\alpha,1}} \leq C r^{2\alpha} \|f\|^2_{\mathcal{E}^{\alpha,p}}.$$

□

Next using Lemmas 1 and 2, we have

Lemma 4. *Let $1 \leq p < \infty$ and $-n/p \leq \alpha < \min(1,\delta)$. Then there exists $C > 0$ such that for any ball $B = B(x_0, r)$, any $x \in B$ and any $f \in \mathcal{E}^{\alpha,p}$*

$$g^*_{\lambda,0}(f_3)(x) \leq C r^\alpha \|f\|_{\mathcal{E}^{\alpha,p}},$$

provided $\lambda > 1$ in the case $\alpha = 0$ and $\lambda > \max(1, \frac{2}{p})$ in the case $-\frac{n}{p} \leq \alpha < 0$, and

$$g^*_{\lambda,0,0}(f_3)(x) \leq C r^\alpha \|f\|_{\mathcal{E}^{\alpha,p}},$$

provided $\lambda > 1$ in the case $0 \leq \alpha < 1$, where $f_3(x) = (f(x) - f_{4B})\chi_{(4B)^c}$,

$$g^*_{\lambda,0}(f_3)(x) := \left(\int_0^r \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \psi_t(x-u-y) f_3(y)\,dy \right|^2 \left(1+\frac{|u|}{t}\right)^{-\lambda n} \frac{dudt}{t^{n+1}} \right)^{\frac{1}{2}}, \text{ and}$$
$$g^*_{\lambda,0,0}(f_3)(x) := \left(\int_0^r \int_{|u|\leq 8r} \left| \int_{\mathbb{R}^n} \psi_t(x-u-y) f_3(y)\,dy \right|^2 \left(1+\frac{|u|}{t}\right)^{-\lambda n} \frac{dudt}{t^{n+1}} \right)^{\frac{1}{2}}.$$

Proof. (i) The case $1 \leq p \leq 2$, $\lambda > \frac{2}{p}$ and $\alpha < \min(\delta, \frac{p}{2}\lambda n - n)/p$. By the Hölder inequality ($\frac{1}{p} + \frac{1}{q} = 1$) and the Minkowski inequality ($2/p \geq 1$),

$$g^*_{\lambda,0}(f_3)(x) \leq \|\psi\|_1^{\frac{1}{q}} \left(\int_0^r \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} |\psi_t(x-u-y)| |f_3(y)|^p\,dy \right|^{\frac{2}{p}} \left(1+\frac{|u|}{t}\right)^{-\lambda n} \frac{dudt}{t^{n+1}} \right)^{\frac{1}{2}}$$
$$\leq \|\psi\|_1^{\frac{1}{q}} \left(\int_{\mathbb{R}^n} \left(\int_0^r \int_{\mathbb{R}^n} |\psi_t(x-u-y)|^{\frac{2}{p}} \left(1+\frac{|u|}{t}\right)^{-\lambda n} \frac{dudt}{t^{n+1}} \right)^{\frac{p}{2}} |f_3(y)|^p\,dy \right)^{\frac{1}{p}}.$$

By Lemma 2 we have

$$\int_0^r \int_{\mathbb{R}^n} |\psi_t(x-u-y)|^{\frac{2}{p}} \left(1+\frac{|u|}{t}\right)^{-\lambda n} \frac{dudt}{t^{n+1}} \leq C \int_0^r t^{-\frac{2n}{p}} \left(1+\frac{|x-y|}{t}\right)^{-\min(\frac{2(n+\delta)}{p},\lambda n)} \frac{dt}{t}$$
$$\leq C|x-y|^{-\min(\frac{2(n+\delta)}{p},\lambda n)} \int_0^r t^{\min(\frac{2(n+\delta)}{p},\lambda n)-\frac{2n}{p}-1}\,dt \leq C \frac{r^{\min(\frac{2(n+\delta)}{p},\lambda n)-\frac{2n}{p}}}{|x-y|^{\min(\frac{2(n+\delta)}{p},\lambda n)}}.$$

We have used here $\lambda > \frac{2}{p}$. For $|x - x_0| < r$ and $|x_0 - y| > 4r$, we have $|x - y| \ge \frac{3}{4}|x_0 - y| \ge \frac{1}{4}(2r + |x_0 - y|)$. Hence, by Lemma 1

$$g^*_{\lambda,0}(f_3)(x) \le C\Big(\int_{\mathbb{R}^n} \frac{r^{\min(n+\delta,\frac{p}{2}\lambda n)-n}|f(y) - f_{4B}|^p}{(r + |x_0 - y|)^{\min(n+\delta,\frac{p}{2}\lambda n)}}\,dy\Big)^{\frac{1}{p}}$$
$$\le C\, r^{(\min(n+\delta,\frac{p}{2}\lambda n)-n)/p} r^{\alpha-(\min(n+\delta,\frac{p}{2}\lambda n)-n)/p}\|f\|_{\mathcal{E}^{\alpha,p}} \le C\, r^{\alpha}\|f\|_{\mathcal{E}^{\alpha,p}}.$$

We have here used $\alpha < \min(\delta, \frac{p}{2}\lambda n - n)/p$.

(ii) The case $2 < p < \infty$, $\lambda > 1$ and $-\frac{n}{p} \le \alpha < 0$. The conclusion in this case follows from (i) for $p = 2$ and the fact $\|f\|_{\mathcal{E}^{\alpha,p_1}} \le \|f\|_{\mathcal{E}^{\alpha,p_2}}$ for $p_1 \le p_2$.

(iii) The case $\alpha = 0$. In this case, it is known that $\mathcal{E}^{\alpha,p}$ norm is equivalent to the usual BMO norm for every $1 \le p < \infty$. Hence the conclusion follows from (i) for $p = 2$.

(iv) The case $0 < \alpha < 1$. In this case, it is known that $\mathcal{E}^{\alpha,p}$ norm is equivalent to the usual Lipschitz norm Lip_α for every $1 \le p < \infty$. So, for $|y - x_0| > 4r$ we have $|f(y) - f_{4B}| \le (|y - x_0|^\alpha + (4r)^\alpha)\|f\|_{\mathrm{Lip}_\alpha} \le 2|y - x_0|^\alpha\|f\|_{\mathrm{Lip}_\alpha}$. Hence

$$g^*_{\lambda,0,0}(f_3)(x) = \Big(\int_0^r\int_{|u|\le 8r}\Big|\int_{(4B)^c}\psi_t(x-u-y)(f(y)-f_{4B})\,dy\Big|^2\Big(1+\frac{|u|}{t}\Big)^{-\lambda n}\frac{dudt}{t^{n+1}}\Big)^{\frac{1}{2}}$$
$$\le C\Big(\int_0^r\int_{|u|\le r}\Big(\int_{|y-x_0|>4r}\frac{|y-x_0|^\alpha\|f\|_{\mathrm{Lip}_\alpha}\,dy}{t^n\big(1+\frac{|x-u-y|}{t}\big)^{n+\delta}}\Big)^2\Big(1+\frac{|u|}{t}\Big)^{-\lambda n}\frac{dudt}{t^{n+1}}$$
$$+\int_0^r\int_{r<|u|\le 8r}\Big(\int_{4r<|y-x_0|<12r}\frac{|y-x_0|^\alpha\|f\|_{\mathrm{Lip}_\alpha}\,dy}{t^n\big(1+\frac{|x-u-y|}{t}\big)^{n+\delta}}\Big)^2\Big(1+\frac{|u|}{t}\Big)^{-\lambda n}\frac{dudt}{t^{n+1}}$$
$$+\int_0^r\int_{r<|u|\le 8r}\Big(\int_{|y-x_0|\ge 12r}\frac{|y-x_0|^\alpha\|f\|_{\mathrm{Lip}_\alpha}\,dy}{t^n\big(1+\frac{|x-u-y|}{t}\big)^{n+\delta}}\Big)^2\Big(1+\frac{|u|}{t}\Big)^{-\lambda n}\frac{dudt}{t^{n+1}}\Big)^{\frac{1}{2}}.$$

For $|u| \le r$, $x \in B$, $y \notin 4B$, we have $|x - y - u| \ge |x_0 - y| - |x - x_0| - |u| \ge |x_0 - y| - 2r \ge \frac{1}{2}|x_0 - y|$, and hence

$$\int_{(4B)^c}\frac{|y-x_0|^\alpha\,dy}{t^n\big(1+\frac{|x-u-y|}{t}\big)^{n+\delta}} \le \int_{|y-x_0|>4r}\frac{C|y-x_0|^\alpha\,dy}{t^n(|y-x_0|/t)^{n+\delta}}$$
$$\le Ct^\delta\int_{|y|>4r}\frac{dy}{|y|^{n+\delta-\alpha}} = C't^\delta r^{\alpha-\delta}.$$

So,

$$I_1 := \int_0^r\int_{|u|\le r}\Big(\int_{|y-x_0|>4r}\frac{|y-x_0|^\alpha}{t^n\big(1+\frac{|x-u-y|}{t}\big)^{n+\delta}}\,dy\Big)^2\Big(1+\frac{|u|}{t}\Big)^{-\lambda n}\frac{dudt}{t^{n+1}}$$
$$\le Cr^{2\alpha-2\delta}\int_0^r\int_{\mathbb{R}^n}\Big(1+\frac{|u|}{t}\Big)^{-\lambda n}\frac{du}{t^n}\,t^{2\delta-1}\,dt = Cr^{2\alpha-2\delta}\int_{\mathbb{R}^n}\frac{1}{(1+|u|)^{\lambda n}}\,du\int_0^r t^{2\delta-1}\,dt$$
$$= C'\,r^{2\alpha-2\delta}r^{2\delta} = C'r^{2\alpha}.$$

For $|u| \le 8r$, $x \in B$ and $|y - x_0| \ge 12r$ we have $|x - u - y| \ge |x_0 - y| - |x - x_0| - |u| \ge \frac{1}{4}|y - x_0|$, and hence as above

$$I_3 := \int_0^r\int_{r<|u|\le 8r}\Big(\int_{|y-x_0|\ge 12r}\frac{|y-x_0|^\alpha}{t^n\big(1+\frac{|x-u-y|}{t}\big)^{n+\delta}}\,dy\Big)^2\Big(1+\frac{|u|}{t}\Big)^{-\lambda n}du\frac{dt}{t^{n+1}} \le Cr^{2\alpha}.$$

Now take $b>0$ so that $0<b<(\lambda n-n)/2$. Then

$$\begin{aligned}
I_2 &:= \int_0^r \int_{r<|u|\le 8r}\bigg(\int_{4r<|y-x_0|<12r}\frac{|y-x_0|^\alpha\,dy}{t^n\big(1+\frac{|x-u-y|}{t}\big)^{n+\delta}}\bigg)^2\Big(1+\frac{|u|}{t}\Big)^{-\lambda n}\frac{dudt}{t^{n+1}}\\
&\le C\int_0^r \int_{r<|u|\le 8r}\bigg(\int_{4r<|y-x_0|<12r}\frac{r^\alpha\,dy}{t^n\big(1+\frac{|x-u-y|}{t}\big)^{n-b}}\bigg)^2\Big(1+\frac{|u|}{t}\Big)^{-\lambda n}\frac{dudt}{t^{n+1}}\\
&\le C\int_0^r \int_{r<|u|\le 8r}\bigg(\int_{4r<|y-x_0|<12r}\frac{r^\alpha\,dy}{t^b|x-u-y|^{n-b}}\bigg)^2\Big(\frac{t}{|u|}\Big)^{\lambda n}\frac{dudt}{t^{n+1}}\\
&= Cr^{2\alpha}\int_0^r t^{\lambda n-n-1-2b}\,dt\int_{r<|u|\le 8r}\frac{du}{|u|^{\lambda n}}\,du\bigg(\int_{|v|<21r}\frac{dv}{|v|^{n-b}}\bigg)^2\\
&= C' r^{2\alpha}r^{\lambda n-n-2b}r^{-\lambda n+n}r^{2b} = C' r^{2\alpha}.
\end{aligned}$$

Thus, we have

$$g^*_{\lambda,0,0}(f_3)(x)\le C(I_1+I_2+I_3)^{\frac12}\|f\|_{\mathrm{Lip}_\alpha}\le Cr^\alpha\|f\|_{\mathrm{Lip}_\alpha}\le C|B|^{\frac{\alpha}{n}}\|f\|_{\mathcal{E}^{\alpha,p}}.$$

□

Lemma 5. *Let $1\le p<\infty$ and $-n/p\le\alpha<\min(1,\delta)$. Then there exists $C>0$ such that for any ball $B=B(x_0,r)$ and any $f\in\mathcal{E}^{\alpha,p}$ satisfying $g^*_{\lambda,\infty}(f_3)(x_0)<+\infty$, it holds $g^*_{\lambda,\infty}(f_3)(x)<+\infty$ for any $x\in B$ and*

$$|g^*_{\lambda,\infty}(f_3)(x)-g^*_{\lambda,\infty}(f_3)(x_0)|\le Cr^\alpha\|f\|_{\mathcal{E}^{\alpha,p}}\ \textit{for any}\ x\in B,$$

provided $\lambda>\max(1,\frac{2}{p})$ in the case $-\frac{n}{p}\le\alpha<0$, $\lambda>1$ in the case $0\le\alpha<\frac12$, and $\lambda>1+\frac{2\alpha}{n}$ in the case $\frac12\le\alpha<\min(\delta,\gamma,\eta)$, where $f_3(x)=(f(x)-f_{4B})\chi_{(4B)^c}$.

Proof. By setting $v=x-u$ we get

$$g^*_{\lambda,\infty}(f_3)(x)=\bigg(\int_r^\infty\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n}\psi_t(v-y)f_3(y)\,dy\Big|^2\Big(1+\frac{|v-x|}{t}\Big)^{-\lambda n}\frac{dudt}{t^{n+1}}\bigg)^{\frac12}.$$

Hence, if we can show

$$\begin{aligned}
I := \bigg(\int_r^\infty\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n}\psi_t(v-y)f_3(y)\,dy\Big|^2\Big|\Big(1+\frac{|v-x|}{t}\Big)^{-\lambda n}-\Big(1+\frac{|v-x_0|}{t}\Big)^{-\lambda n}\Big|\frac{dudt}{t^{n+1}}\bigg)^{\frac12}&\\
\le Cr^\alpha\|f\|_{\mathcal{E}^{\alpha,p}}\quad\text{for } x\in B,&
\end{aligned}$$

then we have by Minkowski's inequality

$$g^*_{\lambda,\infty}(f_3)(x)\le I+g^*_{\lambda,\infty}(f_3)(x_0)\le Cr^\alpha\|f\|_{\mathcal{E}^{\alpha,p}}+g^*_{\lambda,\infty}(f_3)(x_0)<+\infty\quad\text{for } x\in B,$$

and

$$|g^*_{\lambda,\infty}(f_3)(x)-g^*_{\lambda,\infty}(f_3)(x_0)|\le Cr^\alpha\|f\|_{\mathcal{E}^{\alpha,p}}\quad\text{for } x\in B.$$

So, we will estimate I. By the mean value theorem we have

$$\begin{aligned}
&\Big|\Big(1+\frac{|v-x|}{t}\Big)^{-\lambda n}-\Big(1+\frac{|v-x_0|}{t}\Big)^{-\lambda n}\Big|\\
&\qquad=\Big|\int_0^1\sum_{l=1}^n(x_l-x_{0l})\frac{\partial}{\partial x_l}\Big(1+\frac{|x-v|}{t}\Big)^{-\lambda n}\big(x_0+\theta(x-x_0)\big)\,d\theta\Big|\\
&\qquad\le C\frac{r}{t}\int_0^1\Big(1+\frac{|x_0+\theta(x-x_0)-v|}{t}\Big)^{-\lambda n-1}d\theta
\end{aligned}$$

Hence

$$I \le Cr^{\frac{1}{2}}\biggl(\int_0^1\int_r^\infty\int_{\mathbb{R}^n}\biggl|\int_{\mathbb{R}^n}\psi_t(v-y)f_3(y)\,dy\biggr|^2\Bigl(1+\frac{|x_0+\theta(x-x_0)-v|}{t}\Bigr)^{-\lambda n-1}\frac{dvdtd\theta}{t^{n+2}}\biggr)^{\frac{1}{2}}.$$

(i) The case $1\le p\le 2$, $\lambda>\frac{2}{p}$ and $\alpha<\min(\delta/p,\frac{1}{2}+(\frac{\lambda}{2}-\frac{1}{p})n)$. By Hölder's inequality ($1/p+1/q=1$) we have

$$\biggl|\int_{\mathbb{R}^n}\psi_t(v-y)f_3(y)\,dy\biggr| \le \Bigl(\int_{\mathbb{R}^n}|\psi_t(v-y)|\,dy\Bigr)^{\frac{1}{q}}\Bigl(\int_{\mathbb{R}^n}|\psi_t(v-y)||f_3(y)|^p\,dy\Bigr)^{\frac{1}{p}}$$
$$= C\Bigl(\int_{\mathbb{R}^n}|\psi_t(v-y)||f_3(y)|^p\,dy\Bigr)^{\frac{1}{p}}.$$

Hence by Minkowski's inequality

$$I \le Cr^{\frac{1}{2}}\biggl(\int_{\mathbb{R}^n}\Bigl(\int_0^1\int_r^\infty\int_{\mathbb{R}^n}|\psi_t(v-y)|^{\frac{2}{p}}\Bigl(1+\frac{|x_0+\theta(x-x_0)-v|}{t}\Bigr)^{-\lambda n-1}\frac{dvdtd\theta}{t^{n+2}}\Bigr)^{\frac{p}{2}}|f_3(y)|^p\,dy\biggr)^{\frac{1}{p}}.$$

By Lemma 2 we get

$$\int_{\mathbb{R}^n}|\psi_t(v-y)|^{\frac{2}{p}}\Bigl(1+\frac{|x_0+\theta(x-x_0)-v|}{t}\Bigr)^{-\lambda n-1}dv$$
$$\le C\int_{\mathbb{R}^n}t^{-\frac{2}{p}n}\Bigl(1+\frac{|v-y|}{t}\Bigr)^{-\frac{2}{p}(n+\delta)}\Bigl(1+\frac{|x_0+\theta(x-x_0)-v|}{t}\Bigr)^{-\lambda n-1}dv$$
$$\le Ct^{n-\frac{2}{p}n}\Bigl(1+\frac{|x_0+\theta(x-x_0)-y|}{t}\Bigr)^{-\min(\frac{2}{p}(n+\delta),\lambda n+1)}.$$

For $|y-x_0|>4r$ and $|x-x_0|<r$ we have $|x_0+\theta(x-x_0)-y|\ge|y-x_0|-|x-x_0|>\frac{3}{4}|y-x_0|$. So, setting $\eta=\min(\frac{2}{p}(n+\delta),\lambda n+1)$, we get

$$I\le Cr^{\frac{1}{2}}\biggl(\int_{|y-x_0|>4r}\Bigl(\int_r^\infty\Bigl(1+\frac{|y-x_0|}{t}\Bigr)^{-\eta}\frac{dt}{t^{\frac{2}{p}n+2}}\Bigr)^{\frac{p}{2}}|f(y)-f_{4B}|^p\,dy\biggr)^{\frac{1}{p}}.$$

Since

$$\int_r^\infty\Bigl(1+\frac{|y-x_0|}{t}\Bigr)^{-\eta}\frac{dt}{t^{\frac{2}{p}n+2}}\le\int_r^{|y-x_0|}\frac{t^{\eta-\frac{2}{p}n-2}}{|y-x_0|^\eta}\,dt+\int_{|y-x_0|}^\infty\frac{dt}{t^{\frac{2}{p}n+2}}$$
$$=\frac{1}{\eta-\frac{2}{p}-1}\Bigl(\frac{1}{|y-x_0|^{\frac{2}{p}n+1}}-\frac{r^{\eta-\frac{2}{p}n-1}}{|y-x_0|^\eta}\Bigr)+\frac{1}{\frac{2}{p}n+1}\frac{1}{|y-x_0|^{\frac{2}{p}n+1}},$$

we have by using Lemma 1

$$I\le Cr^{\frac{1}{2}}\biggl[\Bigl(\int_{|y-x_0|>4r}\frac{|f(y)-f_{4B}|^p}{|y-x_0|^{\frac{p\eta}{2}}}\,dy\Bigr)^{\frac{1}{p}}r^{\frac{\eta}{2}-\frac{n}{p}-\frac{1}{2}}+\Bigl(\int_{|y-x_0|>4r}\frac{|f(y)-f_{4B}|^p}{|y-x_0|^{n+\frac{p}{2}}}\,dy\Bigr)^{\frac{1}{p}}\biggr]$$
$$\le Cr^{\frac{1}{2}}\Bigl[r^{\alpha-(\frac{p}{2}\eta-n)/p}r^{\frac{\eta}{2}-\frac{n}{p}-\frac{1}{2}}+r^{\alpha-(n+\frac{p}{2}-n)/p}\Bigr]\|f\|_{\mathcal{E}^{\alpha,p}}\le Cr^\alpha\|f\|_{\mathcal{E}^{\alpha,p}}.$$

we have used here $\alpha<1/2$ and $\alpha<\frac{1}{2}\eta-\frac{n}{p}$ (i.e. $\alpha<\frac{1}{2}+(\frac{\lambda}{2}-\frac{1}{p})n$ and $\alpha<\frac{\delta}{p}$).

(ii) The case $2<p<\infty$, $\lambda>1$ and $-\frac{n}{p}\le\alpha<0$. The conclusion in this case follows from (i) for $p=2$ and the fact $\|f\|_{\mathcal{E}^{\alpha,p_1}}\le\|f\|_{\mathcal{E}^{\alpha,p_1}}$ for $p_1\le p_2$.

(iii) The case $\alpha=0$. In this case, it is known that $\mathcal{E}^{\alpha,p}$ norm is equivalent to the usual BMO norm for every $1\le p<\infty$. Hence the conclusion follows from (i) for $p=2$.

(iv) The case $0 < \alpha < \frac{1}{2}$ and $\lambda > 1$. In this case, $\mathcal{E}^{\alpha,p}$ norm is equivalent to the usual Lipschitz norm Lip_α for every $1 \le p < \infty$. Hence, in the case $0 < \alpha < \frac{\delta}{2}$, the conclusion follows from (i) for $p = 2$. So, we treat the case $0 < \delta < 1$. Putting $u = x_0 + \theta(x - x_0) - v$ we get

$$I \le Cr^{\frac{1}{2}} \left(\int_0^1 \int_r^\infty \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \psi_t(x_0 + \theta(x - x_0) - u - y) f_3(y)\, dy \right|^2 \left(1 + \frac{|u|}{t}\right)^{-\lambda n - 1} \frac{du dt d\theta}{t^{n+2}} \right)^{\frac{1}{2}}$$
$$\le Cr^{\frac{1}{2}} \left(\int_0^1 \int_r^\infty \int_{\mathbb{R}^n} \left(\int_{(4B)^c} \frac{|f(y) - f_{4B}|}{t^n \left(1 + \frac{|x_0 - u - y + \theta(x - x_0)|}{t}\right)^{n+\delta}}\, dy \right)^2 \left(1 + \frac{|u|}{t}\right)^{-\lambda n - 1} \frac{du dt d\theta}{t^{n+2}} \right)^{\frac{1}{2}}$$
$$\le Cr^{\frac{1}{2}} \left(\int_0^1 \int_r^\infty \int_{\mathbb{R}^n} \left(\int_{(4B)^c} \frac{|y - x_0|^\alpha \|f\|_{\mathrm{Lip}_\alpha}}{t^n \left(1 + \frac{|x_0 - u - y + \theta(x - x_0)|}{t}\right)^{n+\delta}}\, dy \right)^2 \left(1 + \frac{|u|}{t}\right)^{-\lambda n - 1} \frac{du dt d\theta}{t^{n+2}} \right)^{\frac{1}{2}}.$$

For $|u| \le r$, $x \in B$, $y \notin 4B$, we have $|x_0 - y - u + \theta(x - x_0)| \ge |x_0 - y| - |x - x_0| - |u| \ge |x_0 - y| - 2r \ge \frac{1}{2}|x_0 - y|$, and hence taking $\delta_1 > 0$ with $2\alpha < 2\delta_1 < \min(n + 1, 2\delta)$ we have

$$\int_{(4B)^c} \frac{|y - x_0|^\alpha}{t^n \left(1 + \frac{|x_0 - y - u + \theta(x - x_0)|}{t}\right)^{n+\delta}}\, dy \le C \int_{|y - x_0| > 4r} \frac{|y - x_0|^\alpha\, dy}{t^n \left(1 + \frac{|x_0 - y|}{t}\right)^{n+\delta_1}}$$
$$\le Ct^{\delta_1} \int_{|y - x_0| > 4r} |y - x_0|^{\alpha - n - \delta_1}\, dy = Ct^{\delta_1} \int_{|y| > 4r} |y|^{\alpha - n - \delta_1}\, dy = C' t^{\delta_1} r^{\alpha - \delta_1}.$$

So,

$$\int_r^\infty \int_{|u| \le r} \left(\int_{(4B)^c} \frac{|y - x_0|^\alpha\, dy}{t^n \left(1 + \frac{|x_0 - y - u + \theta(x - x_0)|}{t}\right)^{n+\delta}} \right)^2 \left(1 + \frac{|u|}{t}\right)^{-\lambda n - 1} du \frac{dt}{t^{n+2}}$$
$$\le C \int_r^\infty \int_{|u| \le r} \frac{t^{2\delta_1} r^{2\alpha - 2\delta_1}\, du}{\left(1 + \frac{|u|}{t}\right)^{\lambda n + 1}} \frac{dt}{t^{n+2}} \le Cr^{2\alpha - 2\delta_1} \int_r^\infty \int_{|u| \le r} t^{2\delta_1}\, du \frac{dt}{t^{n+2}}$$
$$\le Cr^{2\alpha - 2\delta_1} \int_r^\infty t^{2\delta_1 - n - 2}\, dt \int_{|u| \le r} du \le C\, r^{2\alpha - 2\delta_1} r^{2\delta_1 - n - 1} r^n \le Cr^{2\alpha - 1}.$$

For the integral on $|u| > r$ we proceed as follows

$$\int_{(4B)^c} \frac{|f(y) - f_{4B}|}{t^n \left(1 + \frac{|x_0 - u - y + \theta(x - x_0)|}{t}\right)^{n+\delta}}\, dy$$
$$\le \int_{(4B)^c} \frac{|f(y) - f(x_0 - u + \theta(x - x_0))|}{t^n \left(1 + \frac{|x_0 - u - y + \theta(x - x_0)|}{t}\right)^{n+\delta}}\, dy + \int_{(4B)^c} \frac{|f(x_0 - u + \theta(x - x_0)) - f(x_0 - u)|}{t^n \left(1 + \frac{|x_0 - u - y + \theta(x - x_0)|}{t}\right)^{n+\delta}}\, dy$$
$$+ \int_{(4B)^c} \frac{|f(x_0 - u) - f_{4B}|}{t^n \left(1 + \frac{|x_0 - u - y + \theta(x - x_0)|}{t}\right)^{n+\delta}}\, dy$$

$$\le C\int_{(4B)^c}\frac{|x_0-u-y+\theta(x-x_0)|^\alpha\|f\|_{\mathrm{Lip}_\alpha}dy}{t^n\big(1+\frac{|x_0-u-y+\theta(x-x_0)|}{t}\big)^{n+\delta}}+C\int_{(4B)^c}\frac{|x-x_0|^\alpha\|f\|_{\mathrm{Lip}_\alpha}dy}{t^n\big(1+\frac{|x_0-u-y+\theta(x-x_0)|}{t}\big)^{n+\delta}}$$
$$+\int_{(4B)^c}\frac{1}{t^n\big(1+\frac{|x_0-u-y+\theta(x-x_0)|}{t}\big)^{n+\delta}}\,dy|f(x_0-u)-f_{4B}|$$
$$\le C\|f\|_{\mathrm{Lip}_\alpha}\bigg(\int_{\mathbb{R}^n}\frac{|x_0-u-y+\theta(x-x_0)|^\alpha}{\big(1+\frac{|x_0-u-y+\theta(x-x_0)|}{t}\big)^{n+\delta}}\,\frac{dy}{t^n}+\int_{\mathbb{R}^n}\frac{r^\alpha}{\big(1+\frac{|x_0-u-y+\theta(x-x_0)|}{t}\big)^{n+\delta}}\,\frac{dy}{t^n}\bigg)$$
$$+\int_{\mathbb{R}^n}\frac{1}{\big(1+\frac{|x_0-u-y+\theta(x-x_0)|}{t}\big)^{n+\delta}}\,\frac{dy}{t^n}\,|f(x_0-u)-f_{4B}|$$
$$\le C\,t^\alpha\|f\|_{\mathrm{Lip}_\alpha}\int_{\mathbb{R}^n}\frac{|y|^\alpha}{(1+|y|)^{n+\delta}}\,dy+C\,r^\alpha\|f\|_{\mathrm{Lip}_\alpha}\int_{\mathbb{R}^n}\frac{1}{(1+|y|)^{n+\delta}}\,dy$$
$$+|f(x_0-u)-f_{4B}|\int_{\mathbb{R}^n}\frac{1}{(1+|y|)^{n+\delta}}\,dy.$$

Now we get

$$\int_r^\infty\int_{|u|>r}\frac{t^{2\alpha}\,du}{\big(1+\frac{|u|}{t}\big)^{\lambda n+1}}\,\frac{dt}{t^{n+2}}\le\int_r^\infty\int_{\mathbb{R}^n}\frac{1}{\big(1+\frac{|u|}{t}\big)^{\lambda n+1}}\,\frac{du}{t^n}\,t^{2\alpha-2}\,dt\le Cr^{2\alpha-1}.$$

Similarly we get

$$\int_r^\infty\int_{|u|>r}\frac{r^{2\alpha}\,du}{\big(1+\frac{|u|}{t}\big)^{\lambda n+1}}\,\frac{dt}{t^{n+2}}\le r^{2\alpha}\int_r^\infty\int_{\mathbb{R}^n}\frac{1}{\big(1+\frac{|u|}{t}\big)^{\lambda n+1}}\,\frac{du}{t^n}\,t^{-2}\,dt\le Cr^{2\alpha-1}.$$

And by change of variable $t=|u|s$ and using Lemma 1 ($p=2$) we have

$$\int_r^\infty\int_{|u|>r}\frac{|f(x_0-u)-f_{4B}|^2\,du}{\big(1+\frac{|u|}{t}\big)^{\lambda n+1}}\,\frac{dt}{t^{n+2}}$$
$$\le\int_{|u|>r}|f(x_0-u)-f_{4B}|^2\int_0^\infty\frac{dt}{\big(1+\frac{|u|}{t}\big)^{\lambda n+1}t^{n+2}}\,du$$
$$\le\int_{|u|>r}\frac{|f(x_0-u)-f_{4B}|^2}{|u|^{n+1}}\,du\int_0^\infty\frac{ds}{\big(1+\frac{1}{s}\big)^{\lambda n+1}s^{n+2}}$$
$$\le C\int_{\mathbb{R}^n}\frac{|f(u)-f_{4B}|^2\,du}{(r+|u-x_0|)^{n+1}}\le C\,r^{2\alpha-1}\|f\|^2_{\mathcal{E}^{\alpha,2}}.$$

Altogether, we have

$$I\le C\,r^{\frac12}\,r^{\alpha-\frac12}\|f\|_{\mathrm{Lip}_\alpha}\le C\,r^\alpha\|f\|_{\mathcal{E}^{\alpha,p}}.$$

(v) The case $\frac12\le\alpha<\min(\delta,\gamma,\eta)$ and $\lambda>1+\frac{2\alpha}{n}$. Since by Minkowski's inequality we get

$$g^*_{\lambda,\infty}(f_3)(x)\le g^*_{\lambda,\infty}(f_3)(x_0)$$
$$+\bigg(\int_r^\infty\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}|\psi_t(x-u-y)-\psi_t(x_0-u-y)||f_3(y)|\,dy\Big)^2\Big(1+\frac{|u|}{t}\Big)^{-\lambda n}\frac{dudt}{t^{n+1}}\bigg)^{\frac12}\|f\|_{\mathrm{Lip}_\alpha},$$

and since $|f_3(y)|=|f(y)-f_{4B}|\chi_{(4B)^c}\le C|y-x_0|^\alpha\|f\|_{\mathrm{Lip}_\alpha}$, it suffices to show

$$J:=\bigg(\int_r^\infty\int_{\mathbb{R}^n}\Big(\int_{y\in(4B)^c}|\psi_t(x-u-y)-\psi_t(x_0-u-y)||y-x_0|^\alpha\,dy\Big)^2$$
$$\times\Big(1+\frac{|u|}{t}\Big)^{-\lambda n}\frac{dudt}{t^{n+1}}\bigg)^{\frac12}\|f\|_{\mathrm{Lip}_\alpha}\le Cr^\alpha\|f\|_{\mathcal{E}^{\alpha,p}}.$$

For $|u|\le r$ and $|y-x_0|>4r$, we have $|x_0-u-y|\ge |y-x_0|-|u|\ge \frac{3}{4}|y-x_0|\ge 3r\ge 3|x-x_0|$. So, for $|u|\le r$ we have by the assumption (iii) for ψ and using $\alpha<\eta$

$$\int_{y\in(4B)^c}|\psi_t(x-u-y)-\psi_t(x_0-u-y)||y-x_0|^\alpha\,dy$$
$$\le C\int_{y\in(4B)^c}\frac{\left(\frac{|x-x_0|}{t}\right)^\gamma|y-x_0|^\alpha}{t^n\left(1+\frac{|y-x_0|}{t}\right)^{n+\eta}}\,dy\le C\Big(\frac{r}{t}\Big)^\gamma\int_{\mathbb{R}^n}\frac{|v|^\alpha}{(1+|v|)^{n+\eta}}\,dv\,t^\alpha\le Cr^\gamma t^{\alpha-\gamma}.$$

For $|u|>r$ and $|y-x_0|>4|u|$ we have $|x_0-u-y|\ge|y-x_0|-|u|\ge\frac{3}{4}|y-x_0|>3r\ge 3|x-x_0|$. So, like as above, we have for $|u|>r$

$$\int_{|y-x_0|>4|u|}|\psi_t(x-u-y)-\psi_t(x_0-u-y)||y-x_0|^\alpha\,dy\le Cr^\gamma t^{\alpha-\gamma}.$$

And for the integration on $4r<|y-x_0|\le 4|u|$, we have, using the property (iii') of ψ

$$\int_{4r<|y-x_0|\le4|u|}|\psi_t(x-u-y)-\psi_t(x_0-u-y)||y-x_0|^\alpha\,dy$$
$$\le C|u|^\alpha\int_{4r<|y-x_0|\le4|u|}|\psi_t(x-u-y)-\psi_t(x_0-u-y)|\,dy$$
$$\le C|u|^\alpha\int_{\mathbb{R}^n}|\psi_t(x-u-y)-\psi_t(x_0-u-y)|\,dy$$
$$\le C|u|^\alpha\Big(\frac{|x-x_0|}{t}\Big)^\gamma\le Cr^\gamma t^{-\gamma}|u|^\alpha.$$

Thus we have

$$J\le C\left(\int_r^\infty\left(\int_{|u|\le r}\frac{r^{2\gamma}t^{2\alpha-2\gamma}}{\left(1+\frac{|u|}{t}\right)^{\lambda n}}\,du+\int_{|u|>r}\frac{r^{2\gamma}t^{2\alpha-2\gamma}+|u|^{2\alpha}r^{2\gamma}t^{-2\gamma}}{\left(1+\frac{|u|}{t}\right)^{\lambda n}}\,du\right)\frac{dt}{t^{n+1}}\right)^{\frac12}\|f\|_{\mathrm{Lip}_\alpha}$$
$$\le C\left(\int_r^\infty\left(\int_{\mathbb{R}^n}\frac{r^{2\gamma}t^{2\alpha-2\gamma}}{\left(1+\frac{|u|}{t}\right)^{\lambda n}}\,du+\int_{|u|>r}\frac{|u|^{2\alpha}r^{2\gamma}t^{-2\gamma}}{\left(1+\frac{|u|}{t}\right)^{\lambda n}}\,du\right)\frac{dt}{t^{n+1}}\right)^{\frac12}\|f\|_{\mathrm{Lip}_\alpha}$$
$$\le Cr^\gamma\left(\int_r^\infty\int_{\mathbb{R}^n}\frac{1}{\left(1+\frac{|u|}{t}\right)^{\lambda n}}\frac{du}{t^n}\frac{dt}{t^{-2\alpha+2\gamma+1}}+\int_r^\infty\int_{\mathbb{R}^n}\frac{|u|^{2\alpha}}{\left(1+\frac{|u|}{t}\right)^{\lambda n}}\frac{du}{t^{n+2\alpha}}\frac{dt}{t^{2\gamma-2\alpha+1}}\right)^{\frac12}\|f\|_{\mathrm{Lip}_\alpha}$$
$$\le Cr^\gamma(r^{2\alpha-2\gamma})^{\frac12}\|f\|_{\mathrm{Lip}_\alpha}\le Cr^\alpha\|f\|_{\mathcal{E}^{\alpha,p}}.$$

We have used here $\alpha<\gamma$ and $\lambda n-2\alpha-n>0$ i.e. $\lambda>1+\frac{2\alpha}{n}$. □

Lemma 6. *Let $\lambda>1$, $1\le p<\infty$ and $0<\alpha<\min(1,\delta)$. Then there exists $C>0$ such that for any ball $B=B(x_0,r)$ and any $f\in\mathcal{E}^{\alpha,p}$ satisfying $g^*_{\lambda,0,\infty}(f_3)(x_0)<+\infty$, it holds $g^*_{\lambda,0,\infty}(f_3)(x)<+\infty$ for any $x\in B$ and*

$$|g^*_{\lambda,0,\infty}(f_3)(x)-g^*_{\lambda,0,\infty}(f_3)(x_0)|\le Cr^\alpha\|f\|_{\mathcal{E}^{\alpha,p}}\ \textit{for any } x\in B,$$

where $f_3(x)=(f(x)-f_{4B})\chi_{(4B)^c}$ and

$$g^*_{\lambda,0,\infty}(f_3)(x):=\left(\int_0^r\int_{|u|>8r}\left|\int_{\mathbb{R}^n}\psi_t(x-u-y)f_3(y)\,dy\right|^2\Big(1+\frac{|u|}{t}\Big)^{-\lambda n}\frac{dudt}{t^{n+1}}\right)^{\frac12}.$$

Proof. By setting $v=x-u$ we see

$$g^*_{\lambda,0,\infty}(f_3)(x)=\left(\int_0^r\int_{|v-x|>8r}\left|\int_{\mathbb{R}^n}\psi_t(v-y)f_3(y)\,dy\right|^2\Big(1+\frac{|v-x|}{t}\Big)^{-\lambda n}\frac{dvdt}{t^{n+1}}\right)^{\frac12}.$$

Hence, for $x \in B$ we have

$$g^*_{\lambda,0,\infty}(f_3)(x) \le \Big(\int_0^r \int_{|v-x_0|>8r} \Big| \int_{\mathbb{R}^n} \psi_t(v-y) f_3(y)\,dy\Big|^2 \Big(1+\frac{|v-x|}{t}\Big)^{-\lambda n} \frac{dvdt}{t^{n+1}}\Big)^{\frac12} + \Big(\int_0^r \int_{|v-x|\le 9r} \Big| \int_{\mathbb{R}^n} \psi_t(v-y) f_3(y)\,dy\Big|^2 \Big(1+\frac{|v-x|}{t}\Big)^{-\lambda n} \frac{dvdt}{t^{n+1}}\Big)^{\frac12}.$$

We see by Lemma 4 (its variant replaced $8r$ by $9r$) that the second term in the right-hand side of the above inequality is bounded by $Cr^\alpha \|f\|_{\mathcal{E}^{\alpha,p}}$. Hence, we have

$$\begin{aligned} &g^*_{\lambda,0,\infty}(f_3)(x) \le Cr^\alpha \|f\|_{\mathcal{E}^{\alpha,p}} + g^*_{\lambda,0,\infty}(f_3)(x_0) \\ &+ \Big(\int_0^r \int_{|v-x_0|>8r} \Big| \int_{\mathbb{R}^n} \psi_t(v-y) f_3(y)\,dy\Big|^2 \Big|\Big(1+\frac{|v-x|}{t}\Big)^{-\lambda n} - \Big(1+\frac{|v-x_0|}{t}\Big)^{-\lambda n}\Big| \frac{dvdt}{t^{n+1}}\Big)^{\frac12} \\ &= Cr^\alpha \|f\|_{\mathcal{E}^{\alpha,p}} + g^*_{\lambda,0,\infty}(f_3)(x_0) + I, \mathit{say}. \end{aligned}$$

By the mean value theorem we get

$$\begin{aligned} I &\le C\Big(\int_0^r \int_{|v-x_0|>8r} \Big| \int_{\mathbb{R}} \psi_t(v-y) f_3(y)\,dy\Big|^2 \\ &\qquad \times \int_0^1 \frac{|x-x_0|}{t}\Big(1+\frac{|v-x_0+\theta(x-x_0)|}{t}\Big)^{-\lambda n-1} d\theta \frac{dvdt}{t^{n+1}}\Big)^{\frac12} \\ &\le Cr^{\frac12}\Big(\int_0^1 \int_0^r \int_{|v-x_0|>8r} \Big(\int_{(4B)^c} \frac{|y-x_0|^\alpha \|f\|_{\mathrm{Lip}_\alpha}\,dy}{t^n\big(1+\frac{|v-y|}{t}\big)^{n+\delta}}\Big)^2 \\ &\qquad \times \Big(1+\frac{|v-x_0+\theta(x-x_0)|}{t}\Big)^{-\lambda n-1} \frac{dvdt}{t^{n+2}}\,d\theta\Big)^{\frac12}. \end{aligned}$$

We take $b>0$ so that $2b<\lambda n-n$. Then noting $\alpha<\delta$ and $|v-y| \ge |y-x_0|-|x_0-v| \ge \frac12|y-x_0|$ for $|y-x_0| \ge 2|v-x_0|$, we have

$$\begin{aligned} &\int_{|y-x_0|\ge 4r} \frac{|y-x_0|^\alpha\,dy}{t^n\big(1+\frac{|v-y|}{t}\big)^{n+\delta}} \\ &\le \int_{4r\le|y-x_0|<2|v-x_0|} \frac{|y-x_0|^\alpha\,dy}{t^n\big(1+\frac{|v-y|}{t}\big)^{n-b}} + \int_{|y-x_0|\ge 2|v-x_0|} \frac{|y-x_0|^\alpha\,dy}{t^n\big(1+\frac{|v-y|}{t}\big)^{n+\delta}} \\ &\le C\int_{|v-y|<3|v-x_0|} \frac{|v-x_0|^\alpha\,dy}{t^b|v-y|^{n-b}} + C\int_{|y-x_0|\ge 2|v-x_0|} \frac{t^\delta\,dy}{|y-x_0|^{n+\delta-\alpha}} \\ &\le Ct^{-b}|v-x_0|^{\alpha+b} + Ct^\delta|v-x_0|^{\alpha-\delta} \end{aligned}$$

Hence noting $2b < \lambda n - n$ and $\alpha < \delta$ we have

$$I \le Cr^{\frac{1}{2}} \left(\int_0^r \int_{|v-x_0|>8r} \frac{|v-x_0|^{2\alpha+2b}}{|v-x_0|^{\lambda n+1}} t^{\lambda n+1-2b-n-2} + \frac{|v-x_0|^{2\alpha-2\delta}}{|v-x_0|^{\lambda n+1}} t^{\lambda n+1+2\delta-n-2}\, dvdt \right)^{\frac{1}{2}} \|f\|_{\mathrm{Lip}_\alpha}$$

$$\le Cr^{\frac{1}{2}} \left(\int_0^r t^{\lambda n-2b-n-1}\, dt \int_{|u|>8r} \frac{du}{|u|^{\lambda n-2\alpha-2b+1}} + \int_0^r t^{\lambda n+2\delta-n-1}\, dt \int_{|u|>8r} \frac{du}{|u|^{\lambda n-2\alpha+2\delta+1}} \right)^{\frac{1}{2}} \|f\|_{\mathrm{Lip}_\alpha}$$

$$\le Cr^{\frac{1}{2}} (r^{\lambda n-2b-n} r^{-\lambda n+2\alpha+2b-1+n} + r^{\lambda n+2\delta-n} r^{-\lambda n+2\alpha-2\delta-1+n})^{\frac{1}{2}} \|f\|_{\mathrm{Lip}_\alpha} \le Cr^\alpha \|f\|_{\mathcal{E}^{\alpha,p}}.$$

Thus, we have

$$g^*_{\lambda,0,\infty}(f_3)(x) \le g^*_{\lambda,0,\infty}(f_3)(x_0) + Cr^\alpha \|f\|_{\mathcal{E}^{\alpha,p}} \text{ for any } x \in B.$$

Reversing the roles of $g^*_{\lambda,0,\infty}(f_3)(x_0)$ and $g^*_{\lambda,0,\infty}(f_3)(x)$, we have

$$g^*_{\lambda,0,\infty}(f_3)(x_0) \le g^*_{\lambda,0,\infty}(f_3)(x) + Cr^\alpha \|f\|_{\mathcal{E}^{\alpha,p}} \text{ for any } x \in B,$$

and hence we have

$$|g^*_{\lambda,0,\infty}(f_3)(x) - g^*_{\lambda,0,\infty}(f_3)(x_0)| \le Cr^\alpha \|f\|_{\mathcal{E}^{\alpha,p}} \text{ for any } x \in B,$$

□

Proof of Theorem 3. We follow the idea by Kurtz [4]. Let $r > 0$ and $B = B(x_0, r)$. Set $f_1 = f_{4B}$, $f_2 = (f - f_{4B})\chi_{4B}$ and $f_3 = (f - f_{4B})\chi_{(4B)^c}$. Then, $f = f_1 + f_2 + f_3$ and $g^*_\lambda(f_1) = 0$.

(i) The case $0 < \alpha < 1$. By assumption, $g^*_\lambda(f)(x_0) < \infty$. So, we have $g^*_{\lambda,\infty}(f)(x_0) + g^*_{\lambda,0,\infty}(f)(x_0) \le 2g^*_\lambda(f)(x_0) < \infty$. Using Lemma 3 we have $g^*_{\lambda,\infty}(f_3)(x_0) + g^*_{\lambda,0,\infty}(f_3)(x_0) \le g^*_{\lambda,\infty}(f)(x_0) + g^*_{\lambda,0,\infty}(f)(x_0) + g^*_{\lambda,\infty}(f_2)(x_0) + g^*_{\lambda,0,\infty}(f_2)(x_0) < \infty$. Hence by Lemmas 4, 5 and 6 we have for $x \in B$

$$g^*_\lambda(f_3)(x) \le g^*_{\lambda,0,0}(f_3)(x) + g^*_{\lambda,0,\infty}(f_3)(x) + g^*_{\lambda,\infty}(f_3)(x) \le 3Cr^\alpha \|f\|_{\mathcal{E}^{\alpha,p}} + g^*_{\lambda,0,\infty}(f_3)(x_0) + g^*_{\lambda,\infty}(f_3)(x_0) < \infty,$$

and

$$|g^*_\lambda(f_3)(x) - g^*_\lambda(f_3)(x_0)| \le |g^*_{\lambda,0,0}(f_3)(x) - g^*_{\lambda,0,0}(f_3)(x_0)| + |g^*_{\lambda,0,\infty}(f_3)(x) - g^*_{\lambda,0,\infty}(f_3)(x_0)| + |g^*_{\lambda,\infty}(f_3)(x) - g^*_{\lambda,\infty}(f_3)(x_0)| \le 4Cr^\alpha \|f\|_{\mathcal{E}^{\alpha,p}}.$$

Using L^p-boundedness of g^*_λ we have $\|g^*_\lambda(f_2)\|_{L^p} \le C\|f_2\|_{L^p}$, and from this it follows that $g^*_\lambda(f_2)(x) < \infty$ for almost all $x \in B$. Thus, we have $g^*_\lambda(f)(x) \le g^*_\lambda(f_2)(x) + g^*_\lambda(f_3)(x) < \infty$ for almost all $x \in B$. Since r is arbitrary, we see that $g^*_\lambda(f)(x) < \infty$ for almost all $x \in \mathbb{R}^n$.

Let $E = \{x \in \mathbb{R}^n;\, g^*_\lambda(f)(x) < \infty\}$. We have only to show that for any ball $B = B(x_0, r)$ with center $x_0 \in E$,

$$\left(\int_B |g^*_\lambda(f)(x) - (g^*_\lambda(f))_B|^p\, dx \right)^{\frac{1}{p}} \le C|B|^{\frac{1}{p}+\frac{\alpha}{n}} \|f\|_{\mathcal{E}^{\alpha,p}}.$$

Set $f = f_1 + f_2 + f_3$ as above. Noting $g^*_\lambda(f_1) = g^*_{\lambda,0}(f_1) = g^*_{\lambda,\infty}(f) = 0$, and using $\|g^*_\lambda(f_2)\|_{L^p} \le C\|f_2\|_{L^p} \le C|B|^{\frac{1}{p}+\frac{\alpha}{n}}\|f\|_{\mathcal{E}^{\alpha,p}}$ and the above inequality for $g^*_\lambda(f_3)$, we have

$$\begin{aligned}\frac{1}{|B|}\int_B |g^*_\lambda(f)(x) - (g^*_\lambda(f))_B|\,dx &\le \frac{2}{|B|}\int_B |g^*_\lambda(f)(x) - g^*_\lambda(f_3)(x_0)|\,dx \\ &= \frac{2}{|B|}\int_B |g^*_\lambda(f_2+f_3)(x) - g^*_\lambda(f_3)(x) + g^*_\lambda(f_3)(x) - g^*_\lambda(f_3)(x_0)|\,dx \\ &\le \frac{2}{|B|}\int_B |g^*_\lambda(f_2)(x)|\,dx + \frac{2}{|B|}\int_B |g^*_\lambda(f_3)(x) - g^*_\lambda(f_3)(x_0)|\,dx \\ &\le C\left(\frac{1}{|B|}\int_{4B}|f_2(x)|^p\,dx\right)^{\frac{1}{p}} + Cr^\alpha\|f\|_{\mathcal{E}^{\alpha,1}} \le Cr^\alpha\|f\|_{\mathcal{E}^{\alpha,p}}.\end{aligned}$$

(ii) The case $-\frac{p}{n} \le \alpha \le 0$. In this case, the proof is simpler than the case (i). We have only to use $g^*_{\lambda,0}$ and $g^*_{\lambda,\infty}$, Lemmas 3, 4 and 5. So, we leave the detailed proof to the reader.

This completes the proof of Theorem 3.

3. Proofs of Theorems 4, 5 and 6

We proceed as in the proof of Theorem 3. For a ball $B = B(x_0, r)$ and a function f we set always $f_1 = f_{4B}$, $f_2 = (f(y) - f_{4B})\chi_{4B}$ and $f_3 = (f(y) - f_{4B})\chi_{(4B)^c}$.

Lemma 7. *Let $\Omega \in L^\infty(S^{n-1})$, $\int_{S^{n-1}}\Omega(x)\,d\sigma(x) = 0$, $-\frac{n}{p} \le \alpha < 1$, and $\rho = \sigma + i\tau$ ($\sigma > 0, \tau \in \mathbb{R}$). Then, if $f \in \mathcal{E}^{\alpha,p}$ and $\mu^\rho(f)(x_0) < +\infty$ for some $x_0 \in \mathbb{R}^n$, there exists $C > 0$ such that for any ball $B = B(x_0, r)$*

$$\mu^\rho_\infty(f_2)(x_0) \le C(\mu^\rho(f)(x_0) + \|\Omega\|_\infty r^\alpha\|f\|_{\mathcal{E}^{\alpha,p}}).$$

Proof. By assumption we have

$$\left(\int_r^{2r}\left|\frac{1}{t^\rho}\int_{|y-x_0|\le t}\frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}f(y)\,dy\right|^2\frac{dt}{t}\right)^{\frac{1}{2}} \le \mu^\rho(f)(x_0) < +\infty.$$

Hence, for some $r \le t_0 \le 2r$ we get

$$\frac{r}{t_0}\left|\frac{1}{t_0^\rho}\int_{|y-x_0|\le t_0}\frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}f(y)\,dy\right| \le \mu^\rho(f)(x_0).$$

Since, in the above integral, the integration domain is contained in $|y - x_0| \le 4r$, we see, using the cancellation property of Ω, that the above integral is equal to

$$\int_{|y-x_0|\le t_0}\frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}(f(y) - f_{4B})\chi_{4B}\,dy.$$

Hence

$$\left|\int_{|y-x_0|\le t_0}\frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}(f(y) - f_{4B})\chi_{4B}\,dy\right| \le Cr^\sigma\mu^\rho(f)(x_0).$$

Thus for $t > r$ we have

$$\begin{aligned}&\left|\int_{|y-x_0|\le t}\frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}f_2(y)\,dy\right| \\ &\qquad\le \left|\int_{|y-x_0|\le t_0}\frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}}f_2(y)\,dy\right| + \int_{t_0<|y-x_0|<\min(t,4r)}\frac{\|\Omega\|_\infty|f(y) - f_{4B}|}{|y-x_0|^{n-\sigma}}\,dy \\ &\qquad\le Cr^\sigma\mu^\rho(f)(x_0) + C\|\Omega\|_\infty r^{\sigma+\alpha}\|f\|_{\mathcal{E}^{\alpha,p}}.\end{aligned}$$

Therefore we have

$$\mu_\infty^\rho(f_2)(x_0) = \left(\int_r^\infty \left|\frac{1}{t^\rho}\int_{|y-x_0|\le t} \frac{\Omega(y-x_0)}{|y-x_0|^{n-\rho}} f_2(y)\,dy\right|^2 \frac{dt}{t}\right)^{\frac12}$$

$$\le Cr^\sigma(\mu^\rho(f)(x_0) + \|\Omega\|_\infty r^\alpha \|f\|_{\mathcal{E}^{\alpha,p}})\left(\int_r^\infty \frac{dt}{t^{2\sigma+1}}\right)^{\frac12} \le C(\mu^\rho(f)(x_0) + \|\Omega\|_\infty r^\alpha\|f\|_{\mathcal{E}^{\alpha,p}}).$$

□

As for $\mu_{S,\infty}^\rho(f_2)$ and $\mu_{\lambda,\infty}^{*,\rho}(f_2)$ we have

Lemma 8. *Let $\Omega \in L^\infty(S^{n-1})$, $\rho = \sigma + i\tau$ $(\sigma > 0, \tau \in \mathbb{R})$, $\max(1, \frac{2n}{n+2\sigma}) < p < +\infty$, and $-\frac{n}{p} \le \alpha < 1$. Then, for any $f \in \mathcal{E}^{\alpha,p}$, any ball $B = B(x_0, r)$ and any $x \in \mathbb{R}^n$*

$$\mu_{S,\infty}^\rho(f_2)(x) = \left(\int_r^\infty \int_{|u-x|\le t}\left|\frac{1}{t^\rho}\int_{|y-u|\le t}\frac{\Omega(u-y)f_2(y)}{|u-y|^{n-\rho}}\,dy\right|^2\frac{dudt}{t^{n+1}}\right)^{\frac12} \le Cr^\alpha\|f\|_{\mathcal{E}^{\alpha,p}}.$$

Lemma 9. *Let $\Omega \in L^\infty(S^{n-1})$, $\rho = \sigma + i\tau$ $(\sigma > 0, \tau \in \mathbb{R})$, $\lambda > 1$. Suppose α and p satisfy (a) $\max(1, \frac{2n}{n+2\sigma}) < p < +\infty$ and $-\frac{n}{p} \le \alpha < 1$ or (b) $1 \le p < +\infty$ and $0 \le \alpha < 1$. Then, for any $f \in \mathcal{E}^{\alpha,p}$, any ball $B = B(x_0, r)$ and any $x \in \mathbb{R}^n$*

$$\mu_{\lambda,\infty}^{*,\rho}(f_2)(x) = \left(\int_r^\infty\int_{\mathbb{R}^n}\left|\frac{1}{t^\rho}\int_{|y-u|\le t}\frac{\Omega(u-y)f_2(y)}{|u-y|^{n-\rho}}\,dy\right|^2\left(\frac{t}{t+|u-x|}\right)^{\lambda n}\frac{dudt}{t^{n+1}}\right)^{\frac12} \le Cr^\alpha\|f\|_{\mathcal{E}^{\alpha,p}}.$$

Since we can prove Lemmas 8 and 9 in similar ways, we only prove Lemma 9.

Proof. (i) The case $0 < \sigma < n$ and $\max(1, \frac{2n}{n+2\sigma}) < p < +\infty$. First we see easily

$$\left|\int_{r<|y-u|\le t}\frac{\Omega(u-y)f_2(y)}{|u-y|^{n-\rho}}\,dy\right| \le \frac{\|\Omega\|_\infty}{r^{n-\sigma}}\int_{|y-x_0|\le 4r}|f(y) - f_{4B}|\,dy \le Cr^{\alpha+\sigma}\|f\|_{\mathcal{E}^{\alpha,1}}.$$

Hence

$$\left(\int_r^\infty\int_{\mathbb{R}^n}\left|\frac{1}{t^\rho}\int_{r<|y-u|\le t}\frac{\Omega(u-y)f_2(y)}{|u-y|^{n-\rho}}\,dy\right|^2\left(1+\frac{|u-x|}{t}\right)^{-\lambda n}\frac{dudt}{t^{n+1}}\right)^{\frac12}$$

$$\le Cr^{\alpha+\sigma}\left(\int_r^\infty\int_{\mathbb{R}^n}\left(1+\frac{|u-x|}{t}\right)^{-\lambda n}\frac{du}{t^n}\frac{dt}{t^{2\sigma+1}}\right)^{\frac12}\|f\|_{\mathcal{E}^{\alpha,1}}$$

$$\le Cr^{\alpha+\sigma}\left(\int_r^\infty\frac{dt}{t^{2\sigma+1}}\right)^{\frac12}\|f\|_{\mathcal{E}^{\alpha,1}} \le Cr^\alpha\|f\|_{\mathcal{E}^{\alpha,p}}.$$

So, we need only to show

$$I := \left(\int_r^\infty\int_{\mathbb{R}^n}\left|\frac{1}{t^\rho}\int_{|y-u|\le r}\frac{\Omega(u-y)f_2(y)}{|u-y|^{n-\rho}}\,dy\right|^2\left(1+\frac{|u-x|}{t}\right)^{-\lambda n}\frac{dudt}{t^{n+1}}\right)^{\frac12} \le Cr^\alpha\|f\|_{\mathcal{E}^{\alpha,p}}.$$

Since $p > \frac{2n}{n+2\sigma}$, we have

$$\frac{n}{2(n-\sigma)} - \left(1 - \frac{n}{n-\sigma}\left(1-\frac1p\right)\right) = \frac{n+2\sigma}{2p(n-\sigma)}\left(p - \frac{2n}{n+2\sigma}\right) > 0.$$

So, we take $p_0 = \min(2, p)$ and choose a real number a so that

$$\frac{1}{p_0} + \frac{1}{p_0'} = 1, \quad \frac{n}{2(n-\sigma)} > a > 1 - \frac{n}{(n-\sigma)p_0'}.$$

Then, noting $0 < (n-\sigma)(1-a)p_0' < n$ we have by Hölder's inequality

$$\begin{aligned}\Big|\int_{|y-u|\le r}&\frac{\Omega(u-y)f_2(y)}{|u-y|^{n-\rho}}\,dy\Big|\\ &\le \|\Omega\|_\infty\Big(\int_{|y-u|\le r}\frac{dy}{|u-y|^{(n-\sigma)(1-a)p_0'}}\Big)^{\frac{1}{p_0'}}\Big(\int_{|y-u|\le r}\frac{|f_2(y)|^{p_0}\,dy}{|u-y|^{(n-\sigma)ap_0}}\Big)^{\frac{1}{p_0}}\\ &\le Cr^{\frac{n}{p_0'}-(n-\sigma)(1-a)}\Big(\int_{|y-u|\le r}\frac{|f_2(y)|^{p_0}\,dy}{|u-y|^{(n-\sigma)ap_0}}\Big)^{\frac{1}{p_0}}.\end{aligned}$$

Hence by Minkowski's inequality ($\frac{2}{p_0}\ge 1$) and by using $2a(n-\sigma) < n$ we get

$$\begin{aligned}I &\le Cr^{\frac{n}{p_0'}-(n-\sigma)(1-a)}\Big(\int_r^\infty\Big(\int_{4B}\Big(\int_{|y-u|\le r}\frac{\big(\frac{t}{t+|u-x|}\big)^{\lambda n}du}{|u-y|^{2(n-\sigma)a}}\Big)^{\frac{p_0}{2}}|f_2(y)|^{p_0}\,dy\Big)^{\frac{2}{p_0}}\frac{dt}{t^{2\sigma+n+1}}\Big)^{\frac12}\\ &\le Cr^{\frac{n}{p_0'}-(n-\sigma)(1-a)}r^{\frac{n}{2}-(n-\sigma)a}\Big(\int_r^\infty\Big(\int_{4B}|f(y)-f_{4B}|^{p_0}\,dy\Big)^{\frac{2}{p_0}}\frac{dt}{t^{2\sigma+n+1}}\Big)^{\frac12}\\ &\le Cr^{\frac{n}{p_0'}+\frac{n}{2}-(n-\sigma)}r^{\alpha+\frac{n}{p_0}}\|f\|_{\mathcal{E}^{\alpha,p_0}}r^{-\sigma-\frac{n}{2}}\le Cr^\alpha\|f\|_{\mathcal{E}^{\alpha,p}}.\end{aligned}$$

(ii) The case $\sigma\ge n$. In this case we see easily

$$\begin{aligned}\mu^{*,\rho}_{\lambda,\infty}(f_2)(x) &\le \Big(\int_r^\infty\int_{\mathbb{R}^n}\Big(\frac{1}{t^\sigma}\int_{|y-u|\le t}t^{\sigma-n}\|\Omega\|_\infty|f_2(y)|\,dy\Big)^2\Big(1+\frac{|u-x|}{t}\Big)^{-\lambda n}\frac{dudt}{t^{n+1}}\Big)^{\frac12}\\ &\le C\Big(\int_r^\infty\Big(\int_{\mathbb{R}^n}\Big(1+\frac{|u-x|}{t}\Big)^{-\lambda n}\frac{du}{t^n}\Big)\frac{dt}{t^{2n+1}}\Big)^{\frac12}\int_{4B}|f(y)-f_{4B}|\,dy\\ &\le Cr^{-n}r^{\alpha+n}\|f\|_{\mathcal{E}^{\alpha,1}}\le Cr^\alpha\|f\|_{\mathcal{E}^{\alpha,p}}.\end{aligned}$$

(iii) The case $\alpha=0$. In this case $\mathcal{E}^{\alpha,p}=\mathrm{BMO}$ $(1\le p<\infty)$, and the norms are equivalent. So, take $p=2$ in the above (i) and (ii).

(iv) The case $0<\alpha<1$. In this case $\mathcal{E}^{\alpha,p}=\mathrm{Lip}_\alpha$ $(1\le p<\infty)$, and the norms are equivalent. So, take $p=2$ in the above (i) and (ii). □

As for $\mu^{*,\rho}_\lambda(f_2)$, we need

Lemma 10. *Let $\Omega\in L^\infty(S^{n-1})$, $\rho=\sigma+i\tau$ $(\sigma>0,\tau\in\mathbb{R})$, $\lambda>1$, $1\le p<+\infty$ and $-\frac{n}{p}\le\alpha<1$. Then, for any $f\in\mathcal{E}^{\alpha,p}$, any ball $B=B(x_0,r)$ and any $x\in B$*

$$\mu^{*,\rho}_{\lambda,0,\infty}(f_2)(x)=\Big(\int_0^r\int_{|u-x|>8r}\Big|\frac{1}{t^\rho}\int_{|y-u|\le t}\frac{\Omega(u-y)f_2(y)}{|u-y|^{n-\rho}}\,dy\Big|^2\Big(\frac{t}{t+|u-x|}\Big)^{\lambda n}\frac{dudt}{t^{n+1}}\Big)^{\frac12}=0.$$

Proof. For $|y-x_0|\le 4r$, $|y-u|\le t\le r$ and $|x-x_0|\le r$, we have $|u-x|\le|u-y|+|y-x_0|+|x_0-x|\le 6r$, and hence the integration u-domain of the above integral is empty. □

Next we investigate $\mu^\rho(f_3)$, $\mu^\rho_S(f_3)$ and $\mu^{*,\rho}_\lambda(f_3)$.

Lemma 11. *Let $\Omega\in L^\infty(S^{n-1})$, $\rho=\sigma+i\tau$ $(\sigma>0,\tau\in\mathbb{R})$, $1\le p<+\infty$ and $-\frac{n}{p}\le\alpha<1$. Then, for any $f\in\mathcal{E}^{\alpha,p}$, for any ball $B=B(x_0,r)$ and any $x\in B$*

$$\mu^\rho_0(f_3)(x)=\Big(\int_0^r\Big|\frac{1}{t^\rho}\int_{|y-x|\le t}\frac{\Omega(x-y)f_3(y)}{|x-y|^{n-\rho}}\,dy\Big|^2\frac{dt}{t}\Big)^{\frac12}=0,$$

and

$$\mu_{S,0}^{\rho}(f_3)(x) = \Big(\int_0^r \int_{|u-x|\le t}\Big|\frac{1}{t^\rho}\int_{|y-u|\le t}\frac{\Omega(u-y)f_3(y)}{|u-y|^{n-\rho}}\,dy\Big|^2\frac{dudt}{t^{n+1}}\Big)^{\frac12} = 0.$$

Proof. For $|x-x_0|\le r$ and $|x-y|\le t\le r$, we have $|x_0-y|\le 2r$, and hence the integration domain with respect to y has no intersection with the support of f_3 in the expression of $\mu_0^\rho(f_3)$. So, we have $\mu_0^\rho(f_3)=0$ for $x\in B$.

For $|x-x_0|\le r$, $|u-x|\le t\le r$ and $|u-y|\le t\le r$, we have $|x_0-y|\le|x_0-x|+|x-u|+|u-y|\le 3r$, and hence the integration domain with respect to y has no intersection with the support of f_3 in the expression of $\mu_{S,0}^\rho(f_3)$. So, we have $\mu_{S,0}^\rho(f_3)=0$ for $x\in B$. □

Lemma 12. *Let* $\Omega\in L^\infty(S^{n-1})$, $\rho=\sigma+i\tau$ $(\sigma>0,\tau\in\mathbb{R})$, $\lambda>1$. *Suppose* α, λ *and* p *satisfiy (a)* $\max(1,\frac{2n}{n+2\sigma})<p<+\infty$, $\lambda>\max(1,\frac{2}{p})$ *and* $-\frac{n}{p}\le\alpha<1$ *or (b)* $1\le p<+\infty$, $\lambda>1+\frac{2\alpha}{n}$ *and* $0\le\alpha<1$. *Then, for any* $f\in\mathcal{E}^{\alpha,p}$, *for any ball* $B=B(x_0,r)$ *and any* $x\in B$

$$\mu_{\lambda,0}^{*,\rho}(f_3)(x) = \Big(\int_0^r\int_{\mathbb{R}^n}\Big|\frac{1}{t^\rho}\int_{|y-u|\le t}\frac{\Omega(u-y)f_3(y)}{|u-y|^{n-\rho}}\,dy\Big|^2\Big(\frac{t}{t+|u-x|}\Big)^{\lambda n}\frac{dudt}{t^{n+1}}\Big)^{\frac12}\le Cr^\alpha\|f\|_{\mathcal{E}^{\alpha,p}}.$$

Proof. (i) The case $0<\sigma<n$ and $\max(1,\frac{2n}{n+2\sigma})<p<+\infty$. Take p_0 and a as in the proof of Lemma 10. Then, by Hölder's inequality we have

$$\begin{aligned}\Big|\int_{|y-u|\le t}&\frac{\Omega(u-y)f_3(y)}{|u-y|^{n-\rho}}\,dy\Big|\\ &\le\|\Omega\|_\infty\Big(\int_{|y-u|\le t}\frac{dy}{|u-y|^{(n-\sigma)(1-a)p_0'}}\Big)^{\frac{1}{p_0'}}\Big(\int_{|y-u|\le t}\frac{|f_3(y)|^{p_0}\,dy}{|u-y|^{(n-\sigma)ap_0}}\Big)^{\frac{1}{p_0}}\\ &\le Ct^{\frac{n}{p_0'}-(n-\sigma)(1-a)}\Big(\int_{|y-u|\le t}\frac{|f_3(y)|^{p_0}\,dy}{|u-y|^{(n-\sigma)ap_0}}\Big)^{\frac{1}{p_0}}.\end{aligned}$$

Hence using Minkowski's inequality ($\frac{2}{p_0}\ge1$) and then noting $|u-x|\ge|y-x_0|-|y-u|-|x_0-x|>\frac14(|y-x_0|+r)$ for $|u-y|\le t\le r$, $|y-x_0|>4r$ and $|x_0-x|\le r$, we have

$$\begin{aligned}&\mu_{\lambda,0}^{*,\rho}(f_3)(x)\\ &\le C\Big(\int_0^r\Big(\int_{\mathbb{R}^n}\Big(\int_{|y-u|\le t}\frac{|f_3(y)|^{p_0}\,dy}{|u-y|^{(n-\sigma)ap_0}}\Big)^{\frac{2}{p_0}}\frac{du}{(\frac{t+|u-x|}{t})^{\lambda n}}\,t^{\frac{2n}{p_0'}-2(n-\sigma)(1-a)-2\sigma-n-1}\,dt\Big)^{\frac12}\\ &\le C\Big(\int_0^r\Big(\int_{4B}\Big(\int_{\mathbb{R}^n}\frac{\chi_{|y-u|\le t}}{|u-y|^{2(n-\sigma)a}}\frac{du}{(\frac{t+|u-x|}{t})^{\lambda n}}\Big)^{\frac{p_0}{2}}|f_3(y)|^{p_0}dy\Big)^{\frac{2}{p_0}}t^{\frac{2n}{p_0'}-2(n-\sigma)(1-a)-2\sigma-n-1}dt\Big)^{\frac12}\\ &\le C\Big(\int_0^r\Big(\int_{4B}\Big(\int_{\mathbb{R}^n}\frac{\chi_{|y-u|\le t}\,du}{|u-y|^{2(n-\sigma)a}}\Big)^{\frac{p_0}{2}}\frac{|f_3(y)|^{p_0}\,dy}{(r+|y-x_0|)^{\frac{p_0\lambda n}{2}}}\Big)^{\frac{2}{p_0}}t^{\lambda n+\frac{2n}{p_0'}-2(n-\sigma)(1-a)-2\sigma-n-1}dt\Big)^{\frac12}\\ &\le C\Big(\int_0^r\Big(\int_{4B}\frac{|f(y)-f_{4B}|^{p_0}\,dy}{(r+|y-x_0|)^{\frac{p_0\lambda n}{2}}}\Big)^{\frac{2}{p_0}}t^{n-2(n-\sigma)a}t^{\lambda n+\frac{2n}{p_0'}-2(n-\sigma)(1-a)-2\sigma-n-1}dt\Big)^{\frac12}\\ &\le C\Big(\int_0^r t^{\lambda n+\frac{2n}{p_0'}-2(n-\sigma)-2\sigma-1}\,dt\Big)^{\frac12}r^{\alpha-(\frac{p_0\lambda}{2}-1)\frac{n}{p_0}}\|f\|_{\mathcal{E}^{\alpha,p_0}}\\ &\le Cr^{\frac12(\lambda n+\frac{2n}{p_0'}-2n)}r^{\alpha-\frac{\lambda n}{2}+\frac{n}{p_0}}\|f\|_{\mathcal{E}^{\alpha,p_0}}\le Cr^\alpha\|f\|_{\mathcal{E}^{\alpha,p}}.\end{aligned}$$

We have used here $\lambda n-\frac{2n}{p_0}>0$, $\alpha<(\frac{\lambda}{2}-\frac{1}{p_0})n$ and Lemma 1.

(ii) The case $\sigma \geq n$. In this case, we take $p_0 = \min(2, p)$ and $a = 0$. Then the reasoning in the step (i) still works.

(iii) The case $\alpha = 0$. In this case $\mathcal{E}^{\alpha,p} = \mathrm{BMO}$ $(1 \leq p < \infty)$, and the norms are equivalent. So, take $p = 2$ in the above (i) and (ii).

(iv) The case $0 < \alpha < 1$. In this case $\mathcal{E}^{\alpha,p} = \mathrm{Lip}_\alpha$ $(1 \leq p < \infty)$, and the norms are equivalent. So, taking $p = p_0 = 2$ in the above (i) and (ii) and noting $\lambda > 1 + \frac{2\alpha}{n}$ implies $\alpha < (\frac{\lambda}{2} - \frac{1}{2})n$, we have the desired inequality. □

Lemma 13. *Let $\Omega \in L^\infty(S^{n-1})$, $\rho = \sigma + i\tau$ $(\sigma > 0, \tau \in \mathbb{R})$, $\lambda > 1$, $1 \leq p < +\infty$ and $0 < \alpha < 1$. Then, for any $f \in \mathcal{E}^{\alpha,p}$, for any ball $B = B(x_0, r)$ and any $x \in B$*

$$\mu^{*,\rho}_{\lambda,0,0}(f_3)(x) \leq Cr^\alpha \|f\|_{\mathcal{E}^{\alpha,p}},$$

where

$$\mu^{*,\rho}_{\lambda,0,0}(f_3)(x) = \left(\int_0^r \int_{|u-x|\leq 8r} \left| \frac{1}{t^\rho} \int_{|y-u|\leq t} \frac{\Omega(u-y) f_3(y)}{|u-y|^{n-\rho}}\, dy \right|^2 \left(\frac{t}{t + |u-x|}\right)^{\lambda n} \frac{du dt}{t^{n+1}}\right)^{\frac{1}{2}},$$

and $f_3(x) = (f(x) - f_{4B})\chi_{(4B)^c}$.

Proof. In this case $\mathcal{E}^{\alpha,p} = \mathrm{Lip}_\alpha$ $(1 \leq p < \infty)$, and the norms are equivalent. So, for $y \in (4B)^c$ we have $|f_3(y)| = |f(y) - f_{4B}| \leq |f(y) - f(x_0)| + |f(x_0) - f_{4B}| \leq \|f\|_{\mathrm{Lip}_\alpha}(|y - x_0|^\alpha + r^\alpha) \leq C\|f\|_{\mathrm{Lip}_\alpha}|y - x_0|^\alpha$. For $|x - x_0| \leq r$, $|u - y| \leq t \leq r$ and $|u - x| \leq 8r$, we have $|y - x_0| \leq |y - u| + |u - x| + |x - x_0| \leq 10r$, and for $|x - x_0| \leq r$, $|u - y| \leq t \leq r$ and $x \in (4B)^c$ we have $|u - x| \geq |y - x_0| - |u - y| - |x_0 - x| > \frac{1}{2}|y - x_0| > 2r$. Hence we have

$$\begin{aligned}
&\mu^{*,\rho}_{\lambda,0,0}(f_3)(x) \\
&\leq C\|\Omega\|_\infty \left(\int_0^r \int_{2r<|u-x|\leq 8r} \left| \frac{1}{t^\sigma} \int_{\substack{|y-u|\leq t \\ 4r<|y-x_0|\leq 10r}} \frac{|y-x_0|^\alpha}{|u-y|^{n-\sigma}}\, dy \right|^2 \left(\frac{t}{t+2r}\right)^{\lambda n} \frac{du dt}{t^{n+1}}\right)^{\frac{1}{2}} \|f\|_{\mathrm{Lip}_\alpha} \\
&\qquad \leq C\left(\int_0^r \int_{2r<|u-x|\leq 8r} \left| \frac{1}{t^\sigma} \int_{|y-u|\leq t} \frac{r^\alpha}{|u-y|^{n-\sigma}}\, dy \right|^2 t^{\lambda n} r^{-\lambda n} \frac{du dt}{t^{n+1}}\right)^{\frac{1}{2}} \|f\|_{\mathrm{Lip}_\alpha} \\
&\qquad\qquad \leq C\left(\int_0^r r^{2\alpha - \lambda n} r^n t^{\lambda n - n - 1}\, dt\right)^{\frac{1}{2}} \|f\|_{\mathrm{Lip}_\alpha} \\
&\qquad\qquad\qquad \leq C(r^{2\alpha - \lambda n + n} r^{\lambda n - n})^{\frac{1}{2}} \|f\|_{\mathrm{Lip}_\alpha} \leq Cr^\alpha \|f\|_{\mathcal{E}^{\alpha,p}}.
\end{aligned}$$

□

Now we prepare three more lemmas.

Lemma 14. *Let $\Omega \in \mathrm{Lip}_\beta(S^{n-1})$ $(0 < \beta \leq 1)$, $\rho = \sigma + i\tau$ $(\sigma > 0, \tau \in \mathbb{R})$, $1 < p < \infty$, and $-n/p \leq \alpha < \beta$. Then there exists $C > 0$ such that for any ball $B = B(x_0, r)$ and any $f \in \mathcal{E}^{\alpha,p}$ satisfying $\mu^\rho(f_3)(x_0) < +\infty$, it holds $\mu^\rho(f_3)(x) < +\infty$ for any $x \in B$ and*

$$|\mu^\rho(f_3)(x) - \mu^\rho(f_3)(x_0)| \leq Cr^\alpha \|f\|_{\mathcal{E}^{\alpha,p}} \text{ for any } x \in B,$$

where $f_3(x) = (f(x) - f_{4B})\chi_{(4B)^c}$.

Lemma 15. *Let $\Omega \in \mathrm{Lip}_\beta(S^{n-1})$ $(0 < \beta \leq 1)$, $\rho = \sigma + i\tau$ $(\sigma > 0, \tau \in \mathbb{R})$, $\max(1, \frac{2n}{n+2\sigma}) < p < \infty$, and $-n/p \leq \alpha < 1/2$ or $1/2 \leq \alpha < \min(\beta, \sigma)$. Then there exists $C > 0$ such that for any ball $B = B(x_0, r)$ and any $f \in \mathcal{E}^{\alpha,p}$ satisfying $\mu^\rho_{S,\infty}(f_3)(x_0) < +\infty$, it holds $\mu^\rho_{S,\infty}(f_3)(x) < +\infty$ for any $x \in B$ and*

$$|\mu^\rho_{S,\infty}(f_3)(x) - \mu^\rho_{S,\infty}(f_3)(x_0)| \leq Cr^\alpha \|f\|_{\mathcal{E}^{\alpha,p}} \text{ for any } x \in B,$$

where $f_3(x) = (f(x) - f_{4B})\chi_{(4B)^c}$.

Lemma 16. *Let* $\Omega \in \mathrm{Lip}_\beta(S^{n-1})$ $(0 < \beta \le 1)$, $\rho = \sigma + i\tau$ $(\sigma > 0, \tau \in \mathbb{R})$, $\max(1, \frac{2n}{n+2\sigma}) < p < \infty$, *and* $-n/p \le \alpha < 1/2$ *or* $1/2 \le \alpha < \min(\beta, \sigma)$. *Then there exists* $C > 0$ *such that for any ball* $B = B(x_0, r)$ *and any* $f \in \mathcal{E}^{\alpha,p}$ *satisfying* $\mu^{*,\rho}_{\lambda,\infty}(f_3)(x_0) < +\infty$, *it holds* $\mu^{*,\rho}_{\lambda,\infty}(f_3)(x) < +\infty$ *for any* $x \in B$ *and*

$$|\mu^{*,\rho}_{\lambda,\infty}(f_3)(x) - \mu^{*,\rho}_{\lambda,\infty}(f_3)(x_0)| \le Cr^\alpha \|f\|_{\mathcal{E}^{\alpha,p}} \text{ for any } x \in B,$$

provided $\lambda > 1$ *in the case* $0 \le \alpha < \frac{1}{2}$, $\lambda > 1 + \frac{2\alpha}{n}$ *in the case* $\frac{1}{2} \le \alpha < 1$ *and* $\lambda > \max(1, \frac{2}{p})$ *in the case* $-\frac{n}{p} \le \alpha < 0$, *where* $f_3(x) = (f(x) - f_{4B})\chi_{(4B)^c}$.

We can prove the above three lemmas modifying the proofs in the cube setting (see, Han [2], Qiu [6], Yabuta [14] and Sakamoto and Yabuta [15]. We give here a way to use the cube setting directly in the case of Lemma 16. Let Q be a cube with center x_0 and side length $2r$, Q' be the cube with center x_0 and side length $16\sqrt{n}r$. Let B be the ball with center x_0 and radius r. Let $f_3(x) = (f(x) - f_{4B})\chi_{(4B)^c}$ and $f_4(x) = (f(x) - f_{Q'})\chi_{(Q')^c}$. Then we have

Lemma 17. *Let* $\Omega \in L^\infty(S^{n-1})$ *and* $\int_{S^{n-1}} \Omega(x)\,d\sigma(x) = 0$. *Let* $\rho = \sigma + i\tau$ $(\sigma > 0, \tau \in \mathbb{R})$, $1 \le p < \infty$, *and* $-n/p \le \alpha < 1$. *Let* x_0, B, Q, Q', f_3 *and* f_4 *be as above. Then, there exists* $C > 0$ *such that for any* $x \in B$

$$\Big(\int_0^\infty \int_{\mathbb{R}^n} \Big|\frac{1}{t^\rho} \int_{|y-u| \le t} \frac{\Omega(u-y)\big(f_3(y) - f_4(y)\big)}{|u-y|^{n-\rho}}\,dy\Big|^2 \Big(\frac{t}{t + |u-x|}\Big)^{\lambda n} \frac{dudt}{t^{n+1}}\Big)^{\frac{1}{2}} \le Cr^\alpha \|f\|_{\mathcal{E}^{\alpha,p}}.$$

Proof. Let I be the left hand side of the above inequality in the statement of Lemma 17. Then by the assumption $\int_{S^{n-1}} \Omega(x)\,d\sigma(x) = 0$ we see that

$$\begin{aligned}
I &= \Big(\int_0^\infty\int_{\mathbb{R}^n} \Big|\frac{1}{t^\rho}\int_{|y-u|\le t} \frac{\Omega(u-y)\big(f(y) - f_{Q'}\big)(\chi_{Q'} - \chi_{(4B)^c})}{|u-y|^{n-\rho}}\,dy\Big|^2 \Big(\frac{t}{t+|u-x|}\Big)^{\lambda n} \frac{dudt}{t^{n+1}}\Big)^{\frac{1}{2}} \\
&\le \|\Omega\|_\infty \Big(\int_0^\infty \int_{\mathbb{R}^n} \Big|\frac{1}{t^\sigma} \int_{\substack{|y-u|\le t \\ 4r<|y-x_0|<8nr}} \frac{|f(y) - f_{Q'}|}{|u-y|^{n-\sigma}}\,dy\Big|^2 \Big(\frac{t}{t+|u-x|}\Big)^{\lambda n} \frac{dudt}{t^{n+1}}\Big)^{\frac{1}{2}} \\
&\le C\Big(\int_0^r \int_{\mathbb{R}^n} \Big|\frac{1}{t^\sigma} \int_{\substack{|y-u|\le t \\ 4r<|y-x_0|<8nr}} \frac{|f(y) - f_{Q'}|}{|u-y|^{n-\sigma}}\,dy\Big|^2 \Big(\frac{t}{t+|u-x|}\Big)^{\lambda n} \frac{dudt}{t^{n+1}}\Big)^{\frac{1}{2}} \\
&\quad + C\Big(\int_r^\infty \int_{\mathbb{R}^n} \Big|\frac{1}{t^\sigma} \int_{\substack{|y-u|\le t \\ 4r<|y-x_0|<8nr}} \frac{|f(y) - f_{Q'}|}{|u-y|^{n-\sigma}}\,dy\Big|^2 \Big(\frac{t}{t+|u-x|}\Big)^{\lambda n} \frac{dudt}{t^{n+1}}\Big)^{\frac{1}{2}} \\
&\hspace{30em} =: I_1 + I_2.
\end{aligned}$$

$I_1 \le Cr^\alpha \|f\|_{\mathcal{E}^{\alpha,p}}$ can be proved in a way quite similar to the proof of Lemma 12, and $I_2 \le Cr^\alpha \|f\|_{\mathcal{E}^{\alpha,p}}$ can be proved in a way quite similar to the proof of Lemma 9. □

Using this lemma, we can use the corresponding result to Lemma 16 in the cube setting. We note here that in Sakamoto and Yabuta [7, pp. 137–141], they really proved $|\mu^{*,\rho}_{\lambda,\infty}(f_3)(x) - \mu^{*,\rho}_{\lambda,\infty}(f_3)(x_0)| \le Cr^\alpha \|f\|_{\mathcal{E}^{\alpha,p}}$ in the case $\frac{1}{2} \le \alpha < 1$ and $\lambda > 1 + \frac{2\alpha}{n}$.

Proofs of Theorems 4, 5 and 6. Using Lemmas 7–16 and L^p boundedness results in [7], we can prove these theorems in the same way as in the proof of Theorem 3, and so we leave the detailed proofs to the reader. □

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