FUZZY IMPLICATIVE LI-IDEALS IN LATTICE IMPLICATION ALGEBRAS

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Abstract. The fuzzification of an implicative LI-ideal is considered and some of their properties are investigated. Characterizations of a fuzzy implicative LI-ideal are established. Conditions for a fuzzy LI-ideal to be a fuzzy implicative LI-ideal are provided. Extension property of a fuzzy implicative LI-ideal is built.

1. Introduction

Non-classical logic has become a considerable formal tool for computer science and artificial intelligence to deal with fuzzy and/or uncertain information. In the field of many-valued logic, lattice-valued logic plays an important role for two aspects: Firstly, it extends the chain-type truth-value field of some well-known presented logic [1] (such as two-valued logic, three-valued logics introduced by Łukasiewicz, Bochvar, Kleene, Heyting, Finn, Hallden, Segerberg, Slupecki and Sobociński, n-valued logics introduced by Łukasiewicz, Post, Slupecki, Sobociński and Gödel, as well as the Łukasiewicz logic with truth value in the interval $[0, 1]$ or Zadeh's infinite-valued logic, etc.) to some relatively general lattices. Secondly, the incompletely comparable property of truth value characterized by general lattice can more efficiently reflect the uncertainty of people's thinking, judging and decision. Hence, lattice-valued logic is becoming a research field which strongly influences the development of Algebraic Logic, Computer Science and Artificial Intelligence Technology. In 1969, Goguen proposed the first lattice-valued logic formal system based on complete-lattice-ordered semigroups [2], where the author did not provide a syntax associated with the given semantics. However, the concept of enriched residuated lattice introduced by Goguen provided a new idea and approach to study the lattice-valued logic. So, in 1979, Pavelka proposed a lattice-valued propositional logic system based on enriched residuated lattices [15]. Although this logic is based on relatively general lattice, its main results are limited to the interval $[0, 1]$ or the finite chain of truth value. In spite of such limitation, these results reflect some fundamental characteristics of fuzzy logic. Pavelka's work is concerned only with propositional fuzzy logic. In 1982, Novak extended it to first-order fuzzy logic based on the interval $[0, 1]$ or a finite chain [14], especially including some additional generalized quantifiers, and proved the soundness theorem and completeness theorem of this formal system. In order to establish a logic system with truth value in a relatively general lattice, in 1990, during the study of the project "The Study of Abstract Fuzzy Logic" granted by National Natural Science Foundation in China, Xu firstly established the lattice implication algebra by combining lattice and implication algebra, and investigated many useful structures

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[11, 12, 16, 17, 18]. Lattice implication algebra provided the foundation to establish the corresponding logic system from the algebraic viewpoint. For the general development of lattice implication algebras, the ideal theory plays an important role as well as the filter theory. Xu and Qin [18] introduced the notions of filter and implicative filter in a lattice implication algebra, and investigated their properties. Jun (together with Xu, Qin, Kim and Roh) studied several filters in lattice implication algebras [3, 4, 5, 8, 10]. In particular, Jun [3] gave an equivalent condition of a filter, and provided some equivalent conditions for a filter to be an implicative filter in a lattice implication algebra. In [6], Jun et al. introduced the notion of LI-ideals in lattice implication algebras and investigated some of its properties. Also, Jun et al. [7, 9] considered the fuzzification of LI-ideals. In [13], Liu, Xu, Qin and Liu introduced the notion of ILI-ideals in lattice implication algebras. In this paper, we discuss the fuzzification of implicative LI-ideals in lattice implication algebras. We five characterizations of a fuzzy implicative LI-ideal. We provide conditions for a fuzzy LI-ideal to be a fuzzy implicative LI-ideal. We build the extension property of a fuzzy implicative LI-ideal.

2. Preliminaries

Definition 2.1. [16] A *lattice implication algebra* is defined to be a bounded lattice (L; $\vee$, $\wedge$, $0, 1$) with order-reversing involution "$'$" and a binary operation "$\to$" satisfying the following axioms:

(I1) $x \to (y \to z) = y \to (x \to z)$,
(I2) $x \to x = 1$,
(I3) $x \to y = y' \to x'$,
(I4) $x \to y = y \to x = 1 \Rightarrow x = y$,
(I5) $(x \to y) \to y = (y \to x) \to x$,
(L1) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$,
(L2) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$,

for all $x, y, z \in L$.

A lattice implication algebra L is called a *lattice H-implication algebra* if it satisfies $x \vee y \vee ((x \wedge y) \to z) = 1$ for all $x, y, z \in L$. We can define a partial ordering $\leq$ on a lattice implication algebra L by $x \leq y$ if and only if $x \to y = 1$. In a lattice implication algebra L, the following hold (see [16]):

(p1) $0 \to x = 1$, $1 \to x = x$ and $x \to 1 = 1$.
(p2) $x \to y \leq (y \to z) \to (x \to z)$.
(p3) $x \leq y$ implies $y \to z \leq x \to z$ and $z \to x \leq z \to y$.
(p4) $x' = x \to 0$.
(p5) $x \vee y = (x \to y) \to y$.
(p6) $((y \to x) \to y')' = x \wedge y = ((x \to y) \to x')'$.
(p7) $x \leq (x \to y) \to y$.
(p8) $((x \to y) \to y) \to y = x \to y$.

In a lattice implication algebra L, the following are equivalent.

(q1) $x \to (x \to y) = x \to y$.
(q2) $x \to (y \to z) = (x \to y) \to (x \to z)$.
(q3) $x \to (y \to z) = (x \wedge y) \to z$.
(q4) $(x \to y) \to x = x$.
(q5) L is a lattice H-implication algebra.

Definition 2.2. [6] A subset A of a lattice implication algebra L is called an *LI-ideal* of L if it satisfies

(Id1) $0 \in A$,
(Id2) $\forall x, y \in L$, $(x \to y)' \in A$, $y \in A \ \Rightarrow\ x \in A$.

Proposition 2.3. [6] *Let A be an LI-ideal of a lattice implication algebra L and let $x, y \in L$. If $y \le x$ and $x \in A$, then $y \in A$.*

Proposition 2.4. [13, Theorem 2.5] *Let I be a nonempty subset of a lattice implication algebra L. Then I is an LI-ideal of L if and only if it satisfies*

$$\forall x, y \in I, \forall z \in L,\ (z \to x)' \le y \ \Rightarrow\ z \in I.$$

Definition 2.5. [7] A fuzzy set $\bar{A}$ in a lattice implication algebra L is called a *fuzzy LI-ideal* of L if it satisfies
(F1) $\bar{A}(0) \ge \bar{A}(x),\ \forall x \in L$.
(F2) $\bar{A}(x) \ge \min\{\bar{A}((x \to y)'),\ \bar{A}(y)\},\ \forall x, y \in L$.

Note that every fuzzy LI-ideal is order reversing (see [7, Proposition 3.4]). Let L and M be lattice implication algebras. A mapping $\mathfrak{f} : L \to M$ is called an *implication homomorphism* if $\mathfrak{f}(x \to y) = \mathfrak{f}(x) \to \mathfrak{f}(y)$ for all $x, y \in L$.

Let $\mathfrak{f}$ be a mapping from a set L to a set M and let $\bar{A}$ and $\bar{B}$ be fuzzy sets in L and M, respectively. Then $\mathfrak{f}(\bar{A})$, the *image* of $\bar{A}$ under $\mathfrak{f}$, is a fuzzy set in M:

$$\mathfrak{f}(\bar{A})(y) := \begin{cases} \sup\limits_{x \in \mathfrak{f}^{-1}(y)} \bar{A}(x) & \text{if } \mathfrak{f}^{-1}(y) \ne \emptyset, \\ 0 & \text{if } \mathfrak{f}^{-1}(y) = \emptyset, \end{cases}$$

for all $y \in M$. The *preimage* of $\bar{B}$ under $\mathfrak{f}$, $\mathfrak{f}^{-1}(\bar{B})$, is a fuzzy set in L given by $\mathfrak{f}^{-1}(\bar{B})(x) = \bar{B}(\mathfrak{f}(x))$ for all $x \in L$. A fuzzy set $\bar{A}$ in L has *sup property* if for every subset T of L there exists $x_0 \in T$ such that $\bar{A}(x_0) = \sup\limits_{u \in T} \bar{A}(u)$.

3. Fuzzy implicative LI-ideals

In what follows, let L denote a lattice implication algebra unless otherwise specified. We first give a characterization of a fuzzy LI-ideal.

Theorem 3.1. *A fuzzy set $\bar{A}$ in L is a fuzzy LI-ideal of L if and only if it satisfies*

$$\forall x, y, z \in L,\ (z \to x)' \le y \ \Rightarrow\ \bar{A}(z) \ge \min\{\bar{A}(x),\ \bar{A}(y)\}.$$

Proof. Suppose that $\bar{A}$ is a fuzzy LI-ideal of L. Let $x, y, z \in L$ be such that $(z \to x)' \le y$. Since every fuzzy LI-ideal is order reversing, it follows that $\bar{A}((z \to x)') \ge \bar{A}(y)$ so from (F2) that

$$\bar{A}(z) \ge \min\{\bar{A}((z \to x)'),\ \bar{A}(x)\} \ge \min\{\bar{A}(x), \bar{A}(y)\}.$$

Conversely, suppose that $\bar{A}$ satisfies the given condition. Since $(0 \to x)' \le x$ for all $x \in L$, we have $\bar{A}(0) \ge \bar{A}(x)$ for all $x \in L$, which is (F1). Note that $(x \to y)' \le (x \to y)'$ for all $x, y \in L$. Hence, by assumption, we get

$$\bar{A}(x) \ge \min\{\bar{A}(y), \bar{A}((x \to y)')\}$$

which is (F2). Hence $\bar{A}$ is a fuzzy LI-ideal of L. $\square$

Definition 3.2. [13] A nonempty subset A of L is called an *implicative LI-ideal* of L if it satisfies (Id1) and
(Id3) $\forall x, y, z \in L$, $(((x \to y)' \to y)' \to z)' \in A$, $z \in A \ \Rightarrow\ (x \to y)' \in A$.

Proposition 3.3. [13, Theorems 3.8 and 3.12] *Let I be an LI-ideal of L. Then the following are equivalent.*

(i) *I is an implicative LI-ideal of L.*
(ii) $\forall x, y \in L,\ (x \to (y \to x)')' \in I \ \Rightarrow\ x \in I.$
(iii) $\forall x, y \in L,\ ((x \to y)' \to y)' \in I \ \Rightarrow\ (x \to y)' \in I.$

Definition 3.4. A fuzzy set $\bar{A}$ in L is called a *fuzzy implicative LI-ideal* of L if it satisfies (F1) and

(F3) $\bar{A}((x \to y)') \geq \min\{\bar{A}((((x \to y)' \to y)' \to z)'),\ \bar{A}(z)\},\ \forall x, y, z \in L.$

Example 3.5. Let $L = \{0, a, b, 1\}$ be a set with Cayley tables as follows:

x	x'
0	1
a	b
b	a
1	0

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Define $\vee$- and $\wedge$-operations on L as follows:

$$x \vee y := (x \to y) \to y,\ x \wedge y := ((x' \to y') \to y')'$$

for all $x, y \in L$. Then $(L, \vee, \wedge, \to, ')$ is a lattice implication algebra (see [13]). Let $\bar{A}$ be a fuzzy set in L defined by $\bar{A}(0) = 0.7$ and $\bar{A}(a) = \bar{A}(b) = \bar{A}(1) = 0.3$. Then $\bar{A}$ is a fuzzy implicative LI-ideal of L. Also a fuzzy set $\bar{B}$ in L given by $\bar{B}(0) = \bar{B}(a) = 0.6$ and $\bar{B}(b) = \bar{B}(1) = 0.2$ is a fuzzy implicative LI-ideal of L.

We give the relation between fuzzy LI-ideals and fuzzy implicative LI-ideals.

Theorem 3.6. *Any fuzzy implicative LI-ideal of L is a fuzzy LI-ideal of L.*

Proof. Let $\bar{A}$ be a fuzzy implicative LI-ideal of L. Taking $y = 0$ and $z = y$ in (F3), we have

$$\begin{aligned} \bar{A}(x) &= \bar{A}((x \to 0)') \\ &\geq \min\{\bar{A}((((x \to 0)' \to 0)' \to y)'),\ \bar{A}(y)\} \\ &= \min\{\bar{A}((x \to y)'),\ \bar{A}(y)\}. \end{aligned}$$

Hence $\bar{A}$ is a fuzzy LI-ideal of L. □

The converse of Theorem 3.6 may not be true as seen in the following example.

Example 3.7. Let $L = \{0, a, b, 1\}$ be a set with Cayley tables as follows:

x	x'
0	1
a	b
b	a
1	0

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

Define $\vee$- and $\wedge$-operations on L as follows:

$$x \vee y := (x \to y) \to y,\ x \wedge y := ((x' \to y') \to y')'$$

for all $x, y \in L$. Then $(L, \vee, \wedge, \to, ')$ is a lattice implication algebra (see [13]). Let $\bar{A}$ be a fuzzy set in L defined by $\bar{A}(0) = 0.8$ and $\bar{A}(a) = \bar{A}(b) = \bar{A}(1) = 0.08$. Then $\bar{A}$ is a fuzzy LI-ideal of L. But it is not a fuzzy implicative LI-ideal of L because

$$\bar{A}((b \to a)') \ngeq \min\{\bar{A}((((b \to a)' \to a)' \to 0)'),\ \bar{A}(0)\}.$$

If we strengthen the condition(s) of L, then we have the following theorem.

Theorem 3.8. *If L is a lattice H-implication algebra, then every fuzzy LI-ideal is a fuzzy implicative LI-ideal.*

Proof. Let $\bar{A}$ be a fuzzy LI-ideal of a lattice H-implication algebra L. Using (I3), (q1), and (F2), we have

$$\begin{aligned}\bar{A}((x \to y)') &= \bar{A}((y' \to x')') = \bar{A}((y' \to (y' \to x'))') \\ &= \bar{A}((y' \to (x \to y))') = \bar{A}(((x \to y)' \to y)') \\ &\geq \min\{\bar{A}((((x \to y)' \to y)' \to z)'),\, \bar{A}(z)\}\end{aligned}$$

for all $x, y, z \in L$, which proves (F3). Hence $\bar{A}$ is a fuzzy implicative LI-ideal of L. □

Proposition 3.9. *Let $\bar{A}$ be a fuzzy set in L satisfying* (F1) *and*
(F4) $\bar{A}(((x \to z)' \to (y \to z)')') \geq \min\{\bar{A}((((x \to y)' \to z)' \to u)'),\, \bar{A}(u)\}$
for all $x, y, z, u \in L$. Then $\bar{A}$ is a fuzzy implicative LI-ideal of L.

Proof. For any $x, y, z \in L$, we have

$$\begin{aligned}\bar{A}((x \to y)') &= \bar{A}((1 \to (x \to y))') && \text{by (p1)} \\ &= \bar{A}((0' \to (x \to y))') && \text{by (I2) and (p4)} \\ &= \bar{A}(((x \to y)' \to 0)') && \text{by (I3)} \\ &= \bar{A}(((x \to y)' \to 1')') && \text{by (p1) and (p4)} \\ &= \bar{A}(((x \to y)' \to (y \to y)')') && \text{by (I2)} \\ &\geq \min\{\bar{A}((((x \to y)' \to y)' \to z)'),\, \bar{A}(z)\} && \text{by (F4)}\end{aligned}$$

which proves (F3). Hence $\bar{A}$ is a fuzzy implicative LI-ideal of L. □

Proposition 3.10. *Let $\bar{A}$ be a fuzzy LI-ideal of L satisfying*
(F5) $\bar{A}(((x \to z)' \to (y \to z)')') \geq \bar{A}(((x \to y)' \to z)')$
for all $x, y, z \in L$. Then $\bar{A}$ is a fuzzy implicative LI-ideal of L.

Proof. Let $x, y, z, u \in L$. Using (F5) and (F2), we have

$$\begin{aligned}\bar{A}(((x \to z)' \to (y \to z)')') &\geq \bar{A}(((x \to y)' \to z)') \\ &\geq \min\{\bar{A}((((x \to y)' \to z)' \to u)'),\, \bar{A}(u)\}.\end{aligned}$$

It follows from Proposition 3.9 that $\bar{A}$ is a fuzzy implicative LI-ideal of L. □

Proposition 3.11. *Let $\bar{A}$ be a fuzzy LI-ideal of L satisfying*
(F6) $\bar{A}((x \to y)') \geq \bar{A}(((x \to y)' \to y)')$
for all $x, y \in L$. Then $\bar{A}$ is a fuzzy implicative LI-ideal of L.

Proof. For any $x, y, z \in L$, we get

$$\begin{aligned}\bar{A}((x \to y)') &\geq \bar{A}(((x \to y)' \to y)') \\ &\geq \min\{\bar{A}((((x \to y)' \to y)' \to z)'),\, \bar{A}(z)\}, && \text{by (F2)}\end{aligned}$$

and so $\bar{A}$ is a fuzzy implicative LI-ideal of L. □

Proposition 3.12. *Every fuzzy implicative LI-ideal $\bar{A}$ of L satisfies the inequality* (F6).

Proof. For any $x, y \in L$, we obtain

$$\begin{aligned}\bar{A}((x \to y)') &\geq \min\{\bar{A}((((x \to y)' \to y)' \to 0)'),\, \bar{A}(0)\} \\ &= \bar{A}((((x \to y)' \to y)' \to 0)') \\ &= \bar{A}((0' \to ((x \to y)' \to y))') \\ &= \bar{A}(((x \to y)' \to y)'),\end{aligned}$$

completing the proof. □

Combining Propositions above, we give a characterization of a fuzzy implicative LI-ideal.

Theorem 3.13. *Let $\bar{A}$ be a fuzzy LI-ideal of L. Then the following statements are equivalent.*

(i) *$\bar{A}$ is a fuzzy implicative LI-ideal of L.*
(ii) *$\bar{A}$ satisfies* (F4).
(iii) *$\bar{A}$ satisfies* (F5).
(iv) *$\bar{A}$ satisfies* (F6).

Theorem 3.14. *Let $\bar{A}$ be a fuzzy LI-ideal of L. Then the following are equivalent.*

(i) *$\bar{A}$ is a fuzzy implicative LI-ideal of L.*
(ii) $\bar{A}(x) \geq \bar{A}((x \to (y \to x)')'),\ \forall x, y \in L.$
(iii) $\bar{A}(x) \geq \min\{\bar{A}(((x \to (y \to x)')' \to z)'),\ \bar{A}(z)\},\ \forall x, y, z \in L.$

Proof. (i) $\Rightarrow$ (ii). Suppose that $\bar{A}$ is a fuzzy implicative LI-ideal of L and let $x, y, z \in L$. Note that

$$\begin{aligned}
&((y \to (y \to x)')' \to (y \to x)')' \to (x \to (y \to x)')' \\
= \;& (x \to (y \to x)') \to ((y \to (y \to x)')' \to (y \to x)') && \text{by (I3)} \\
\geq \;& (y \to (y \to x)')' \to x && \text{by (p2)} \\
= \;& x' \to (y \to (y \to x)') && \text{by (I3)} \\
= \;& x' \to ((y \to x) \to y') && \text{by (I3)} \\
= \;& (y \to x) \to (x' \to y') && \text{by (I1)} \\
= \;& (y \to x) \to (y \to x) = 1. && \text{by (I3) and (I2)}
\end{aligned}$$

Since $x \leq 1$ for all $x \in L$, it follows from (I4) that

$$((y \to (y \to x)')' \to (y \to x)')' \to (x \to (y \to x)')' = 1,$$

i.e., $((y \to (y \to x)')' \to (y \to x)')' \leq (x \to (y \to x)')'$. Since every fuzzy LI-ideal is order reversing, we have

$$\begin{aligned}
\bar{A}((x \to (y \to x)')') &\leq \bar{A}(((y \to (y \to x)')' \to (y \to x)')') \\
&\leq \bar{A}((y \to (y \to x)')'). && \text{by Proposition 3.12}
\end{aligned}$$

Note that

$$\begin{aligned}
(x \to (y \to (y \to x)')')' &= (x \to ((y \to x) \to y')')' && \text{by (I3)} \\
&= (x \to ((x' \to y') \to y')')' && \text{by (I3)} \\
&= (((x' \to y') \to y') \to x')' && \text{by (I3)} \\
&= (((y' \to x') \to x') \to x')' && \text{by (I5)} \\
&= (y' \to x')' && \text{by (p8)} \\
&= (x \to y)' && \text{by (I3)}
\end{aligned}$$

and

$$\begin{array}{rll} & (x \to y)' \to (x \to (y \to x)')' & \\ = & (x \to (y \to x)') \to (x \to y) & \text{by (I3)} \\ \geq & (y \to x)' \to y & \text{by (I1) and (p2)} \\ = & y' \to (y \to x) & \text{by (I3)} \\ = & (y \to 0) \to (y \to x) & \text{by (p4)} \\ \geq & 0 \to x = 1. & \text{by (I1), (p2) and (p1)} \end{array}$$

Hence $(x \to y)' \to (x \to (y \to x)')' = 1$, i.e., $(x \to y)' \leq (x \to (y \to x)')'$, and so

$$\bar{A}((x \to (y \to x)')') \leq \bar{A}((x \to y)') = \bar{A}((x \to (y \to (y \to x)')')').$$

It follows from (F2) that

$$\begin{array}{rcl} \bar{A}(x) & \geq & \min\{\bar{A}((x \to (y \to (y \to x)')')'),\, \bar{A}((y \to (y \to x)')')\} \\ & \geq & \bar{A}((x \to (y \to x)')'), \end{array}$$

which proves (ii).

(ii) $\Rightarrow$ (iii). It is straightforward by (F2).

(iii) $\Rightarrow$ (i). Assume that $\bar{A}$ satisfies the condition (iii) and let $x, y \in L$. Taking $z = 0$ in (iii) and using (F1) and (p4), we get

$$\begin{array}{rcl} \bar{A}(x) & \geq & \min\{\bar{A}(((x \to (y \to x)')' \to 0)'),\, \bar{A}(0)\} \\ & = & \bar{A}(((x \to (y \to x)')' \to 0)') \\ & = & \bar{A}((((x \to (y \to x)')')')') \\ & = & \bar{A}((x \to (y \to x)')'). \end{array} \tag{1}$$

Note that

$$\begin{array}{rll} & ((x \to y)' \to (x \to (x \to y)')')' & \\ = & ((x \to (x \to y)') \to (x \to y))' & \text{by (I3)} \\ = & (((x \to y) \to x') \to (y' \to x'))' & \text{by (I3)} \\ = & (y' \to (((x \to y) \to x') \to x'))' & \text{by (I1)} \\ = & (y' \to ((x' \to (x \to y)) \to (x \to y)))' & \text{by (I5)} \\ = & (y' \to ((x' \to (y' \to x')) \to (y' \to x')))' & \text{by (I3)} \\ = & ((x' \to (y' \to x')) \to (y' \to (y' \to x')))' & \text{by (I1)} \\ = & ((y' \to (x' \to x')) \to (y' \to (y' \to x')))' & \text{by (I1)} \\ = & ((y' \to 1) \to (y' \to (y' \to x')))' & \text{by (I2)} \\ = & (y' \to (y' \to x'))' & \text{by (p1)} \\ = & ((x \to y)' \to y)'. & \text{by (I3)} \end{array}$$

Using (1), we have

$$\bar{A}((x \to y)') \geq \bar{A}(((x \to y)' \to (x \to (x \to y)')')') = \bar{A}(((x \to y)' \to y)'),$$

and so $\bar{A}$ is a fuzzy implicative LI-ideal of L by Proposition 3.11. This completes the proof. □

For any fuzzy set $\bar{A}$ in L, let us denote $U(\bar{A}; t)$, $t \in [0, 1]$, the level set of $\bar{A}$, that is, $U(\bar{A}; t) := \{x \in L \mid \bar{A}(x) \geq t\}$.

Lemma 3.15. [7, Proposition 3.10] *A fuzzy set $\bar{A}$ in L is a fuzzy LI-ideal of L if and only if the nonempty level set $U(\bar{A};t)$ of $\bar{A}$ is an LI-ideal of L, where $t \in [0,1]$.*

Theorem 3.16. *A fuzzy set $\bar{A}$ in L is a fuzzy implicative LI-ideal of L if and only if the nonempty level set $U(\bar{A};t)$ of $\bar{A}$ is an implicative LI-ideal of L, where $t \in [0,1]$.*

Proof. If $\bar{A}$ is a fuzzy implicative LI-ideal of L, then it is a fuzzy LI-ideal of L (see Theorem 3.6). Hence $U(\bar{A};t)$ $(\neq \emptyset)$, $t \in [0,1]$, is an LI-ideal of L by Lemma 3.15. Let $x, y \in L$ be such that $(x \to (y \to x)')' \in U(\bar{A};t)$. Then, by Theorem 3.14(ii), we have $\bar{A}(x) \geq \bar{A}((x \to (y \to x)')') \geq t$, and so $x \in U(\bar{A};t)$. Hence $U(\bar{A};t)$ is an implicative LI-ideal of L by Proposition 3.3. Conversely, suppose that $U(\bar{A};t)$, $t \in [0,1]$, is a nonempty implicative LI-ideal of L. Then $U(\bar{A};t)$ is a nonempty LI-ideal, and so $\bar{A}$ is a fuzzy LI-ideal of L by Lemma 3.15. Now, assume that there exists $x_0, y_0 \in L$ such that $\bar{A}(x_0) \ngeq \bar{A}((x_0 \to (y_0 \to x_0)')')$. Taking

$$t_0 := \frac{1}{2}(\bar{A}(x_0) + \bar{A}((x_0 \to (y_0 \to x_0)')')),$$

we get $\bar{A}(x_0) < t_0 < \bar{A}((x_0 \to (y_0 \to x_0)')')$. Hence $(x_0 \to (y_0 \to x_0)')' \in U(\bar{A};t_0)$ and $x_0 \notin U(\bar{A};t_0)$. This is a contradiction. Therefore $\bar{A}(x) \geq \bar{A}((x \to (y \to x)')')$ for all $x, y \in L$. Using Theorem 3.14, we know that $\bar{A}$ is a fuzzy implicative LI-ideal of L. □

Lemma 3.17. [13, Theorem 3.9] *Let I and J be LI-ideals of L such that $I \subseteq J$. If I is an implicative LI-ideal of L, then so is J.*

Theorem 3.18. (Extension property for fuzzy implicative LI-ideals) *Let $\bar{A}$ and $\bar{B}$ be fuzzy LI-ideals of L such that $\bar{A} \subseteq \bar{B}$, that is, $\bar{A}(x) \leq \bar{B}(x)$ for all $x \in L$. If $\bar{A}$ is a fuzzy implicative LI-ideal of L, then so is $\bar{B}$.*

Proof. Note that the inclusion $\bar{A} \subseteq \bar{B}$ implies that $U(\bar{A};t) \subseteq U(\bar{B};t)$ for every $t \in [0,1]$. If $\bar{A}$ is a fuzzy implicative LI-ideal of L, then $U(\bar{A};t)$ $(\neq \emptyset)$ is an implicative LI-ideal of L for $t \in [0,1]$. Using Lemma 3.17, $U(\bar{B};t)$ $(\neq \emptyset)$ is an implicative LI-ideal of L for $t \in [0,1]$. It follows from Theorem 3.16 that $\bar{B}$ is a fuzzy implicative LI-ideal of L. □

Theorem 3.19. *Let I be a subset of L. For a fixed element $e \in L$, let $\bar{A}_e$ be a fuzzy set in L given by*

$$\bar{A}_e(x) := \begin{cases} t_1 & \text{if } (x \to e)' \in I, \\ t_2 & \text{otherwise,} \end{cases}$$

for all $x \in L$, where $t_1 > t_2$ in $[0,1]$. If I is an implicative LI-ideal of L, then $\bar{A}_e$ is a fuzzy LI-ideal of L.

Proof. Assume that I is an implicative LI-ideal of L. Since $(0 \to e)' = 1' = 0 \in I$, we get $\bar{A}_e(0) = t_1 \geq \bar{A}_e(x)$ for all $x \in L$. Let $x, y \in L$. If $((x \to y)' \to e)' \notin I$ or $(y \to e)' \notin I$, then $\bar{A}_e((x \to y)') = t_2$ or $\bar{A}_e(y) = t_2$. Hence

$$\bar{A}_e(x) \geq t_2 = \min\{\bar{A}_e((x \to y)'),\ \bar{A}_e(y)\}.$$

Assume that $((x \to y)' \to e)' \in I$ and $(y \to e)' \in I$. Note that

$$\begin{aligned} &((x \to e)' \to e)' \to ((x \to y)' \to e)' \\ &= ((x \to y)' \to e) \to ((x \to e)' \to e) \qquad \text{by (I3)} \\ &\geq (x \to e)' \to (x \to y)' \qquad \text{by (p2)} \\ &= (x \to y) \to (x \to e) \qquad \text{by (I3)} \\ &\geq y \to e \qquad \text{by (I1) and (p2)} \end{aligned}$$

so that $(((x \to e)' \to e)' \to ((x \to y)' \to e)')' \leq (y \to e)'$. It follows from Proposition 2.4 that $((x \to e)' \to e)' \in I$, and so $(x \to e)' \in I$ by Proposition 3.3. Therefore $\bar{A}_e(x) = t_1 = \min\{\bar{A}_e((x \to y)'), \bar{A}_e(y)\}$. Consequently, $\bar{A}_e$ is a fuzzy LI-ideal of L. □

Theorem 3.20. *For every LI-ideal I of L and every element e of L, if the fuzzy set $\bar{A}_e$ which is given in Theorem 3.19 is a fuzzy LI-ideal of L, then I is an implicative LI-ideal of L.*

Proof. Suppose that for each $e \in L$, $\bar{A}_e$ is a fuzzy LI-ideal of L. Assume that $((x \to y)' \to y)' \in I$ for all $x, y \in L$. Then $\bar{A}_y((x \to y)') = t_1$. Since $(y \to y)' = 1' = 0 \in I$, we get $\bar{A}_y(y) = t_1$. Using (F2), we have

$$\bar{A}_y(x) \geq \min\{\bar{A}_y((x \to y)'),\, \bar{A}_y(y)\} = t_1,$$

and so $\bar{A}_y(x) = t_1$ which shows that $(x \to y)' \in I$. Therefore I is an implicative LI-ideal of L by Proposition 3.3. □

Theorem 3.21. *Let $\mathfrak{f} : L \to M$ be an onto implication homomorphism of lattice implication algebras such that $\mathfrak{f}(0) = 0$.*

(i) *If $\bar{A}$ is a fuzzy implicative LI-ideal of L with sup property, then $\mathfrak{f}(\bar{A})$ is a fuzzy implicative LI-ideal of M.*
(ii) *If $\bar{B}$ is a fuzzy implicative LI-ideal of M, then $\mathfrak{f}^{-1}(\bar{B})$ is a fuzzy implicative LI-ideal of L.*

Proof. (i) Note that $\mathfrak{f}(\bar{A})(0) = \sup_{z \in \mathfrak{f}^{-1}(0)} \bar{A}(z) = \bar{A}(0) \geq \bar{A}(x)$ for all $x \in L$. Moreover, we have $\mathfrak{f}(\bar{A})(a) = \sup_{x \in \mathfrak{f}^{-1}(a)} \bar{A}(x)$ for all $a \in M$. Thus $\mathfrak{f}(\bar{A})(0) \geq \sup_{x \in \mathfrak{f}^{-1}(a)} \bar{A}(x) = \mathfrak{f}(\bar{A})(a)$ for all $a \in M$. For any $a, b, c \in M$, let $x_0 \in \mathfrak{f}^{-1}(a)$, $y_0 \in \mathfrak{f}^{-1}(b)$, and $z_0 \in \mathfrak{f}^{-1}(c)$ be such that $\bar{A}((x_0 \to y_0)') = \sup_{u \in \mathfrak{f}^{-1}((a \to b)')} \bar{A}(u)$, $\bar{A}(z_0) = \sup_{v \in \mathfrak{f}^{-1}(c)} \bar{A}(v)$, and

$$\bar{A}((((x_0 \to y_0)' \to y_0)' \to z_0)') = \sup_{w \in \mathfrak{f}^{-1}((((a \to b)' \to b)' \to c)')} \bar{A}(w).$$

Then

$$\begin{aligned}
\mathfrak{f}(\bar{A})((a \to b)') &= \sup_{u \in \mathfrak{f}^{-1}((a \to b)')} \bar{A}(u) = \bar{A}((x_0 \to y_0)') \\
&\geq \min\{\bar{A}((((x_0 \to y_0)' \to y_0)' \to z_0)'),\, \bar{A}(z_0)\} \\
&= \min\{\sup_{w \in \mathfrak{f}^{-1}((((a \to b)' \to b)' \to c)')} \bar{A}(w),\, \sup_{v \in \mathfrak{f}^{-1}(c)} \bar{A}(v)\} \\
&= \min\{\mathfrak{f}(\bar{A})((((a \to b)' \to b)' \to c)'),\, \mathfrak{f}(\bar{A})(c)\}.
\end{aligned}$$

Hence $\mathfrak{f}(\bar{A})$ is a fuzzy implicative LI-ideal of L.

(ii) Note that $\mathfrak{f}^{-1}(\bar{B})(0) = \bar{B}(\mathfrak{f}(0)) = \bar{B}(0) \geq \bar{B}(x)$ for all $x \in M$. Since $\mathfrak{f}$ is onto, there exists $u_x \in L$ such that $\mathfrak{f}(u_x) = x$. Hence

$$\mathfrak{f}^{-1}(\bar{B})(0) \geq \bar{B}(x) = \bar{B}(\mathfrak{f}(u_x)) = \mathfrak{f}^{-1}(\bar{B})(u_x).$$

Since x is arbitrary, we know that $\mathfrak{f}^{-1}(\bar{B})(0) \geq \mathfrak{f}^{-1}(\bar{B})(y)$ for all $y \in L$. Now for any $x, y \in L$, we have

$$\begin{aligned} &\min\{\mathfrak{f}^{-1}(\bar{B})((x \to y)'),\, \mathfrak{f}^{-1}(\bar{B})(y)\} \\ &= \min\{\bar{B}(\mathfrak{f}((x \to y)')),\, \bar{B}(f(y))\} \\ &= \min\{\bar{B}(\mathfrak{f}((x \to y) \to 0)),\, \bar{B}(f(y))\} \\ &= \min\{\bar{B}(\mathfrak{f}(x \to y) \to \mathfrak{f}(0)),\, \bar{B}(f(y))\} \\ &= \min\{\bar{B}((\mathfrak{f}(x) \to \mathfrak{f}(y)) \to 0),\, \bar{B}(f(y))\} \\ &= \min\{\bar{B}((\mathfrak{f}(x) \to \mathfrak{f}(y))'),\, \bar{B}(f(y))\} \\ &\leq \bar{B}(\mathfrak{f}(x)) = \mathfrak{f}^{-1}(\bar{B})(x), \end{aligned}$$

and

$$\begin{aligned} \mathfrak{f}^{-1}(\bar{B})((x \to y)') &= \bar{B}(\mathfrak{f}((x \to y)')) = \bar{B}((\mathfrak{f}(x) \to \mathfrak{f}(y))') \\ &\geq \bar{B}(((\mathfrak{f}(x) \to \mathfrak{f}(y))' \to \mathfrak{f}(y))') = \bar{B}(((\mathfrak{f}(x \to y))' \to \mathfrak{f}(y))') \\ &= \bar{B}((\mathfrak{f}((x \to y)') \to \mathfrak{f}(y))') = \bar{B}((\mathfrak{f}((x \to y)' \to y))') \\ &= \bar{B}(\mathfrak{f}(((x \to y)' \to y)')) = \mathfrak{f}^{-1}(\bar{B})(((x \to y)' \to y)'). \end{aligned}$$

Hence $\mathfrak{f}^{-1}(\bar{B})$ is a fuzzy implicative LI-ideal of L. □

Theorem 3.22. *Let $\bar{A}$ be a fuzzy implicative LI-ideal of L with $\mathrm{Im}(\bar{A}) = \{t_i \mid i \in \Lambda\}$, where $\emptyset \neq \Lambda \subset [0,1]$. Let $\Omega := \{U(\bar{A}; t) \mid t \in \mathrm{Im}(\bar{A})\}$. Then Ω contains all level implicative LI-ideals of $\bar{A}$ if and only if $\bar{A}$ attains its infimum on all implicative LI-ideals of L.*

Proof. Suppose that Ω contains all level implicative LI-ideals of $\bar{A}$. Let I be an implicative LI-ideal of L. If $\bar{A}$ is constant on I, then we are done. Assume that $\bar{A}$ is not constant on I. If $I = L$, we let $\beta := \inf \mathrm{Im}(\bar{A})$. Then $\beta \leq t$ for all $t \in \mathrm{Im}(\bar{A})$, and so $U(\bar{A}; t) \subseteq U(\bar{A}; \beta)$. But $U(\bar{A}; 0) = L \in \Omega$ because Ω contains all level implicative LI-ideals of $\bar{A}$. Hence there exists $\alpha \in \mathrm{Im}(\bar{A})$ such that $U(\bar{A}; \alpha) = L$. It follows that $L = U(\bar{A}; \alpha) \subseteq U(\bar{A}; \beta)$ so that $U(\bar{A}; \beta) = U(\bar{A}; \alpha) = L$. Now it is sufficient to show that $\alpha = \beta$. If $\beta < \alpha$, then there exists $\gamma \in \mathrm{Im}(\bar{A})$ such that $\beta \leq \gamma < \alpha$. Thus $U(\bar{A}; \gamma) \supsetneqq U(\bar{A}; \alpha) = L$, a contradiction. Therefore $\alpha = \beta$. If $I \subsetneqq L$, then we consider the fuzzy set $\bar{A}_I$ in L defined by

$$\bar{A}_I(x) := \begin{cases} \delta \in (0,1] & \text{for } x \in I, \\ 0 & \text{otherwise.} \end{cases}$$

It is routine to check that $\bar{A}_I$ is a fuzzy implicative LI-ideal of L. Let

$$\Gamma := \{i \in \Lambda \mid \bar{A}(y) = t_i \text{ for some } y \in I\}$$

and $\Omega_I := \{U(\bar{A}_I; t_i) \mid i \in \Gamma\}$. Noticing that Ω_I contains all level implicative LI-ideals, then there exists $x_0 \in I$ such that $\bar{A}(x_0) = \inf\{\bar{A}_I(x) \mid x \in I\}$, which implies that $\bar{A}(x_0) = \inf\{\bar{A}(x) \mid x \in I\}$. This proves that $\bar{A}$ attains its infimum on all implicative LI-ideals of L. To prove the converse, let $U(\bar{A}; \alpha)$ be a level implicative LI-ideal of $\bar{A}$. If $\alpha = t$ for some $t \in \mathrm{Im}(\bar{A})$, then clearly $U(\bar{A}; \alpha) \in \Omega$. If $\alpha \neq t$ for all $t \in \mathrm{Im}(\bar{A})$, then there does not exist $x \in L$ such that $\bar{A}(x) = \alpha$. Let $I = \{x \in L \mid \bar{A}(x) > \alpha\}$. Obviously $0 \in I$. Let $x, y, z \in L$ be such that $(((x \to y)' \to y)' \to z)' \in I$ and $z \in I$. Then $\bar{A}((((x \to y)' \to y)' \to z)') > \alpha$ and $\bar{A}(z) > \alpha$. It follows from (F3) that

$$\bar{A}((x \to y)') \geq \min\{\bar{A}((((x \to y)' \to y)' \to z)'),\, \bar{A}(z)\} > \alpha$$

so that $(x \to y)' \in I$. Hence I is an implicative LI-ideal of L. By hypothesis, there exists $y \in I$ such that $\bar{A}(y) = \inf\{\bar{A}(x) \mid x \in I\}$. But $\bar{A}(y) \in \mathrm{Im}(\bar{A})$ implies $\bar{A}(y) = s$ for some $s \in \mathrm{Im}(\bar{A})$. Hence

$$\inf\{\bar{A}(x) \mid x \in I\} = \bar{A}(y) = s > \alpha.$$

Note that there does not exist $z \in L$ such that $\alpha \leq \bar{A}(z) < s$, which gives $U(\bar{A};\alpha) = U(\bar{A};s) \in \Omega$. Therefore Ω contains all level implicative LI-ideals of $\bar{A}$. This completes the proof. □

Theorem 3.23. *Let $\bar{A}$ be a fuzzy set in L with $\mathrm{Im}(\bar{A}) = \{t_0, t_1, \cdots, t_n\}$, where $t_i < t_j$ whenever $i > j$. Let $\{I_k \mid k = 0, 1, \cdots, n\}$ be a family of implicative LI-ideals of L such that*

(i) $I_0 \subset I_1 \subset \cdots \subset I_n = L$,
(ii) $\bar{A}(I_k^*) = t_k$, *where* $I_k^* = I_k \setminus I_{k-1}$, $I_{-1} = \emptyset$ *for* $k = 0, 1, \cdots, n$.

Then $\bar{A}$ is a fuzzy implicative LI-ideal of L.

Proof. Since $0 \in I_0$, we have $\bar{A}(0) = t_0 \geq \bar{A}(x)$ for all $x \in L$. Let $x, y, z \in L$. If $(((x \to y)' \to y)' \to z)' \in I_k^*$ and $z \in I_k^*$, then $(x \to y)' \in I_k$. Hence

$$\bar{A}((x \to y)') \geq t_k = \min\{\bar{A}((((x \to y)' \to y)' \to z)'),\, \bar{A}(z)\}.$$

If $(((x \to y)' \to y)' \to z)' \notin I_k^*$ and $z \notin I_k^*$, then the following four cases arise:

- $(((x \to y)' \to y)' \to z)' \in L \setminus I_k$ and $z \in L \setminus I_k$.
- $(((x \to y)' \to y)' \to z)' \in I_{k-1}$ and $z \in I_{k-1}$.
- $(((x \to y)' \to y)' \to z)' \in L \setminus I_k$ and $z \in I_{k-1}$.
- $(((x \to y)' \to y)' \to z)' \in I_{k-1}$ and $z \in L \setminus I_k$.

But, in either case, we know that

$$\bar{A}((x \to y)') \geq \min\{\bar{A}((((x \to y)' \to y)' \to z)'),\, \bar{A}(z)\}.$$

If $(((x \to y)' \to y)' \to z)' \in I_k^*$ and $z \notin I_k^*$, then either $z \in I_{k-1}$ or $z \in I_r$ for some $r > k$. It follows that either $(x \to y)' \in I_k$ or $(x \to y)' \in I_r$. Hence

$$\bar{A}((x \to y)') \geq \min\{\bar{A}((((x \to y)' \to y)' \to z)'),\, \bar{A}(z)\}.$$

If $(((x \to y)' \to y)' \to z)' \notin I_k^*$ and $z \in I_k^*$, then by similar process we have

$$\bar{A}((x \to y)') \geq \min\{\bar{A}((((x \to y)' \to y)' \to z)'),\, \bar{A}(z)\}.$$

Summarizing the above results, we obtain

$$\bar{A}((x \to y)') \geq \min\{\bar{A}((((x \to y)' \to y)' \to z)'),\, \bar{A}(z)\}$$

for all $x, y, z \in L$. Hence $\bar{A}$ is a fuzzy implicative LI-ideal of L. □

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