G-FREE COLORABILITY AND THE BOOLEAN PRIME IDEAL THEOREM

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Abstract. We show that the compactness of G-free k-colorability is equivalent to the Boolean prime ideal theorem for any graph G with more than two vertices and any $k \geq 2$.

1 Introduction In 1951, de Bruijn and Erdös [4] proved that a graph is k-colorable if every finite subgraph is k-colorable. We shall refer to this statement as P_k. In 1961, Mycielski [14] asked about the strength of P_k as a set theory axiom, pointing out that it follows from BPI, the prime ideal theorem for Boolean algebras. Ten years later, in 1971, Läuchli [13] showed that P_k is in fact one of the equivalent forms of BPI, even for fixed k, $k \geq 3$. (P_2 is much weaker.) BPI is weaker than the Axiom of Choice or AC([10]), but like AC has a large number of equivalent formulations and can often be substituted for AC in mathematical proofs (see [11]).

Many other graph coloring notions for both vertices and edges have been introduced in the last 30 years, leading Borowiecki and Mihók [3] to formulate their notion of generalized colorings. In a recent paper [8], we have shown that several of these colorings lead to compactness theorems which are also equivalent to BPI. In this paper, we continue this work by showing the very general notion of G-free k-colorability, introduced by Achlioptas [1] has a compactness theorem which is equivalent to BPI for any graph G with more than two vertices and any $k \geq 2$. (A graph is G-free k-colorable if its vertex set can be partitioned into k classes, each of which induces a a graph which does not contain G.) Thus we see that the result of Läuchli is far from unique; in fact it seems that almost any vertex coloring compactness theorem is equivalent to BPI. The inspiration for this work is the paper of Achlioptas [1], in which it is proved that G-free k-colorability is NP-Complete for any graph G with more than two vertices and any $k \geq 2$. We have noticed this connection between NP-Completeness and BPI before ([6], [7], ([8]), and it remains a fruitful source of new results.

2 Background A *hypergraph*, $\mathcal{H} =< V, E >$, is a set, V, called the *vertices*, together with a collection E, of finite subsets of V, called *edges*. If each edge is an ordered set, we shall refer to the hypergraph as an *ordered hypergraph*. If each edge has exactly k elements, the hypergraph will be called *k-uniform*. Thus a (simple) graph is a 2-uniform hypergraph. If v is a vertex of a hypergraph, $N(v)$ shall denote the set of neighbors of v, that is the set of all other vertices which belong to an edge containing v. A *k-coloring* of a hypergraph is an assignment of the k "colors", {1,...,k}, to its vertices so that no edge is monochromatic; that is, each edge has vertices of differing colors.

Let G, H be graphs; then H is *G-free* if it contains no induced subgraph isomorphic to G. A *k-coloring* of a graph H is, in this paper, just a partition of its vertex set V into k sets, $V_1, ..., V_k$. The coloring is said to be *G-free* if each color class, V_i, induces a

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subgraph $H[V_i]$ which is G-free. Equivalently, a k-coloring is G-free if no induced copy of G is monochromatic under the coloring. Note that if G is K_2, the complete graph on 2 vertices, the usual vertex k-coloring notion is obtained.

Let $\bar{G}$ denote the complement of graph G; that is, the graph with the same set of vertices as G and whose edges are those edges absent from G. It is an easily established fact that a graph H is G-free if and only if $\bar{H}$ is $\bar{G}$-free. Moreover a G-free k-coloring of H is also a $\bar{G}$-free k-coloring of $\bar{H}$ (and vice versa). Since one of G, $\bar{G}$ must be connected, these facts will allow us to assume that G is connected in proofs involving G-free properties of graphs.

The following definition of "G^k-distinguishers" is due to D. Achlioptas [1](where "G^k-distinguishers" are called "G^k-gadgets").

Definition 2.1 *A G^k-distinguisher is a G-free k-colorable graph H with two designated vertices s, t, such that in every G-free k-coloring of H,*
(1) s and t receive different colors
(2) Each vertex in $N(s)$ receives a different color than that of s
(3) Each vertex in $N(t)$ receives a different color than that of t.

It is a theorem of [1] that G^k-distinguishers exist for all G with more than two vertices. This theorem depends heavily on results in [2] on the existence of uniquely G-free k-colorable graphs with n vertices, $n > 2$.

A G^2-distinguisher will be called a G-distinguisher. Assume G is a connected graph. A sequence of G-distinguishers can be "chained" together by identifying the "s" vertex of one with the "t" vertex of its predecessor. (A more precise definition of this process of chaining distinguishers can be found in [1]). We shall only chain together two distinguishers and we call the result a *G-identifier*, since in any G-free 2-coloring, the initial "s" vertex and final "t" vertex must receive identical colors. The fact that G is connected is needed to insure that a G-identifier is G-free 2-colorable. For example, if G consisted of two disjoint triangles, this need not be the case, since one triangle could be colored "red" and the other "blue" in each distinguisher and then the identifier would have two disjoint red triangles! Since G-distinguishers exist for all G with more than two vertices, G-identifiers exist for all connected G with more than two vertices.

3 G-free colorings and BPI .

Definition 3.1 *For any graph G and any $k > 1$, Let $\Gamma(G,k)$ denote the statement: If every finite induced subgraph of a graph, H, is G-free k-colorable, then H is G-free k-colorable.*

Remark 3.2 It is easily seen, by complementation, that $\Gamma(G,k) \leftrightarrow \Gamma(\bar{G},k)$, and hence when we assume that $\Gamma(G,k)$, we can also assume that G is a connected graph.

Theorem 3.3 $\Gamma(G,k) \leftrightarrow$ *BPI if G has at least 3 vertices and* $k > 1$ *or G has 2 vertices and* $k > 2$.

Proof. This will follow from the lemmas we present below.

That BPI implies $\Gamma(G,k)$ follows immediately from results in [8](see Theorem 3.4 of [8]).

Lemma 3.4 $\Gamma(G,k) \rightarrow P_k$, *if G has at least 2 vertices and* $k > 1$.

Proof. We note that we may also assume that G is a connected graph, as explained above. Furthermore, if G is connected and has exactly 2 vertices, then $\Gamma(G,k)$ is just P_k. Thus we need only consider the case where G is connected and has more than two vertices. Now assume $\Gamma(G,k)$. Let H be an infinite graph such that every finite subgraph is

k-colorable. ("k-colorablity" here means, of course, that adjacent vertices receive different colors.) We will show that H is k-colorable. Replace each edge of H by a separate copy of a G^k-distinguisher with the vertices of the edge being identified with the designated vertices of the distinguisher. Call this new graph H'. We note that each vertex of H' is either a vertex of H or a vetex of a unique distinguisher which has been added to H. For each new vertex v which has been added to H, let D_v be the entire distinguisher, containing v, which has been added. Now consider any finite subgraph K' of H'. Let K'' be the graph obtained by adding to K' all elements of every D_v with $v \in K' - H$. We claim that K'' can be G-free k-colored. This is the case because the vertices of K'' derived from H induce a finite subgraph, K, of H, which is, by assumption, k-colorable. Extend this coloring to a G-free k-coloring of the added distinguishers, whose designated vertices can now receive different colors. We note that coloring the distinguishers cannot create a monochromatic copy of G because of conditions 2 and 3 in the definition of distinguisher and the fact that G is connected. Of course, this coloring induces a G-free k-coloring of K' and by $\Gamma(G, k)$, H' can be G-free k-colored. Finally, this coloring induces a k-coloring of H, since the designated vertices of each distinguisher must receive differing colors.

Remark 3.5 It now follows from Läuchli's result [13] that Theorem 3.3 holds for $k > 2$.

Instead of proving BPI from $\Gamma(G, 2)$ directly, we prove that $\Gamma(G, 2)$ implies a coloring compactness theorem for k-uniform ordered hypergraphs, where k is the order of G, $k > 2$; then we show that these hypergraph compactness theorems imply BPI.

Definition 3.6 *Let $\Delta(k)$ denote the statement: if every finite subhypergraph of a k-uniform ordered hypergraph is 2-colorable, then the ordered hypergraph is 2-colorable.*

Lemma 3.7 *If G has k vertices, then $\Gamma(G, 2) \to \Delta(k)$, $k > 2$.*

Proof. We can assume that G is a connected graph by Remark 3.2.

Let $\mathcal{H}$ be a k-uniform ordered hypergraph and suppose that every finite subhypergraph of $\mathcal{H}$ is 2-colorable. Let G be a graph with k vertices, $k > 2$. We can assume the vertices of G are ordered.

We construct, for each ordered edge, E_ν, of the hypergraph, G_ν, a new copy of G. The edge, E_ν, of the hypergraph and the vertex set of G_ν are of the same cardinality, k, and since both are ordered, we can assume, without using the Axiom of Choice, that there is a family of one-to-one mappings, ϕ_ν, mapping E_ν onto the vertices in G_ν; let $\bar{e}_{\nu i}$ denote $\phi_\nu(e_{\nu i})$, $e_{\nu i} \in E_\nu$.

We now construct a graph H as follows.

(1) For each ordered edge of the hypergraph, E_ν, we include in H, G_ν. All of the vertices of G_ν thus added to H will be said to *depend* on the edge, E_ν.

(2) If elements $e_{\nu i}$ in E_ν and $e_{\mu j}$ in E_μ are equal, we join the corresponding vertices $\bar{e}_{\nu i}$ in G_ν and $\bar{e}_{\mu j}$ in G_μ by a G-identifier with these two vertices as endpoints. The vertices of the identifier thus added to H will be said to *depend* on both E_ν and E_μ. This completes the construction of graph H.

Let H' be a finite subgraph of H. Since the vertices of H' are finite in number and each depends on only one or two edges, they depend on a finite set of edges, $\mathcal{E}'$ of hypergraph, $\mathcal{H}$. Let H'' be the full finite subgraph of H generated from $\mathcal{E}'$ by (1) and (2). $\mathcal{E}'$ is 2-colorable, by assumption. Select one such coloring and using this coloring, color each vertex $\bar{e}_{\nu i}$ in H'' that belongs to some G_ν, the color received by $e_{\nu i}$ in E_ν. No copy of G thus colored can be monochromatic, since no edge in $\mathcal{E}'$ is monochromatic under the hypergraph 2-coloring of $\mathcal{E}'$.

Finally the coloring can be extended to a G-free 2-coloring of the vertices in the G-identifiers, since the endpoints of each have already received the same color. Since G is connected, conditions (2) and (3) in the definition of G distinguisher guarantee that no new monochromatic copies of G can be created by the G-free coloring of the identifiers. Thus H'' is G-free 2-colorable, and hence, so is its subgraph, H'.

Therefore, by $\Gamma(G, 2)$, H is G-free 2-colorable. Color each $e_{\nu i}$ in edge E_ν of the hypergraph, the color received by $\bar{e}_{\nu i}$; since $\bar{e}_{\nu i}$ and $\bar{e}_{\mu j}$ are connected by a G-identifier if $e_{\nu i} = e_{\mu j}$, this is really a coloring of the hypergraph vertices, not just a coloring of the edge elements. Also no edge can be monochromatic, since it has the same colors as its copy of G under the G-free 2 coloring of H. Hence $\mathcal{H}$ is a 2-colorable hypergraph as required.

Lemma 3.8 *For all* $k > 3$, $\Delta(k) \to \Delta(3)$.

Proof. This follows immediately from the fact, which we show next, that $\Delta(k+1) \to \Delta(k)$, $k \geq 3$. Assume $\Delta(k+1)$ and suppose $\mathcal{H}$ is a k-uniform ordered hypergraph such that every finite subhypergraph is 2-colorable.We must show that $\mathcal{H}$ is 2-colorable. We construct a (k+1)-uniform ordered hypergraph, $\mathcal{H}'$ as follows.

1) The vertices of $\mathcal{H}'$ are the vertices of $\mathcal{H}$ together with new vertices: $x_i, 1 \leq i \leq k+1$.
2) The edges of $\mathcal{H}'$ are,
   a) $\{e_1, ..., e_k, x_i\}, 1 \leq i \leq k+1$, for each edge $e = \{e_1, ..., e_k\}$ in $\mathcal{H}$.
   b) $\{x_1, ..., x_{k+1}\}$.

Every finite subhypergraph, $\mathcal{K}'$ of $\mathcal{H}'$, is 2-colorable since the restriction $\mathcal{K}$ of $\mathcal{K}'$ to $\mathcal{H}$ is 2-colorable, by assumption, and then the vertex x_1 can be colored '1', while $x_2, ..., x_{k+1}$ can be colored '0'.

Then, $\mathcal{H}'$ is 2-colorable, by $\Delta(k+1)$. The colors assigned to the vertices, other than the x_i, constitute a 2-coloring of $\mathcal{H}$, since if x_i receives the color '1' and x_j receives the color '0' (the edge, $\{x_1, ..., x_{k+1}\}$ cannot be monochromatic), any edge e of $\mathcal{H}$ must have a vertex which receives the color '0' ($e \cup \{x_i\}$ is an edge of $\mathcal{H}'$) and e must have a vertex which receives the color '1' (($e \cup \{x_j\}$ is an edge of $\mathcal{H}'$).

The compactness theorem for propositional logic states that a set of propositional formulas is satisfiable if every finite subset is satisfiable. This theorem, which is equivalent to BPI, is often referred to as SAT. Let 3SAT denote the restricted version of SAT where each propositional formula is a disjunction of exactly 3 literals (a literal is a statement letter or a negated statement letter). In fact 3SAT is also one of the equivalent forms of BPI (see [6]).

Lemma 3.9 $\Delta(3) \to 3SAT$.

Proof. Suppose then that a set of propositional formulas, each of which is a disjunction of 3 literals, is given and that each finite subset is satisfiable. Let $\mathcal{H}$ be the 3-uniform hypergraph whose vertex set consists of the union of the following four sets.

(1) all literals in any of the propositional formulas
(2) a set of copies of these literals, that is, for each literal, l_i, we have a new element, u_i.
(3) new elements: v_1, v_2, v_3.
(4) a new element, f.

The edges of $\mathcal{H}$ consist of all of the following ordered sets:

(1) $\{v_1, v_2, v_3\}$
(2) For each occurring propositional letter p: $\{p, \neg p, v_i\}, i = 1, 2, 3$.
(3) All four the following, for each clause, $l_1 \vee l_2 \vee l_3$:
   (a) $\{l_1, f, u_1\}$

(b) $\{l_2, f, u_2\}$
(c) $\{l_3, f, u_3\}$
(d) $\{u_1, u_2, u_3\}$

It is easy to see that $l_1 \vee l_2 \vee l_3$ is satisfiable if and only if these associated hyperedges are 2-colorable, with colors, '1', '0', with f getting the color '0'. Since finite subsets of these formulas are assumed satisfiable, it follows that every finite subhypergraph will be 2-colorable. Hence, by $\Delta(3)$, the entire hypergaph is 2-colorable. We can assume f gets the color '0'. Assign truth values to the literals based on this coloring; that is, assign a literal the value true if and only if it is colored '1'. Since f and some u_i must be colored '0', the corresponding l_i must be colored '1', that is, at least one literal in every clause is true, and, because of (1) and (2), a literal and its negation will receive opposite truth values. Therefore the entire set of propositional formulas is satisfiable, as required.

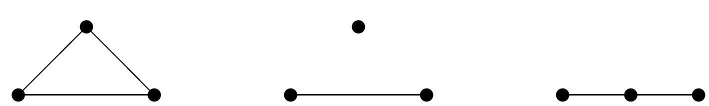
Remark 3.10 Since 3SAT is known to imply BPI, as mentioned above, we have completed the proof of Theorem 3.3.

Open Problem. Let $G_1, ..., G_k$ be finite graphs. A graph H is $(G_1, ..., G_k)$*-free k-colorable* if its vertex set V can be partitioned into sets $(V_1, ..., V_k)$ such that $H[V_i]$ is G_i-free, $1 \leq i \leq k$. Let $\Gamma(G_1, ..., G_k)$ stand for the statement: a graph is $(G_1, ..., G_k)$-free k-colorable if every finite subgraph is $(G_1, ..., G_k)$-free k-colorable. If each G_i has more than two vertices and $k > 1$ is $\Gamma(G_1, ..., G_k) \leftrightarrow BPI$?

4 $\mathcal{G}$-free colorings Let $\mathcal{G}$ be a set of graphs, and let H be a graph. H is *$\mathcal{G}$-free* if no induced subgraph of H is a member of $\mathcal{G}$. H is *$\mathcal{G}$-free n-colorable* if the vertices of H can be partitioned into n sets, such that the graph induced by each of these sets is $\mathcal{G}$-free.

It might be expected that $\mathcal{G}$-free n-colorability would be at least as complex as ordinary G-free n-colorability. In particular, we might expect that for finite graphs, it would be NP-complete and for infinite graphs, the compactness would be equivalent to BPI. However this is not always the case, as we shall show.

Let $\mathcal{G}^*$ be the set of the three graphs:



We shall show that the question of whether or not a finite graph is $\mathcal{G}^*$-free 2-colorable is polynomial (of low degree), and for infinite graphs, the compactness statement is weaker than BPI, requiring only C_2, the axiom of choice for pairs.

We first note that the only $\mathcal{G}^*$-free graphs are the discrete graphs and the graph consisting of two vertices connected by an edge. Thus a graph of order greater than 4 is $\mathcal{G}^*$-free 2-colorable if and only if it is 2-colorable or it contains two adjacent vertices, and every edge contains at least one of these vertices. Clearly, in the latter case, we need only color the two vertices one color and the remaining vertices the other color to obtain a $\mathcal{G}^*$-free 2-coloring. It is easy to show that a graph which contains an odd cycle of length greater than three cannot be $\mathcal{G}^*$-free 2-colored. Also, if we have two adjacent vertices each connected to every vertex of some discrete graph, the result can be $\mathcal{G}^*$-free 2-colored by coloring the adjacent vertices one color and the remaining vertices, the other color.

Let H be a finite graph. If the order of H were less than five, its vertices could be divided into two sets of size less than three and so could obviously be $\mathcal{G}^*$-free 2-colored. We assume, then, that the order of H is at least five. We first attempt a ordinary 2-coloring of H, using a standard polynomial-time algorithm. If we succeed, we also have a $\mathcal{G}^*$-free 2-coloring of H. If not, it is because H has an odd cycle. If this cycle is of length five or more, H cannot be $\mathcal{G}^*$-free 2-colored, as mentioned above. For a cycle of degree three, it is easy to check (in polynomial-time) whether or not there are two vertices in this triangle such that every edge in the graph contains at least one of them.

Finally, to prove compactness in the infinite case, we use a similar argument. Assume H is any infinite graph such that each of its finite subgraphs is $\mathcal{G}^*$-free 2-colorable. We consider three cases.

Case 1. H contains an odd cycle of length five or greater. This is not possible because the finite graph consisting of this cycle is not $\mathcal{G}^*$-free 2-colorable.

Case 2. H contains no odd cycles. Then H is 2-colorable and therefore $\mathcal{G}^*$-free 2-colorable. (If H has infinitely many components, C_2 is needed to obtain a coloring for the entire graph.)

Case 3. H contains a cycle C of length three but no odd cycles of greater length. Suppose the vertices of C are v_1, v_2, v_3. Now assume that H is not $\mathcal{G}^*$-free 2-colorable. Then H must contain an edge e_{12} which contains neither v_1 nor v_2, an edge e_{13} that contains neither v_1 nor v_3, and an edge e_{23} that contains neither v_2 nor v_3. Let F be the finite subgraph of H which consists of the union of C and the edges e_{ij}. However, F would be a finite subgraph of H which is not $\mathcal{G}^*$-free 2-colorable, contrary to the assumption that every finite subgraph of H is $\mathcal{G}^*$-free 2-colorable. Hence, again, H is $\mathcal{G}^*$-free 2-colorable.

Open Problems.

1. Characterize those sets $\mathcal{G}$ for which $\mathcal{G}$-free 2-colorablility is NP-Complete.
2. Characterize those sets $\mathcal{G}$ for which the compactness of $\mathcal{G}$-free 2-colorablility is equivalent to BPI.
3. The same problems for $\mathcal{G}$-free k-colorablility, $k > 2$.

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