ON TRANSFER PRINCIPLE OF FUZZY BCK/BCI-ALGEBRAS

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ABSTRACT. In this paper we establish the transfer principle for fuzzy BCK/BCI-algebras, that is, some concepts about BCK/BCI-algebras expressed by a certain formula can be extended to those of fuzzy BCK/BCI-algebras uniformly and many results concerning to fuzzy BCK/BCI-algebras are obtained immediately by the transfer principle.

1. INTRODUCTION

Many results about BCK/BCI-algberas are extended to those of *fuzzy* BCK/BCI-algebras. For example, we can list the following :

(1) For any ideal A of a BCI-algebra X, the following three conditions are equivalent.
   - A : associative, that is, $(x*z)*(0*y) \in A$ and $z \in A$ imply $y*x \in A$,
   - $(x*z)*(0*y) \in A$ implies $y*(x*z) \in A$,
   - $x*(0*y) \in A$ implies $y*x \in A$,

   which is extended to the fuzzy setting, that is, the following three statements are equivalent.
   - $\bar{A}$: fuzzy associative, that is, $\bar{A}(y*x) \geq \bar{A}((x*z)*(0*y)) \wedge \bar{A}(z)$,
   - $\bar{A}(y*(x*z)) \geq \bar{A}((x*z)*(0*y))$,
   - $\bar{A}(y*x) \geq \bar{A}(x*(0*y))$.

(2) For any BCK-algebras X, Y and a surjective BCK-homomorphism f from X to Y, if A is an ideal of X then $f(A)$ is an ideal of Y,

which is applied to the fuzzy setting as follows: Let X, Y and f be the same as above. If $\bar{A}$ is a fuzzy ideal of X, then $f[\bar{A}]$ is a fuzzy ideal of Y.

Many results obtained in BCK/BCI-algebras can be applied to the fuzzy setting by similar method. In this paper we prove that some concepts of BCK/BCI-algebras expressed by a certain formula can be naturally extended to the fuzzy setting and that many results are obtained immediately with the use of our method. Moreover we show that these results can be extended to fuzzy IS-algebras.

2. TRANSFER PRINCIPLE

To express our method exactly we first of all define terms of BCK/BCI-algebras. Let X be any BCK/BCI-algebra, V a countable set $\{x, y, z, \cdots\}$ of syntactic variables which range over the elements of X. A *term* of X is defined recursively :

- $0 \in X$ is a *term*;
- each variable of X is a *term*;
- if s and t are *terms*, then $s*t$ is a *term*.

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Thus, for example, $x, 0 * x$, and $x * (0 * y)$ are terms of X. Let A be a subset of X and satisfy the following property $\mathcal{P}$ expressed by a first-order formula :

$$\mathcal{P} : \frac{t_1(x, \cdots, y) \in A, \cdots, t_n(x, \cdots, y) \in A}{t(x, \cdots, y) \in A},$$

where $t_1(x, \cdots, y), \cdots, t_n(x, \cdots, y)$, and $t(x, \cdots, y)$ are terms of X constructed by variables $x, \cdots, y$. We note that the subset A satisfies the property $\mathcal{P}$ if, for all elements $a, \cdots, b \in X$, $t(a, \cdots, b) \in A$ whenever $t_1(a, \cdots, b), \cdots, t_n(a, \cdots, b) \in A$. For the subset A we define a fuzzy subset $\bar{A}$ which satisfies the following property

$$\bar{\mathcal{P}} : \bar{A}(t(x, \cdots, y) \geq \bar{A}(t_1(x, \cdots, y)) \wedge \cdots \wedge \bar{A}(t_n(x, \cdots, y))$$

For any $\alpha \in [0, 1]$, we put $U(\bar{A}; \alpha) = \{x \in X | \bar{A}(x) \geq \alpha\}$. The closed interval $[0, 1]$ has a lattice structure with a usual order, so we have $\alpha \wedge \beta = \min\{\alpha, \beta\}$ for $\alpha, \beta \in [0, 1]$. If we use a lattice L instead of $[0, 1]$, then we have a so-called *L-fuzzy subset*.

Now we establish a statement, called the *transfer principle*.

Theorem 2.1 (Transfer Principle)**.** *A fuzzy subset $\bar{A}$ satisfies the property $\bar{\mathcal{P}}$ if and only if for any $\alpha \in [0, 1]$ if $U(\bar{A}; \alpha) \neq \phi$ then the crisp set $U(\bar{A}; \alpha)$ satisfies the property $\mathcal{P}$. We simply denote this result by*

$$\bar{A}; \bar{\mathcal{P}} \Leftrightarrow \forall \alpha (U(\bar{A}; \alpha) \neq \phi \Rightarrow U(\bar{A}; \alpha) : \mathcal{P}$$

Proof. Suppose that $\bar{A}$ satisfies $\bar{\mathcal{P}}$. If there is $\alpha \in [0, 1]$ such that $U(\bar{A}; \alpha) \neq \phi$ but $U(\bar{A}; \alpha)$ does not satisfy $\mathcal{P}$, then we have $t_1(a, \cdots, b) \in U(\bar{A}; \alpha), \cdots, t_n(a, \cdots, b) \in U(\bar{A}; \alpha)$ but $t(a, \cdots, b) \notin U(\bar{A}; \alpha)$ for some $a, \cdots, b \in X$. Since $\bar{A}(t_1(a, \cdots, b)) \geq \alpha, \cdots, \bar{A}(t_n(a, \cdots, b)) \geq \alpha$ but $\bar{A}(t(a, \cdots, b)) \ngeqq \alpha$, it follows that $\bar{A}(t(a, \cdots, b)) \ngeq \bar{A}(t_1(a, \cdots, b)) \wedge \cdots \wedge \bar{A}(t_n(a, \cdots, b))$ for some $a, \cdots, b \in X$. This means that $\bar{A}$ does not satisfy $\bar{\mathcal{P}}$. This is a contradiction.

Conversely, suppose that $\bar{A}$ does not satisfy $\bar{\mathcal{P}}$. Then $\bar{A}(t(a, \cdots, b)) \ngeq \bar{A}(t_1(a, \cdots, b)) \wedge \cdots \wedge \bar{A}(t_n(a, \cdots, b))$ for some $a, \cdots, b \in X$, if we put $\alpha = \bigwedge_i \bar{A}(t_i(a, \cdots, b))$ then we have $\alpha \in [0, 1]$ and $U(\bar{A}; \alpha) \neq \phi$ because $t_i(a, \cdots, b) \in U(\bar{A}; \alpha)$. On the other hand, in this case we have $t(a, \cdots, b) \notin U(\bar{A}; \alpha)$. This means that $U(\bar{A}; \alpha)$ does not satisfy $\mathcal{P}$, completing the proof. □

The above implies that if we define a fuzzy subset $\bar{A}$ which satisfies $\bar{\mathcal{P}}$ whenever a crisp subset A does $\mathcal{P}$ then the transfer principle holds generally.

Conversely we show that we can define a fuzzy subset $\bar{A}$ as of the form $\bar{\mathcal{P}}$ if the Transfer Principle holds.

Theorem 2.2. *If the transfer principle holds for a subset A with the property $\mathcal{P}$, then the fuzzy subset $\bar{A}$ has the property $\bar{\mathcal{P}}$.*

Proof. Suppose that the transfer principle holds for A but the fuzzy subset $\bar{A}$ does not have the property $\bar{\mathcal{P}}$. Then there exist $a, \cdots, b \in X$ such that

$$\bar{A}(t(a, \cdots, b)) \ngeq \bar{A}(t_1(a, \cdots, b)) \wedge \cdots \wedge \bar{A}(t_n(a, \cdots, b)).$$

We take $\alpha = \bigwedge_i \bar{A}(t_i(a, \cdots, b))$. It is clear that $\alpha \in [0, 1]$ and $U(\bar{A}; \alpha) \neq \phi$ because of $\bar{A}(t_i(a, \cdots, b)) \geq \alpha$. Since $\bar{A}(t(a, \cdots, b)) \ngeq \alpha$, we have $t(a, \cdots, b) \notin U(\bar{A}; \alpha)$ but $t_i(a, \cdots, b) \in U(\bar{A}; \alpha)$. This means that $U(\bar{A}; \alpha)$ does not satisfy the property $\mathcal{P}$. This is a contradiction. □

Since any concept defined for BCK/BCI-algebras so far has a form $\mathcal{P}$ and the corresponding fuzzy subsets are defined by the form $\bar{\mathcal{P}}$, it is easy to get the relation between a crisp set A with $\mathcal{P}$ and its fuzzy set $\bar{A}$ with $\bar{\mathcal{P}}$. The following examples show that many results are obtained immediately from the theory of crisp BCK/BCI-algebras.

3. Examples

In this section, we deal with some examples.

Example 3.1. (Subalgebras and fuzzy subalgebras) A subset A of a BCK/BCI-algebra X is called a *subalgebra* if it satisfies

$$\mathcal{P} \,:\, \frac{x \in A,\ y \in A}{x * y \in A}.$$

Hence if we define a *fuzzy subalgebra* $\bar{A}$ of X by

$$\bar{\mathcal{P}} \,:\, \bar{A}(x * y) \geq \bar{A}(x) \wedge \bar{A}(y),$$

then it is obvious from the Transfer Principle that

$$\bar{A} \,:\, \text{fuzzy subalgebra} \Leftrightarrow \forall\alpha(U(\bar{A};\alpha) \neq \phi \Rightarrow U(\bar{A};\alpha) \,:\, \text{subalgebra}).$$

Therefore many properties of fuzzy BCK/BCI-subalgebras are obtained from those of crisp BCK/BCI-subalgebras immediately.

In some cases we have simpler form of the theorem. These proofs are very easy, so we omit the proofs.

Theorem 3.2. $\forall\alpha(U(\bar{A};\alpha) \neq \phi \Rightarrow \forall x, \cdots, \forall y(t(x, \cdots, y) \in U(\bar{A};\alpha) \Rightarrow \mathcal{P}))$
$\Longleftrightarrow \forall\alpha\forall x \cdots \forall y(t(x, \cdots, y) \in U(\bar{A};\alpha) \Rightarrow \mathcal{P})$.

Theorem 3.3. *Two statements* $\forall\alpha(F(\alpha) \Rightarrow G \wedge H)$ *and* $\forall\alpha(F(\alpha) \Rightarrow G) \wedge \forall\alpha(F(\alpha) \Rightarrow H)$ *are equivalent, where* G *and* H *do not contain the symbol* α.

Example 3.4. (Ideals) A non-empty subset A of a BCK/BCI-algebra X is called an *ideal* of X if

- $0 \in A$.
- $\frac{x*y \in A,\ y \in A}{x \in A}$.

The first formula, $0 \in A$, does not have the form of our theorem, but it is easy to see that the following two statements "for a non-empty set A, $0 \in A$" and "$\forall x(x \in A \Rightarrow 0 \in A)$" are equivalent, so we can define an ideal A of X as follows :

- $\forall x(x \in A \Rightarrow 0 \in A)$.
- $\frac{x*y \in A,\ y \in A}{x \in A}$.

Hence if we define a fuzzy ideal $\bar{A}$ by

- $\bar{A}(0) \geq \bar{A}(x)$,
- $\bar{A}(x) \geq \bar{A}(x * y) \wedge \bar{A}(y)$,

then it follows from Transfer Principle that

$$\bar{A} \,:\, \text{fuzzy ideal of } X \Longleftrightarrow \forall\alpha(U(\bar{A};\alpha) \neq \phi \Rightarrow U(\bar{A};\alpha) :\ \text{ideal of } X),$$

which is considered in [7, Thorem 3].

For other ideals, e.g., *implicative ideal*, we can do similarly. A non-empty subset A of X is called an *implicative ideal* of X if it satisfies

- $0 \in A$
- $\frac{(x*y)*z \in A,\ y*z \in A}{x*z \in A}$.

Hence, as above, if we define a *fuzzy implicative ideal* $\bar{A}$ of X by

- $\bar{A}(0) \geq \bar{A}(x)$
- $\bar{A}(x * z) \geq \bar{A}((x * y) * z) \wedge \bar{A}(y * z)$,

then it follows from Transfer Principle that

$$\bar{A}\ :\ \text{fuzzy implicative ideal} \iff \forall\alpha(U(\bar{A};\alpha)\neq\phi\Rightarrow U(\bar{A};\alpha):\ \text{implicative ideal}),$$

which is discussed in [7, Theorem 5]. We prove a fundamental result which plays an important role in the rest of paper.

Lemma 3.5. *Let X be a BCK/BCI-algebra and $\bar{A}$ be a fuzzy ideal of X. For $x, y \in X$, if $x \le y$ then $\bar{A}(x) \ge \bar{A}(y)$*

Proof. Suppose that $\bar{A}$ is a fuzzy ideal and $x \le y$. Since $x * y = 0$, we have $\bar{A}(x) \ge \bar{A}(x*y)\wedge\bar{A}(y) = \bar{A}(0)\wedge\bar{A}(y) = \bar{A}(y)$. □

Example 3.6. (Stable Subsets) As the last example we list the case of stability of IS-algebras. Let X be an IS-algebra (See [4, 5, 6]). A subset A of X is said to be *stable* if it satisfies

- $\frac{x\in A}{xy\in A}$.
- $\frac{x\in A}{yx\in A}$.

Thus, if we define a *fuzzy stable subset* $\bar{A}$ of X by

- $\bar{A}(xy) \ge \bar{A}(x)$
- $\bar{A}(yx) \ge \bar{A}(x)$,

then we have

$$\bar{A}\ :\ \text{fuzzy stable} \iff \forall\alpha(U(\bar{A};\alpha)\neq\phi\Rightarrow U(\bar{A};\alpha):\text{stable}).$$

In particular, an $\mathcal{I}$-ideal A of an IS-algebra X is defined by

$$A:\mathcal{I}\text{-ideal} \iff A:\text{ideal and stable},$$

so if we define a *fuzzy $\mathcal{I}$-ideal* $\bar{A}$ of X by

- $\bar{A}(0) \ge \bar{A}(x)$
- $\bar{A}(x) \ge \bar{A}(x*y)\wedge\bar{A}(y)$
- $\bar{A}(xy) \ge \bar{A}(x)$
- $\bar{A}(yx) \ge \bar{A}(x)$,

then it follows from Transfer Principle that

$$\bar{A}:\text{fuzzy }\mathcal{I}\text{-ideal} \iff \forall\alpha(U(\bar{A};\alpha)\neq\phi\Rightarrow U(\bar{A};\alpha):\mathcal{I}\text{-ideal}).$$

4. Applications

In this section we consider some results obtained in the theory of fuzzy BCK/BCI-algebras by using our method. The next result is proved in [7].

Theorem 4.1. *Let X, Y be BCK-algebras and $f : X \to Y$ be a BCK-homomorphism. If A is an ideal of Y, then $f^{-1}(A)$ is an deal of X.*

It follows from Transfer Principle that we can extend the result to the case of fuzzy ideal.

Theorem 4.2. *Let X and Y be BCK-algebras. For a BCK-homomorphism $f : X \to Y$, if $\bar{A}$ is a fuzzy ideal of Y then $f^{-1}(\bar{A})$ is a fuzzy ideal of X, where $f^{-1}(\bar{A})$ is defined by $f^{-1}(\bar{A})(x) = \bar{A}(f(x))$.*

Proof. Using Transfer Principle, we have

$$f^{-1}(\bar{A}):\text{fuzzy ideal} \iff \forall\alpha(U(f^{-1}(\bar{A});\alpha)\neq\phi\Rightarrow U(f^{-1}(\bar{A});\alpha):\text{ideal of }X).$$

It is sufficient to prove that $U(f^{-1}(\bar{A});\alpha)$ is an ideal of X whenever $U(f^{-1}(\bar{A});\alpha)\neq\phi$. Since $\bar{A}$ is a fuzzy ideal, it follows from Transfer Principle that $U(\bar{A};\alpha)$ is an ideal whenever $U(\bar{A};\alpha)\neq\phi$. It is clear that $U(f^{-1}(\bar{A});\alpha) = f^{-1}(U(\bar{A};\alpha))$. Hence if $U(f^{-1}(\bar{A});\alpha)\neq\phi$

then $f^{-1}(U(\bar{A};\alpha)) \neq \phi$, $U(\bar{A};\alpha) \neq \phi$, and $U(\bar{A};\alpha)$ is an ideal by assumption and hence $f^{-1}(U(\bar{A};\alpha))$ is an ideal by the above. This means that $U(f^{-1}(\bar{A});\alpha)$ is an ideal if $U(f^{-1}(\bar{A});\alpha) \neq \phi$. □

Moreover the following is proved in [4] when f is surjective :

Theorem 4.3. *Let X and Y be BCK-algebras and $f : X \to Y$ be a surjective BCK-homomorphism. If A is an ideal of X, then $f(A)$ is an deal of Y.*

To extend this theorem to the case of fuzzy ideals, we need a lemma. Let X, Y be BCK-algebras and $f : X \to Y$ be a BCK-homomorphism. For a fuzzy set $\bar{A}$ we define a fuzzy set $f[\bar{A}]$ by $f[\bar{A}](x) = \bigvee_{u \in f^{-1}(x)} \bar{A}(u)$.

Lemma 4.4. *For all $\alpha \in [0,1]$, $U(f[\bar{A}];\alpha) = \bigcap_{\varepsilon>0} f(U(\bar{A};\alpha-\varepsilon))$*

Proof. We have that

$$\begin{aligned}
y \in U(f[\bar{A}];\alpha)) &\iff f[\bar{A}](y) \geq \alpha \iff \bigvee_{u\in f^{-1}(y)} \bar{A}(u) \geq \alpha \\
&\iff \forall \varepsilon > 0\, \exists u \in f^{-1}(y)\, \text{s.t. } \bar{A}(u) \geq \alpha - \varepsilon \\
&\iff \forall \varepsilon > 0\ \exists u \in f^{-1}(y) \text{ s.t. } u \in U(\bar{A};\alpha-\varepsilon) \\
&\iff \forall \varepsilon > 0\ y = f(u) \in f(U(\bar{A};\alpha-\epsilon)) \\
&\iff y \in \bigcap_{\varepsilon>0} f(U(\bar{A};\alpha-\varepsilon).)
\end{aligned}$$

This completes the proof. □

Using this lemma we can show the next theorem.

Theorem 4.5. *Let X and Y be BCK-algebras. For a surjective BCK-homomorphism $f : X \to Y$, if $\bar{A}$ is a fuzzy ideal of X then $f[\bar{A}]$ is a fuzzy ideal of Y.*

Proof. It follows from Transfer Principle that

$$f[\bar{A}] : \text{fuzzy ideal of } Y \iff \forall\alpha(U(f[\bar{A}];\alpha) \neq \phi \Rightarrow U(f[\bar{A}];\alpha) : \text{ideal of } Y).$$

It is sufficient to show that $U(f[\bar{A}];\alpha)$ is an ideal of Y if $U(f[\bar{A}];\alpha) \neq \phi$. Since $\bar{A}$ is a fuzzy ideal, it follows that $U(\bar{A};\alpha-\varepsilon)$ is also an ideal for all $\varepsilon > 0$ if $U(\bar{A};\alpha-\varepsilon) \neq \phi$. From f being surjective, we have $f(U(\bar{A};\alpha-\varepsilon))$ is an ideal of Y for all $\varepsilon > 0$ by the theorem of the crisp theory of BCK-algebras. Hence $\bigcap_{\varepsilon>0} f(U(\bar{A};\alpha-\varepsilon)) = U(f[\bar{A}];\alpha)$ is an ideal of Y. □

Using the Transfer Principle we can show the next results without difficulty, so we omit the proofs.

Theorem 4.6. *Let X and Y be IS-algebras and f be an IS-homomorphism, that is, $f(x * y) = f(x) * f(y)$ and $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in X$.*

(1) *If $\bar{A}$ is fuzzy stable in Y then $f^{-1}(\bar{A})$ is fuzzy stable in X.*

Moreover if f is surjective, then

(2) *$\bar{A}$: fuzzy stable in $Y \iff f^{-1}(\bar{A})$: fuzzy stable in X.*

(3) *$\bar{A}$: fuzzy stable in $X \implies f[\bar{A}]$: fuzzy stable in Y.*

Hence in particular

(4) *$\bar{A}$: fuzzy $\mathcal{I}$-ideal of $Y \implies f^{-1}(\bar{A})$: fuzzy $\mathcal{I}$-ideal in X.*

Moreover if f is surjective

(5) $\bar{A}$ *: fuzzy* $\mathcal{I}$*-ideal in* $Y \Longleftrightarrow f^{-1}(\bar{A})$ *: fuzzy* $\mathcal{I}$*-ideal in* X.
(6) $\bar{A}$ *: fuzzy* $\mathcal{I}$*-ideal in* $X \Longrightarrow f[\bar{A}]$ *: fuzzy* $\mathcal{I}$*-ideal in* Y.

Finally, we give an alternative proof of the following result (Theorem 3.5 in [1]) about fuzzy BCC-algebras by using the Transfer Principle.

Theorem 4.7. *A fuzzy set* $\bar{A}$ *in a* BCC*-algebra is a fuzzy* g*-ideal if and only if it is a* BCC*-ideal.*

For the definitions of BCC-algebras, BCC-ideals, and g-ideals, we refer to [1]. The following is well known.

Lemma 4.8. *Let* A *be a non-empty subset of a* BCC*-algebra* G*. Then* A *is a* BCC*-ideal if and only if it is a* g*-ideal.*

Now in general we can prove the next result.

Lemma 4.9. *Let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be properties with respect to a subset* A*. If it is equivalent for* A *to have the property* $\mathcal{P}$ *and to have* $\mathcal{Q}$*, then so is for a fuzzy set* $\bar{A}$ *to have* $\bar{\mathcal{P}}$ *and to have* $\bar{\mathcal{Q}}$*.*

Proof. We have that

$$\begin{aligned} \bar{A} : \bar{\mathcal{P}} &\Longleftrightarrow \forall \alpha (U(\bar{A};\alpha) \neq \phi \Longrightarrow U(\bar{A};\alpha) : \mathcal{P}) \\ &\Longleftrightarrow \forall \alpha (U(\bar{A};\alpha) \neq \phi \Longrightarrow U(\bar{A};\alpha) : \mathcal{Q}) \\ &\Longleftrightarrow \bar{A} : \bar{\mathcal{Q}}. \end{aligned}$$

This completes the proof. □

Thus from the lemma above, we conclude the Theorem 4.7.

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