SATISFACTORY FILTERS OF BCK-ALGEBRAS

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Received May 4, 2003

ABSTRACT. The notion of satisfactory filters in BCK-algebras is introduced. Characterizations of satisfactory filters are discussed. Extension property for a satisfactory filter is established.

1. INTRODUCTION.

For the general development of BCK-algebra, the filter theory plays an important role as well as ideal theory. In [3], Meng introduced the notion of (prime) filters in BCK-algebras, and then gave a description of the filter generated by a set, and obtained some of fundamental properties of prime filters. In this paper, we introduce the notion of satisfactory filters in BCK-algebras, and investigate some of its properties. We give characterizations of satisfactory filters. We build the extension property of a satisfactory filter.

2. PRELIMINARIES

We review some definitions and properties that will be useful in our results.

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions:

- $\forall x, y, z \in X\ (((x * y) * (x * z)) * (z * y) = 0)$,
- $\forall x, y \in X\ ((x * (x * y)) * y = 0)$,
- $\forall x \in X\ (x * x = 0)$,
- $\forall x, y \in X\ (x * y = 0,\ y * x = 0\ \Rightarrow\ x = y)$.

A BCI-algebra X satisfying $0 * x = 0$ for all $x \in X$ is called a *BCK-algebra*. In any BCK/BCI-algebra X one can define a partial order "$\leq$" by putting $x \leq y$ if and only if $x * y = 0$. In a BCK-algebra X, the following hold.

(p1) $\forall x \in X\ (x * 0 = x)$.
(p2) $\forall x, y, z \in X\ ((x * y) * z = (x * z) * y)$.
(p3) $\forall x, y, z \in X\ (x \leq y\ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x)$.
(p4) $\forall x, y, z \in X\ ((x * z) * (y * z) \leq x * y)$.
(p5) $\forall x, y, z \in X\ (x * (x * (x * y)) = x * y)$.
(p6) $\forall x, y \in X\ (x * y \leq x)$.

A BCK-algebra X is said to be *positive implicative* if $(x * z) * (y * z) = (x * y) * z$ for all $x, y, z \in X$. A BCK-algebra X is positive implicative if and only if it satisfies $x*y = (x*y)*y$ for all $x, y \in X$ (see [1]). A BCK-algebra X is said to be *commutative* if $x*(x*y) = y*(y*x)$ for all $x, y \in X$. A nonempty subset I of a BCK-algebra X is called an *ideal* of X it it satisfies

(I1) $0 \in I$.

2000 *Mathematics Subject Classification.* 06F35, 03G25.
Key words and phrases. (Positive implicative) ideal, (satisfactory) filter.
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(I2) $\forall x, y \in X\ (x * y \in I,\ y \in I\ \Rightarrow\ x \in I)$.

A nonempty subset I of a BCK-algebra X is called a *positive implicative ideal* of X it it satisfies (I1) and

(I3) $\forall x, y, z \in X\ ((x * y) * z \in I,\ y * z \in I\ \Rightarrow\ x * z \in I)$.

Note that every positive implicative ideal is an ideal (see [2, Proposition 2]). A BCK-algebra X is said to be *bounded* if there exists a special element $e \in X$ such that $x \leq e$ for all $x \in X$. In this case, we call e the *bound* of X. In what follows let X denote a bounded BCK-algebra unless otherwise specified, and we will use the notation $e(x)$ instead of $e * x$ for all $x \in X$ and the bound e of X. Note that, in a bounded commutative BCK-algebra, the equalities $e(e(x)) = x$ and $e(x) * e(y) = y * x$ hold.

Definition 2.1. [3] A nonempty subset F of X is called a *filter* of X if it satisfies:

(F1) F contains the bound e of X,
(F2) $\forall x, y \in X\ (e(e(x) * e(y)) \in F,\ y \in F\ \Rightarrow\ x \in F)$.

Proposition 2.2. [3, Theorem 11] *Assume that X is commutative. Then a nonempty subset F of X is a filter of X if and only if it satisfies.* (F1) *and*

(F3) $\forall x, y \in X\ (e(x * y) \in F,\ x \in F\ \Rightarrow\ y \in F)$.

3. Satisfactory filters

Lemma 3.1. *Assume that X is commutative. Let F be a filter of X. If $x \leq y$ and $x \in F$, then $y \in F$.*

Proof. Suppose $x \leq y$ and $x \in F$. Since $e(x * y) = e(0) = e \in F$, it follows from (F3) that $y \in F$. □

Theorem 3.2. *Assume that X is commutative. Let F be a nonempty subset of X. Then F is a filter of X if and only if it satisfies*

(F4) $\forall x, y \in F, \forall z \in X\ (y \leq e(x * z)\ \Rightarrow\ z \in F)$.

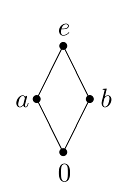
Proof. Suppose that F is a filter of X. Let $x, y \in F$ and $z \in X$ be such that $y \leq e(x * z)$. Then $e(x*z) \in F$ by Lemma 3.1, and so $z \in F$ by (F3). Conversely assume that (F4) holds. We can select $x \in F$ because F is nonempty. Since $x \leq e(x * e)$, it follows from (F4) that $e \in F$. Let $x, y \in X$ be such that $e(x * y) \in F$ and $x \in F$. The inequality $e(x * y) \leq e(x * y)$ implies that $y \in F$ by (F4). Hence, by Proposition 2.2, F is a filter of X. This completes the proof. □

Definition 3.3. A nonempty subset F of X is called a *satisfactory filter* of X if it satisfies (F1) and

(F5) $\forall x, y, z \in X\ (e(x * e(y * e(y * z))) \in F,\ x \in F\ \Rightarrow\ e(y * z) \in F)$.

Example 3.4. Let $X = \{0, a, b, e\}$ be a bounded BCK-algebra with Cayley table and Hasse diagram:

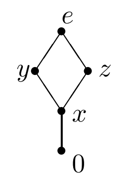
$*$	0	a	b	e
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
e	e	b	a	0



Then $F_1 := \{e\}$, $F_2 := \{e, a\}$, and $F_3 := \{e, b\}$ are satisfactory filters of X.

Example 3.5. Let $X = \{0, x, y, z, e\}$ be a bounded BCK-algebra with Cayley table and Hasse diagram:

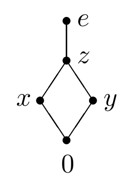
$*$	0	x	y	z	e
0	0	0	0	0	0
x	x	0	0	0	0
y	y	x	0	x	0
z	z	z	z	0	0
e	e	z	z	x	0



Then $F := \{z, e\}$ is a satisfactory filter of X.

Example 3.6. Let $X = \{0, x, y, z, e\}$ be a bounded BCK-algebra with Cayley table and Hasse diagram:

$*$	0	x	y	z	e
0	0	0	0	0	0
x	x	0	x	0	0
y	y	y	0	0	0
z	z	z	z	0	0
e	e	z	e	x	0



Then $G := \{z, e\}$ is a satisfactory filter of X.

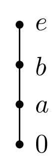
Theorem 3.7. *In a bounded commutative BCK-algebra, every satisfactory filter is a filter.*

Proof. Let F be a satisfactory filter of a bounded commutative BCK-algebra X and let $x, y \in X$ be such that $e(x * y) \in F$ and $x \in F$. Since $e(e(x)) = x$ for all $x \in X$, we have $e(x * y) = e(x * e(e(e(e(y))))) \in F$. It follows from (F5) that $y = e(e(y)) \in F$. Hence F is a filter of X by Proposition 2.2. □

The converse of Theorem 3.7 may not be true as seen in the following example.

Example 3.8. Let $X = \{0, a, b, e\}$ be a set with the following Cayley table and Hasse diagram.

$*$	0	a	b	e
0	0	0	0	0
a	a	0	0	0
b	b	a	0	0
e	e	b	a	0



Then X is a bounded commutative BCK-algebra (see [4]). It is easy to check that $\{e\}$ is a filter of X, but not a satisfactory filter of X because $e(e(e(b * e(b * a)))) = e(b * e(a)) = e(b * b) = e(0) = e \in \{e\}$, but $e(b * a) = e(a) = b \notin \{e\}$.

Theorem 3.9. *If X is commutative and positive implicative, then every filter of X is a satisfactory filter of X.*

Proof. Let F be a filter of X and let $x, y, z \in X$ be such that $e(x * e(y * e(y * z))) \in F$ and $x \in F$. Using (F3), we get

$$e(y * z) = e(e(z) * e(y)) = e((e(z) * e(y)) * e(y)) = e((y * z) * e(y)) = e(y * e(y * z)) \in F.$$

Hence F is a satisfactory filter of X. □

Corollary 3.10. *If X is implicative, then the notion of filters and satisfactory filters coincide.*

Theorem 3.11. *Assume that X is commutative and let F be a filter of X. Then F is a satisfactory filter of X if and only if it satisfies*

(F6) $\forall x, y \in X \ (e(x * e(x * y)) \in F \ \Rightarrow \ e(x * y) \in F)$.

Proof. Assume that F is a satisfactory filter of X and $e(x * e(x * y)) \in F$ for all $x, y \in X$. Then $e(e(e(x * e(x * y)))) = e(x * e(x * y)) \in F$. Since $e \in F$, it follows from (F5) that $e(x * y) \in F$. Conversely, let F be a filter of X that satisfies (F6). Let $x, y, z \in X$ be such that $e(x * e(y * e(y * z))) \in F$ and $x \in F$. Then, by (F3), we get $e(y * e(y * z)) \in F$. Hence $e(y * z) \in F$ by (F6), which shows that (F5) holds. Therefore F is a satisfactory filter of X. □

Lemma 3.12. [2, Theorem 2] *Let I be an ideal of a BCK-algebra X. Then the following are equivalent.*

(i) *I is a positive implicative ideal of X.*
(ii) $\forall x, y \in X \ ((x * y) * y \in I \ \Rightarrow \ x * y \in I)$.
(iii) $\forall x, y, z \in X \ ((x * y) * z \in I \ \Rightarrow \ (x * z) * (y * z) \in I)$.

Theorem 3.13. *Let X be commutative and let $e(G) := \{e(x) \mid x \in G\}$ for every nonempty subset G of X. Then $e(G)$ is a satisfactory filter of X if and only if G is a positive implicative ideal of X.*

Proof. Assume that G is a positive implicative ideal of X. Since $0 \in G$, it follows that $e = e(0) \in e(G)$. Let $x, y, z \in X$ be such that $e(x * e(y * e(y * z))) \in e(G)$ and $x \in e(G)$. Then there exist $u, v \in X$ such that $e(x * e(y * e(y * z))) = e(u)$ and $x = e(v)$. It follows that $e(x) = e(e(v)) = v \in G$ and

$$x * e(y * e(y * z)) = e(e(x * e(y * e(y * z)))) = e(e(u)) = u \in G,$$

so that $e(x) \in G$ and $(y * e(y * z)) * e(x) \in G$. Since G is a positive implicative ideal and hence an ideal, it follows that

$$(e(z) * e(y)) * e(y) = y * e(e(z) * e(y)) = y * e(y * z) \in G$$

so from Lemma 3.12(ii) that $y * z = e(z) * e(y) \in G$. Thus $e(y * z) \in e(G)$, and so $e(G)$ is a satisfactory filter of X. Conversely suppose that $e(G)$ is a satisfactory filter of X. Since $e \in e(G)$, we have $0 = e(e) \in e(e(G)) = G$. Let $x, y \in X$ be such that $x * y \in G$ and $y \in G$. Then $e(e(y) * e(x)) = e(x * y) \in e(G)$ and $e(y) \in e(G)$. Using (F3), we get $e(x) \in e(G)$ and thus $x \in G$. Hence G is an ideal of X. Now let $x, y \in X$ be such that $(x * y) * y \in G$. Then $e((x * y) * y) \in e(G)$, which implies that $e(e(e(e(y) * e(e(y) * e(x))))) \in e(G)$. Using (F5), we have $e(x * y) = e(e(y) * e(x)) \in e(G)$, and so $x * y \in G$. Therefore, by Lemma 3.12(ii), G is a positive implicative ideal of X. □

Theorem 3.14. *Let X be commutative and F a filter of X. Then the following are equivalent.*

(i) *F is a satisfactory filter of X.*
(ii) $\forall x, y, z \in X \ (e(z * e(x * y)) \in F \ \Rightarrow \ e(e(z * x) * e(z * y)) \in F)$.
(iii) $\forall x, y, z, u \in X \ (e(u * e(z * e(x * y))) \in F,\ u \in F \ \Rightarrow \ e(e(z * x) * e(z * y)) \in F)$.

Proof. (i) $\Rightarrow$ (ii). Let F be a satisfactory filter of X and let $x, y, z \in X$ be such that $e(z * e(x * y)) \in F$. Then $(e(y) * e(x)) * e(z) = (x * y) * e(z) = z * e(x * y) \in e(F)$. It follows from Lemma 3.12(iii) that

$$e(z * x) * e(z * y) = (z * y) * (z * x) = (e(y) * e(z)) * (e(x) * e(z)) \in e(F)$$

so that $e(e(z * x) * e(z * y)) \in e(e(F)) = F$.

(ii) $\Rightarrow$ (iii). Trivial.

(iii) $\Rightarrow$ (i). Let $x, y, z \in X$ be such that $x \in F$ and $e(x * e(y * e(y * z))) \in F$. Using (iii), we have $e(y * z) = e(e(0) * e(y * z)) = e(e(y * y) * e(y * z)) \in F$. Hence F is a satisfactory filter of X. □

Theorem 3.15. (Extension property for a satisfactory filter) *Assume that X is commutative and let F and G be filters of X such that $F \subseteq G$. If F is a satisfactory filter of X, then so is G.*

Proof. Suppose that F is a satisfactory filter of X and $e(x * e(x * y)) \in G$ for all $x, y \in X$. If we put $w = e(x * e(x * y))$, then

$$\begin{aligned} & e(x * e(x * e(w * y))) = e((x * e(w * y)) * e(x)) \\ = \quad & e(((w * y) * e(x)) * e(x)) = e((((e(y) * e(w)) * e(x)) * e(x)) \\ = \quad & e(((e(y) * e(x)) * e(x)) * e(w)) = e((x * e(x * y)) * e(w)) \\ = \quad & e(w * e(x * e(x * y))) = e(0) = e \in F. \end{aligned}$$

It follows from (F6) that $e(x * e(w * y)) \in F \subseteq G$ so that

$$\begin{aligned} & e(w * e(x * y)) = e((x * y) * e(w)) = e((x * e(w)) * y) \\ = \quad & e((w * e(x)) * y) = e((w * y) * e(x)) = e(x * e(w * y)) \in G. \end{aligned}$$

Since G is a filter, we get $e(x * y) \in G$ by using (F3). Hence, by Theorem 3.11, G is a satisfactory filter of X. □

Theorem 3.16. *Let X be commutative and F a filter of X. Then F is a satisfactory filter of X if and only if for every $w \in X$, the set $F_w := \{x \in X \mid e(w * x) \in F\}$ is a filter of X.*

Proof. Assume that F is a satisfactory filter of X. Obviously, $e \in F_w$. Let $x, y \in X$ be such that $e(x * y) \in F_w$ and $x \in F_w$. Then $e(w * e(x * y)) \in F$ and $e(w * x) \in F$. Note that

$$\begin{aligned} & e(w * e(x * y)) * e(w * e(w * y)) = (w * e(w * y)) * (w * e(x * y)) \\ \leq \quad & e(x * y) * e(w * y) = (w * y) * (x * y) \leq w * x. \end{aligned}$$

It follows from (p3) that $e(w * x) \leq e(e(w * e(x * y)) * e(w * e(w * y)))$ so from Theorem 3.2 that $e(w * e(w * y)) \in F$. By Theorem 3.11, we get $e(w * y) \in F$, that is, $y \in F_w$. Hence F_w is a filter of X by Proposition 2.2. Conversely, suppose that for any $w \in X$, the set F_w is a filter of X. Let $x, y \in X$ be such that $e(x * e(x * y)) \in F$. Then $e(x * y) \in F_x$. Since $x \in F_x$, it follows from (F3) that $y \in F_x$, that is, $e(x * y) \in F$. Hence, by Theorem 3.11, F is a satisfactory filter of X. □

Corollary 3.17. *Let X be commutative and F a satisfactory filter of X. For any $w \in X$, the set F_w is the least filter of X containing F and w.*

Proof. By Theorem 3.16, F_w is a filter of X. Obviously $w \in F_w$. Let $u \in F$. Then

$$u * e(w * u) = (w * u) * e(u) \leq w * e = 0,$$

and so $u * e(w * u) = 0$, i.e., $u \leq e(w * u)$. Using Lemma 3.1, we get $e(w * u) \in F$, and so $u \in F_w$. This shows that $F \subseteq F_w$. Let G be a filter of X containing F and w. If $x \in F_w$, then $e(w * x) \in F \subseteq G$ and thus $x \in G$ by (F3). Hence $F_w \subseteq G$, completing the proof. □

Theorem 3.18. *Let X be commutative and F a filter of X. Then the following are equivalent.*

(i) *F is a satisfactory filter of X.*
(ii) $\forall x, y \in X \ (e(e(x * y) * x) \in F \ \Rightarrow \ x \in F)$.
(iii) $\forall x, y, z \in X \ (e(x * e(e(y * z) * y)) \in F, \, x \in F \ \Rightarrow \ y \in F)$.

Proof. (i) ⇒ (ii). Suppose that F is a satisfactory filter of X and let $x, y \in X$ be such that $e(e(x*y)*x) \in F$. Note that

$$\begin{aligned} & e(e(x*y)*x)*e(e(x*y)*e(e(x*y)*y)) \\ = \;& (e(x*y)*e(e(x*y)*y))*(e(x*y)*x) \\ \leq \;& x*e(e(x*y)*y) = (e(x*y)*y)*e(x) \\ = \;& (e(y)*(x*y))*e(x) = (e(y)*e(x))*(x*y) = 0. \end{aligned}$$

Thus $e(e(x*y)*x)*e(e(x*y)*e(e(x*y)*y)) = 0$, that is,

$$e(e(x*y)*x) \leq e(e(x*y)*e(e(x*y)*y)),$$

which implies from Lemma 3.1 that $e(e(x*y)*e(e(x*y)*y)) \in F$. Using Theorem 3.11, we have $e(e(x*y)*y) \in F$. Note that

$$\begin{aligned} & e(e(e(x*y)*y)*x) = e(e(e(y)*(e(y)*e(x)))*x) \\ = \;& e(e(e(x)*(e(x)*e(y)))*x) = e(e(x)*(e(x)*(e(x)*e(y)))) \\ = \;& e(e(x)*e(y)) = e(y*x) \end{aligned}$$

and

$$\begin{aligned} & e(e(x*y)*x)*e(y*x) = (y*x)*(e(x*y)*x) \\ \leq \;& y*e(x*y) = (x*y)*e(y) \leq x*e = 0. \end{aligned}$$

Hence $e(e(x*y)*x) \leq e(e(e(x*y)*y)*x)$, and so $e(e(e(x*y)*y)*x) \in F$ by Lemma 3.1. Using Proposition 2.2, we get $x \in F$.

(ii) ⇒ (iii). Assume that (ii) is true. Let $x, y, z \in X$ be such that $e(x*e(e(y*z)*y)) \in F$ and $x \in F$. Then $e(e(y*z)*y) \in F$ by Proposition 2.2, and hence $y \in F$ by (ii).

(iii) ⇒ (i). Suppose that (iii) holds. We first show that

$$\forall x, y \in X \; (e(e(x*y)*x) \in F \;\Rightarrow\; x \in F). \tag{1}$$

In fact, if $e(e(x*y)*x) \in F$, then since $e(e(x*y)*x) = e(e(e(e(x*y)*x)))$ it follows from (iii) that $x \in F$. Now let $x, y \in X$ be such that $e(x*e(x*y)) \in F$. Note that

$$\begin{aligned} & e(e(e(x*y)*y)*e(x*y)) = e((x*y)*(e(x*y)*y)) \\ = \;& e((e(y)*e(x))*(e(y)*(x*y))) = e((e(y)*(e(y)*(x*y)))*e(x)) \\ = \;& e(((x*y)*((x*y)*e(y)))*e(x)) \\ = \;& e(((e(y)*e(x))*((e(y)*e(x))*e(y)))*e(x)) \\ = \;& e(((e(y)*e(x))*e(x))*((e(y)*e(x))*e(y))) \\ = \;& e(((e(y)*e(x))*e(x))*((e(y)*e(y))*e(x)) \\ = \;& e(((e(y)*e(x))*e(x)) = e(x*e(x*y)) \in F, \end{aligned}$$

which implies from (1) that $e(x*y) \in F$. Thus, by Theorem 3.11, F is a satisfactory filter of X. □

Theorem 3.19. *If X is commutative, then the following are equivalent.*

(i) *X is positive implicative.*
(ii) *Every filter of X is a satisfactory filter.*
(iii) *The trivial filter $\{e\}$ of X is a satisfactory filter.*
(iv) *For every $w \in X$, the set $X(w) := \{x \in X \mid e(w*x) = e\}$ is a filter of X.*
(v) *For every filter F of X and $w \in X$, the set $F_w := \{x \in X \mid e(w*x) \in F\}$ is a filter of X.*

Proof. (i) $\Rightarrow$ (ii). It follows from Theorem 3.9.

(ii) $\Rightarrow$ (iii). Trivial.

(iii) $\Rightarrow$ (iv). Assume that $\{e\}$ is a satisfactory filter of X. For $w \in X$, let $x, y \in X$ be such that $e(x*y) \in X(w)$ and $x \in X(w)$. Then $e(w*e(x*y)) = e \in \{e\}$ and $e(w*x) = e \in \{e\}$. Using the similar method to the proof of Theorem 3.16, we obtain $e(w*e(w*y)) \in \{e\}$. Since $\{e\}$ is a satisfactory filter, it follows from Theorem 3.11 that $e(w*y) \in \{e\}$, i.e., $y \in X(w)$. Hence $X(w)$ is a filter of X by Proposition 2.2.

(iv) $\Rightarrow$ (i). Assume that (iv) holds and let $x, y \in X$ be such that $x*y \leq y$. Then $e(y) \leq e(x*y)$, that is, $e(y)*e(x*y) = 0$. Hence $e = e(0) = e(e(y)*e(x*y))$, and so $e(e(y)*e(x)) = e(x*y) \in X(e(y))$. Since $X(e(y))$ is a filter of X and $e(y) \in X(e(y))$, it follows from (F3) that $e(x) \in X(e(y))$, that is, $e(x*y) = e(e(y)*e(x)) = e$. Thus $x*y = e(e) = 0$, and thus $x \leq y$. This shows that

$$\forall x, y \in X \ (x*y \leq y \ \Rightarrow \ x \leq y). \tag{2}$$

To show that $(x*y)*y = x*y$ for all $x, y \in X$, let $u = (x*y)*y$. Then $((x*u)*y)*y = ((x*y)*y)*u = 0$, and so $(x*y)*((x*y)*y) = (x*((x*y)*y))*y = (x*u)*y = 0$ by (2). Since $((x*y)*y)*(x*y) = 0$ by (p6), we conclude that $(x*y)*y = x*y$. Therefore X is positive implicative.

(i) $\Rightarrow$ (v). This is by Theorems 3.9 and 3.16.

(v) $\Rightarrow$ (iii). Suppose that (v) is true. Let $x, y \in X$ be such that $e(x*e(x*y)) \in \{e\}$. Note that the set $\{e\}_x := \{z \in X \mid e(x*z) \in \{e\}\}$ is a filter of X by assumption. Since $e(x*y) \in \{e\}_x$ and $x \in \{e\}_x$, it follows from (F3) that $y \in \{e\}_x$ so that $e(x*y) \in \{e\}$. Therefore $\{e\}$ is a satisfactory filter of X by Theorem 3.11. This completes the proof. $\square$

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