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N-MAPS OF BCK-ALGEBRAS

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Abstract. In this paper, we introduce the concept of n-maps of BCK-algebras and study some ideals of n-fold positive implicative BCK-algebras. Moreover, we show that if X is n-fold positive implicative BCK-algebras, then X is isomorphic to $N(X)$

1 Introduction and Preliminaries.

By a BCK-algebras we mean an algebra $(X; *, 0)$ of type $(2, 0)$ satisfying the following axioms:

(I) $(x * y) * (x * z) \leq (z * y)$,

(II) $x * (x * y) \leq y$,

(III) $x \leq x$,

(IV) $x \leq y$ and $y \leq x$ implies $x = y$,

(V) $0 \leq x$.

where $x \leq y$ is defined by $x * y = 0$.

A BCK-algebra X is said to be n-fold positive implicative (briefly, PI^n) if there exists a natural number n such that $(x * y) * z^n = (x * z^n) * (y * z^n)$ for all $x, y, z \in X$. For any elements x and y of a BCK-algebras, $x * y^n$ denotes $(\cdots(x * y) * \cdots) * y$ in which y occurs n times. A nonempty subset A of a BCK-algebras X is called an ideal of X if (i) $0 \in A$ and (ii) $y, x * y \in A$ implies $x \in A$.

Definition 1.1 ([2])Let X be a BCK-algebras and n a natural number. A self-map N_x over X defined by $N_x(t) = x * t^n$ for all $t \in X$ is called an n-map over X.

Let A be a subset of a BCK-algebra X. Denote $N_A = \{N_x \mid x \in A\}$ and $N(X) = \{N_x \mid x \in X\}$, we define $*$ on $N(X)$ by $(N_x * N_y)(t) = N_x(t) * N_y(t)$ for all $t \in X$.

It's clear that A BCK-algebras X is PI^n if and only if $N_{x*y} = N_x * N_y$ for all $x, y \in X$.

Definition 1.2 ([1])A nonempty subset A of a BCK-algebras X is called an n-fold positive implicative ideal (briefly, PI^n-ideal) if it satisfies:

(i) $0 \in A$,

(ii) $(x * y) * z^n \in A, y * z^n \in A$ imply $x * z^n \in A$ for all $x, y, z \in X$.

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2 Characterizations of some ideals by n-maps.

Theorem 2.1 A BCK-algebras X is PI^n, then so $N(X)$ is.

Proof. If X is a PI^n-BCK-algebra, then for every $t \in X$, we have $((N_x * N_y) * (N_z^n))(t) = ((x*t^n)*(y*t^n))*(z*t^n)^n = ((x*t^n)*(z*t^n)^n)*((y*t^n)*(z*t^n)^n) = ((N_x*N_z^n)*(N_y*N_z^n))(t)$ that is, $(N_x * N_y) * N_z^n = (N_x * N_z^n) * (N_y * N_z^n)$. Hence $N(X)$ is a PI^n-BCK-algebras.

Theorem 2.2 Let X be a PI^n-BCK-algebra and A a subset of X. Then A is a PI^n-ideal if and only if N_A is PI^n-ideal in $N(X)$.

Proof. Assume that A is a PI^n-ideal of X, $0 \in A$ implies $N_0 \in N_A$. Let $(N_x * N_y) * N_z^n \in N_A$ and $N_y * N_z^n \in N_A$ for all $x, y, z \in X$. Since X is PI^n, we have $N_{(x*y)*z^n} \in N_A$ and $N_{y*z^n} \in N_A$, and so $(x * y) * z^n \in A$ and $y * z^n \in A$. Since A is a PI^n-ideal, it follows that $x * z^n \in A$, and hence $N_{x*z^n} \in N_A$.

Conversely, let N_A be a PI^n-ideal of $N(X)$, then $N_0 \in N_A$ implies $0 \in A$. Let $x, y, z \in A$ be such that $(x*y)*z^n \in A$ and $y*z^n \in A$. Then we have $N_{(x*y)*z^n} \in N_A$ and $N_{y*z^n} \in N_A$. Since X is PI^n-BCK-algebra and N_A is a PI^n-ideal, it follows that $N_{x*z^n} \in N_A$ and so $x * z^n \in A$. The proof is complete.

Corollary 2.3 Let X be a PI^n-BCK-algebras and A a subset of X. Then A is an ideal in X if and only if N_A is an ideal in $N(X)$.

Definition 2.4 ([3]) A proper ideal A of BCK-algebra X is said to be obstinate if $x \in A$ and $y \notin A$ implies $x * y \in A$.

Theorem 2.5 Let X be a PI^n-BCK-algebra and A a subset of X. Then A is an obstinate ideal in X if and only if N_A is an obstinate ideal in $N(X)$.

Proof. Assume that A is an obstinate ideal of X. Then by corollary 2.3, N_A is an ideal of $N(X)$. Let $N_x, N_y \in N(X) - N_A$. Then $x, y \notin A$, and so $x * y \in A$ because A is obstinate. Since X is PI^n-BCK-algebra, we have $N_x * N_y = N_{x*y} \in N_A$. Hence N_A is an obstinate ideal of $N(X)$.

Conversely suppose that N_A is an obstinate ideal of $N(X)$. Using corollary 2.3, A is an ideal of $N(X)$. If $x, y \in X - A$, then $N_x, N_y \in N(X) - N_A$. Since N_A is an obstinate ideal, it follows that $N_x * N_y = N_{x*y} \in N_A$. Hence $x * y \in A$, which shows that A is an obstinate ideal of X. The proof is complete.

Let X be a BCK-algebra. Then for any N_x, N_y in L, we define $(N_x \wedge L_y)(t) = N_x(t) \wedge N_y(t)$. where $x \wedge y = y * (y * x)$.

In general, $N_x \wedge N_y \neq N_{x \wedge y}$

Theorem 2.6 If X is a PI^n-BCK-algebra, then we have $N_x \wedge N_y = N_{x \wedge y}$.

Proof. For any $t \in X$, we have $(N_x \wedge N_y)(t) = N_x(t) \wedge N_y(t) = (x * t^n) \wedge (y * t^n) = (y*t^n)*((y*t^n)*(x*t^n)) = (y*(y*x))*t^n = (x \wedge y)*t^n = N_{x \wedge y}(t)$. The proof is complete.

Definition 2.7([4]) An ideal A of a BCK-algebra X is said to be prime if for all $x, y \in X, x \wedge y \in X, x \wedge y \in A$ implies $x \in A$ or $y \in A$.

Theorem 2.8 Let X be a PI^n-BCK-algebra and A a subset of X. Then A is a prime ideal in X if and only if N_A is a prime ideal in $N(X)$.

Proof. Assume that A is a prime ideal of X. Then by corollary 2.3, N_A is an ideal of $N(X)$. Let $N_x \wedge N_y \in N_A$. Then by Theorem 2.6 we have $N_{x*y} \in N_A$, and so $x * y \in A$ Since A is a prime ideal, it follows that $x \in A$ or $y \in A$, and hence $N_x \in N_A$ of $N_y \in N_A$. Hence N_A is a prime ideal of $N(X)$.

Conversely suppose that N_A is a prime ideal of $N(X)$. Using corollany 2.3, A is an ideal of X If $x \wedge y \in A$, then $N_{x\wedge y} \in N_A$. By Theorem 2.6, we have $N_x \wedge N_y \in N_A$. Since N_A is a prime ideal of $N(X)$. we have $N_x \in N_A$ or $N_y \in N_A$, which imply that $x \in A$ or $y \in A$. Thus A is a prime ideal of X. The proof is complete.

Definition 2.9 ([7]) A nonempty subset F of a BCK-algebra X is called a filter of X if it satisfies:

(i) $x \in F$ and $y \leq y$ imply $y \in F$.

(ii) $x \in F$ and $y \in F$ imply $x \wedge y \in F$.

Theorem 2.10 Let X be a PI^n-BCK-algebra and F a subset of X. If N_F is a filter of $N(X)$, then F is a filter of X.

Proof. Assume that N_F is a filter of $N(X)$. Let $x \in F$. If $x \leq y$, then $(N_x * N_y)(t) = N_x(t) * N_y(t) = (x * t^n) * (y * t^n) = (x * y) * t^n = 0 * t^n = N_0(t)$ for all $t \in X$. Hence $N_x \leq N_y$ since N_F is a filter of $N(X)$. It follows that $N_y \in N_F$ and so $y \in F$. Let $x, y \in F$. Then $N_x, N_y \in N_F$. Using Theorem 2.6, we have $N_x \wedge N_y \in N_F$. Hence $x \wedge y \in F$, and so F is a filter of X. The proof is complete.

Definition 2.11 ([8]) Let X be a BCK-algebra and let a, b be any fixed elements of X. we suppose that there is a greatest element X satisfying $x * a \leq b$. Then the BCK-algebra X is said to be with condition (S). In this case X is denoted by $a \circ b$.

Lemma 2.12 Let X be a BCK-algebra with condition (S). Then X is PI^n if and only if $(x \circ y) * z^n = (x * z^n) \circ (y * z^n)$ for all $x, y, z \in X$.

Let X be a BCK-algebra with condition (S), then for any $N_x, N_y \in N(X)$ we define:

$$(N_x \circ N_y)(t) = N_x(t) \circ N_y(t)$$

Theorem 2.13 If X is a PI^n-BCK-algebra with condition (S), then we have $N_x \circ N_y = N_{x\circ y}$.

Proof. For any $t \in X$. we have $(N_x \circ N_y)(t) = N_x(t) \circ N_y(t) = (x * t^n) \circ (y * t^n) = (x \circ y) * t^n = N_{x\circ y}(t)$. The proof is complete.

Theorem 2.14 If X is a PI^n-BCK-algebra with condition (S), then $N(X)$ is a PI^n-BCK-algebra with condition (S).

Proof. It is easy to see that $N(x)$ is a PI^n-BCK-algebra. For every $N_a, N_b \in N(x)$, $(N_a \circ N_b) * N_a = N_{a\circ b} * N_a = N_{(a\circ b)*a} \leq N_b$ because $(a \circ b) * a \leq b$. Let $N_x * N_a \leq N_b$. Then by Theorem 2.1, we have $N_{(a\circ b)*a} = N_0$. Thus, we have $(x * a) * b = 0$ Since X is with condition (S) we obtain $x \leq a \circ b$. Hence by Theorem 2.13, we have $N_x \leq N_{a\circ b} = N_a \circ N_b$. Therefore $N(X)$ is also with condition (S). The proof is complete.

3 On homomorphism of n-maps.

Definition 3.1 Let X and X' be BCK-algebras. A mapping $f: X \to X'$ is called a homomorphism if for any x and y in $X, f(x*y) = f(x)*' f(y)$, where $*$ and $*'$ are operators in X and X', respectively. Now if we consider a natural map $f: X \to N(X)$ as $f(x) = N_x$, then we have:

Theorem 3.2 If X is a PI^n-BCK-algebra, then X is isomorphic to $N(X)$.

Proof. By Theorem 2.1, $N(X)$ is a PI^n-BCK-algebra. We have to show that a map $f: X \to N(X)$ as $f(x) = N_x$ is a bijective homomorphism. First of all we prove that f is an injective map. Suppose that $f(x) = f(y)$, that is, $N_x = N_y$. For every t in X. $N_x(t) = N_y(t)$ and hence $x * t^n = y * t^n$. If we set $t = y$, we have $x = y$. This implies that f is an injective map. Clearly f is a surjective map.

Finally, we show that f is a homomorphism. Since $f(x*y) = N_{x*y}, f(x)*' f(y) = N_x *' N_y$ and $N_x *' N_y = N_{x*y}$ because X is PI^n, we have $f(x*y) = f(x*y) = f(x)*' f(y)$, that is f is a homomorphism. The proof is complete.

Theorem 3.3 N_x is a homomorphism if and only if $x = 0$.

Proof. First we suppose that N_x is a homomorphism. Then $N_x(0) = N_x(0*0) = N_x(0)*N_x(0) = (x*0^n)*(x*0^n) = 0$ and hence $x = x*0^n = 0$. This implies that $x = 0$. Conversely if $x = 0$, then clearly that we obtain N_0 is a homomorphism.

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