

RANKS OF DIRECT PRODUCTS OF C^* -ALGEBRAS

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ABSTRACT. We show that the stable rank, connected stable rank, general stable rank and real rank of direct products of C^* -algebras are equal to supremums of these ranks of their direct factors.

Introduction. The theory of the stable rank, connected stable rank and general stable rank for C^* -algebras was initiated by M.A. Rieffel [Rf1] as a noncommutative analogue to the dimension theory for spaces, and for study of stability of C^* -algebras such as the cancellation of projections. On the other hand, the real rank for C^* -algebras was introduced by Brown and Pedersen [BP] as a real version of the stable rank. These ranks are ones of the most important concepts in some recent topics of C^* -algebras such as classification of C^* -algebras by K-theory (cf. [Bl]).

In this paper it is shown that the ranks of direct products of C^* -algebras are equal to supremums of the ranks of their direct factors. This formula is one of the most fundamental formulas for the ranks so that it would be useful in the theory of the ranks, but it has remained open till now except the case of the restricted (or c_0 -) direct sum ([Rf1, Theorem 5.2]).

Notation. Let \mathfrak{A} be a C^* -algebra. Denote by $\text{sr}(\mathfrak{A})$, $\text{csr}(\mathfrak{A})$, $\text{gsr}(\mathfrak{A})$ and $\text{RR}(\mathfrak{A})$ the stable rank, connected stable rank, general stable rank and the real rank respectively ([Rf1], [BP]). By definition, $\text{sr}(\mathfrak{A}), \text{csr}(\mathfrak{A}), \text{gsr}(\mathfrak{A}) \in \{1, 2, \dots, \infty\}$ and $\text{RR}(\mathfrak{A}) \in \{0, 1, 2, \dots, \infty\}$. If \mathfrak{A} is nonunital, we define its ranks by those of its unitization \mathfrak{A}^+ .

(F): For an exact sequence of C^* -algebras: $0 \rightarrow \mathfrak{J} \rightarrow \mathfrak{A} \rightarrow \mathfrak{A}/\mathfrak{J} \rightarrow 0$,

$$\text{sr}(\mathfrak{J}) \vee \text{sr}(\mathfrak{A}/\mathfrak{J}) \leq \text{sr}(\mathfrak{A}), \quad \text{and} \quad \text{RR}(\mathfrak{J}) \vee \text{RR}(\mathfrak{A}/\mathfrak{J}) \leq \text{RR}(\mathfrak{A}),$$

where \vee is the maximum ([Rf1, Theorem 4.3 and 4.4], [Eh2, Theorem 1.4]).

The direct product $\prod_{j \in J} \mathfrak{A}_j$ of C^* -algebras $\{\mathfrak{A}_j\}_{j \in J}$ indexed by a set J consists of all elements $a = (a_j)_{j \in J}$ with $a_j \in \mathfrak{A}_j$ and the norm $\|a\| = \sup_{j \in J} \|a_j\|$ finite. Then the following main theorem is obtained:

Theorem 1. *Let $\{\mathfrak{A}_\lambda\}_{\lambda \in \Lambda}$ be a family of C^* -algebras. Then*

$$\text{sr}(\prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda) = \sup_{\lambda \in \Lambda} \text{sr}(\mathfrak{A}_\lambda), \quad \text{and} \quad \text{RR}(\prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda) = \sup_{\lambda \in \Lambda} \text{RR}(\mathfrak{A}_\lambda).$$

Moreover, it is obtained that

$$\text{csr}(\prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda) = \sup_{\lambda \in \Lambda} \text{csr}(\mathfrak{A}_\lambda), \quad \text{and} \quad \text{gsr}(\prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda) = \sup_{\lambda \in \Lambda} \text{gsr}(\mathfrak{A}_\lambda).$$

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Proof. If \mathfrak{A}_λ is nonunital for some $\lambda \in \Lambda$, we consider \mathfrak{A}_λ^+ . Then $\prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda$ is a closed ideal of $\prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda^+$. Hence, it follows by (F) that

$$\text{sr}(\prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda) \leq \text{sr}(\prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda^+), \quad \text{RR}(\prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda) \leq \text{RR}(\prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda^+).$$

By (F), $\text{sr}(\prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda) \geq \sup_{\lambda \in \Lambda} \text{sr}(\mathfrak{A}_\lambda)$ and $\text{RR}(\prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda) \geq \sup_{\lambda \in \Lambda} \text{RR}(\mathfrak{A}_\lambda)$.

Now suppose that

$$\sup_{\lambda \in \Lambda} \text{sr}(\mathfrak{A}_\lambda) = \sup_{\lambda \in \Lambda} \text{sr}(\mathfrak{A}_\lambda^+) \equiv M < \infty.$$

For any $\varepsilon > 0$ and any $(a_{\lambda,j}) \in \prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda^+$ ($1 \leq j \leq M$), there exists $(b_{\lambda,j}) \in \prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda^+$ such that $\|a_{\lambda,j} - b_{\lambda,j}\| < \varepsilon_{\lambda,j} < \varepsilon$ and $c_\lambda \equiv \sum_{j=1}^M b_{\lambda,j}^* b_{\lambda,j}$ is invertible in \mathfrak{A}_λ^+ . For a large constant $L > 0$, we may assume that $c_\lambda \geq \varepsilon/L > 0$ for any $\lambda \in \Lambda$ if necessary, by taking $\varepsilon_{\lambda,j}$ small enough, and replacing $b_{\lambda,j}$ with its suitable perturbation and $\varepsilon_{\lambda,j}$ with $\varepsilon'_{\lambda,j} < \varepsilon$, when $c_\lambda \geq \delta_\lambda > 0$ and $\delta_\lambda < \varepsilon/L$ for some $\lambda \in \Lambda$.

In fact, for a unital C^* -algebra \mathcal{A} , we have a continuous map Φ from $L_n(\mathcal{A}) = \{(a_j) \in \mathcal{A}^n \mid \sum_{j=1}^n a_j^* a_j \in \mathcal{A}^{-1}\}$ to the positive part \mathcal{A}_+ of \mathcal{A} , defined by $(a_j) \mapsto \sum_{j=1}^n a_j^* a_j$. Let $\mathcal{S} = \{b \in \mathcal{A}_+ \mid \|\sum_{j=1}^n a_j^* a_j - b\| < \eta, \text{ and } b > \sum_{j=1}^n a_j^* a_j + \eta' 1\}$ for some $\eta, \eta' > 0$. Then \mathcal{S} is open in \mathcal{A}_+ since for $b' \in \mathcal{A}_+$ with $\|b - b'\|$ small, we can make the distance of their spectrums small. Taking η, η' suitably, the distance between $\sum_{j=1}^n a_j^* a_j$ and \mathcal{S} can be small enough. Then there exists a small open neighborhood of (a_j) such that its image under Φ has the nonzero intersection with \mathcal{S} .

Note that

$$\sum_{j=1}^M (b_{\lambda,j})^* (b_{\lambda,j}) = \left(\sum_{j=1}^M b_{\lambda,j}^* b_{\lambda,j} \right) = (c_\lambda) \in \prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda^+.$$

Since $\sup_{\lambda \in \Lambda} \|c_\lambda^{-1}\| \leq (\varepsilon/L)^{-1}$ from the above argument, we have $(c_\lambda^{-1}) \in \prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda^+$. Therefore, (c_λ) is invertible in $\prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda^+$.

The proof for the real rank case is the same as above.

Next, note that

$$\text{csr}(\prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda) = \text{csr}(\mathfrak{A}_\mu \oplus (\prod_{\lambda \in \Lambda, \lambda \neq \mu} \mathfrak{A}_\lambda)) = \text{csr}(\mathfrak{A}_\mu) \vee \text{csr}(\prod_{\lambda \in \Lambda, \lambda \neq \mu} \mathfrak{A}_\lambda).$$

Hence, $\text{csr}(\prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda) \geq \sup_{\lambda \in \Lambda} \text{csr}(\mathfrak{A}_\lambda)$.

Now suppose that every \mathfrak{A}_λ for $\lambda \in \Lambda$ is unital, and that

$$\sup_{\lambda \in \Lambda} \text{csr}(\mathfrak{A}_\lambda) \equiv N < \infty.$$

Then, for any $(a_{\lambda,j}) \in \prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda$ ($1 \leq j \leq N$) such that

$$\sum_{j=1}^N (a_{\lambda,j})^* (a_{\lambda,j}) = \left(\sum_{j=1}^N a_{\lambda,j}^* a_{\lambda,j} \right) = I = (I_\lambda)$$

where I_λ is the unit of \mathfrak{A}_λ , there exists a unitary matrix $U = (U_{i,j})_{i,j=1}^N$ over $\prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda$ such that $U_{i,j} = (U_{i,j,\lambda})_{\lambda \in \Lambda}$ and $V_\lambda = (U_{i,j,\lambda})_{i,j=1}^N$ is a unitary matrix over \mathfrak{A}_λ in the connected component of the unit $\bigoplus_{k=1}^N I_\lambda$ of $GL_N(\mathfrak{A}_\lambda)$ such that $(a_{\lambda,j})_{j=1}^N$ is mapped to $(I_\lambda, 0_\lambda, \dots, 0_\lambda)$ by V_λ , where 0_λ is the zero of \mathfrak{A}_λ (cf. [Rf2, Proposition 5.3]). This shows $\text{csr}(\prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda) \leq N$.

When \mathfrak{A}_λ for some $\lambda \in \Lambda$ is nonunital, we consider the split embedding from $\prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda$ to $\prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda^+$. Then it is obtained that (cf. [Eh1, Theorem 2.11])

$$\text{csr}(\prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda) = \text{csr}(\prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda^+).$$

The proof for the general stable rank is the same as that for the connected stable rank given above (cf. [Rf2, Proposition 5.2]). \square

Let $\bigoplus_{\lambda \in \Lambda} \mathfrak{A}_\lambda$ be the (restricted) direct sum of C^* -algebras $\{\mathfrak{A}_\lambda\}_{\lambda \in \Lambda}$ (cf. [Pd, 1.2.4]). Then it follows that

Corollary 2. *Let $\{\mathfrak{A}_\lambda\}_{\lambda \in \Lambda}$ be a family of C^* -algebras. Then*

$$\text{sr}(\oplus_{\lambda \in \Lambda} \mathfrak{A}_\lambda) = \sup_{\lambda \in \Lambda} \text{sr}(\mathfrak{A}_\lambda), \text{ and } \text{RR}(\oplus_{\lambda \in \Lambda} \mathfrak{A}_\lambda) = \sup_{\lambda \in \Lambda} \text{RR}(\mathfrak{A}_\lambda).$$

Moreover, for any closed ideal \mathfrak{J} of $\prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda$ containing \mathfrak{A}_λ for any $\lambda \in \Lambda$,

$$\text{sr}(\mathfrak{J}) = \sup_{\lambda \in \Lambda} \text{sr}(\mathfrak{A}_\lambda), \text{ and } \text{RR}(\mathfrak{J}) = \sup_{\lambda \in \Lambda} \text{RR}(\mathfrak{A}_\lambda).$$

Proof. Note that $\oplus_{\lambda \in \Lambda} \mathfrak{A}_\lambda$ is a closed ideal of $\prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda$. \square

Corollary 3. *Let $\{\mathfrak{A}_\lambda\}_{\lambda \in \Lambda}$ be a family of C^* -algebras. Then*

$$\text{csr}(\oplus_{\lambda \in \Lambda} \mathfrak{A}_\lambda) = \sup_{\lambda \in \Lambda} \text{csr}(\mathfrak{A}_\lambda), \text{ and } \text{gsr}(\oplus_{\lambda \in \Lambda} \mathfrak{A}_\lambda) = \sup_{\lambda \in \Lambda} \text{gsr}(\mathfrak{A}_\lambda).$$

For any closed ideal \mathfrak{J} of $\prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda$ containing \mathfrak{A}_λ for any $\lambda \in \Lambda$, we have

$$\begin{cases} \sup_{\lambda \in \Lambda} \text{csr}(\mathfrak{A}_\lambda) \leq \text{csr}(\mathfrak{J}) \leq 1 + \sup_{\lambda \in \Lambda} \text{sr}(\mathfrak{A}_\lambda), \\ \sup_{\lambda \in \Lambda} \text{gsr}(\mathfrak{A}_\lambda) \leq \text{gsr}(\mathfrak{J}) \leq 1 + \sup_{\lambda \in \Lambda} \text{sr}(\mathfrak{A}_\lambda). \end{cases}$$

Proof. The first follows from the same inductive process as [Rf1, Theorem 5.1 and 5.2]. We have $\text{gsr}(\mathfrak{A}) \leq \text{csr}(\mathfrak{A}) \leq \text{sr}(\mathfrak{A}) + 1$ for any C^* -algebra \mathfrak{A} by [Rf1, Corollary 4.10 and Section 10]. \square

Remark. Note that $\text{sr}(C(\mathbb{T})) = 1$ while $\text{csr}(C(\mathbb{T})) = 2$ and $\text{gsr}(C(\mathbb{T})) = 1$ (cf.[Sh, p.381], [Rf2, p.247]), where $C(\mathbb{T})$ is the C^* -algebra of continuous functions on the torus.

Corollary 4. *Let $\mathfrak{A} = \prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda$, $\mathfrak{B} = \prod_{\mu \in M} \mathfrak{B}_\mu$. Suppose that for any $\lambda \in \Lambda, \mu \in M$, we have $\text{rk}(\mathfrak{A}_\lambda \otimes \mathfrak{B}_\mu) \leq \text{rk}(\mathfrak{A}_\lambda) + \text{rk}(\mathfrak{B}_\mu)$, where rk means either sr , csr , gsr or RR , and \otimes means the minimal C^* -tensor product. Then*

$$\text{rk}(\mathfrak{A} \otimes \mathfrak{B}) \leq \text{rk}(\mathfrak{A}) + \text{rk}(\mathfrak{B}).$$

Proof. Note that

$$\mathfrak{A} \otimes \mathfrak{B} \cong \prod_{\lambda \in \Lambda} (\mathfrak{A}_\lambda \otimes \mathfrak{B}) \cong \prod_{\lambda \in \Lambda} (\prod_{\mu \in M} \mathfrak{A}_\lambda \otimes \mathfrak{B}_\mu).$$

By Theorem 1 and the assumption, we have

$$\begin{aligned} \text{rk}(\mathfrak{A} \otimes \mathfrak{B}) &= \sup_{\lambda \in \Lambda} \text{rk}(\mathfrak{A}_\lambda \otimes \mathfrak{B}) = \sup_{\lambda \in \Lambda} \sup_{\mu \in M} \text{rk}(\mathfrak{A}_\lambda \otimes \mathfrak{B}_\mu) \\ &\leq \sup_{\lambda \in \Lambda} \sup_{\mu \in M} (\text{rk}(\mathfrak{A}_\lambda) + \text{rk}(\mathfrak{B}_\mu)) = \text{rk}(\mathfrak{A}) + \text{rk}(\mathfrak{B}). \quad \square \end{aligned}$$

Remark. The assumption of the product formula for the ranks is crucial. However, this would be affirmative for some general C^* -algebras (cf.[KO], [Ns1,2], [Sd1-5] and [ST1,2]).

Remark. The equality $(\prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda) \otimes (\prod_{\mu \in M} \mathfrak{B}_\mu) = \prod_{(\lambda, \mu) \in \Lambda \times M} \mathfrak{A}_\lambda \otimes \mathfrak{B}_\mu$ is not true in general. For example, $l^\infty(\mathbb{N}) \otimes l^\infty(\mathbb{N}) \subsetneq l^\infty(\mathbb{N}^2)$ (cf.[APT, Theorem 3.8]).

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