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SEQUENTIAL $\sigma(E(X), E'(X^*))$-COMPACTNESS AND COMPLETENESS IN KÖTHE-BOCHNER SPACES $E(X)$

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Abstract. Let E be a Banach function space over a complete finite measure space (Ω, Σ, μ), E'-the Köthe dual of E and let X be a Banach space, X^*-the topological dual of X. We give a criterion for relative sequential $\sigma(E(X), E'(X^*))$-compactness in Köthe-Bochner spaces $E(X)$. We characterize Banach spaces X having the Radon-Nikodym Property in terms of relatively sequentially $\sigma(E(X), E'(X^*))$-compact subsets of $E(X)$. Moreover, we show that $E(X)$ is sequentially $\sigma(E(X), E'(X^*))$-complete if and only if E is sequentially $\sigma(E, E')$-complete, X has the Radon-Nikodym Property (with respect to μ) and X is sequentially weakly complete. We generalize F. Bomball, J. Batt and W. Hiermayer's results concerning weak compactness and sequential weak completeness in Lebesgue-Bochner spaces $L^p(X) (1 \le p < \infty)$ and Orlicz-Bochner spaces $L^\varphi(X)$.

1. Introduction and preliminaries

Given a dual pair $\langle L, M\rangle$ a subset A of L is said to be *conditionally* $\sigma(L, M)$*-compact* (resp. *relatively sequentially* $\sigma(L, M)$*-compact*) whenever each sequence in A contains a $\sigma(L, M)$-Cauchy subsequence (resp. each sequence in A contains a subsequence which is $\sigma(L, M)$-convergent to some element of L).

The problem of characterizing of relatively sequentially $\sigma(L^p(X), L^q(X^*))$-compact subsets of Lebesque-Bochner spaces $L^p(X)$ for $1 \le p < \infty$ and q conjugate to p over a finite measure space (Ω, Σ, μ) has been considered in [B$_1$], [BD], [BH], [D].

In particular, F. Bombal [B$_1$] showed that if $1 < p < \infty$ and the Banach space X has the Radon-Nikodym Property (RNP) with respect to μ, then a subset H of $L^p(X)$ is relatively sequentially $\sigma(L^p(X), L^q(X^*))$-compact iff the following conditions are satisfied:

(i) H is norm bounded,

(ii) the set $\left\{\int_A f(\omega) d\mu : f \in H\right\}$ is relatively weakly compact in X for every $A \in \Sigma$,

(iii) $\lim_{\mu(A)\to 0} \sup\left\{\int_A \langle f(\omega), g(\omega)\rangle \, d\mu : f \in H\right\} = 0$ for every $g \in L^q(X^*)$.

Moreover, in [B$_1$] it is shown that the condition on X to have the RNP is also necessary in order that relatively sequentially $\sigma(L^p(X), L^q(X^*))$-compact subset of $L^p(X)$ $(1 < p < \infty)$ be exactly those satisfying the above conditions (i)–(iii) for each finite measures. As a consequence, a characterization of sequential $\sigma(L^p(X), L^q(X^*))$-completeness of $L^p(X)$ is obtained.

Next, in [B$_2$] the above results are extended to the class of Orlicz-Bochner spaces $L^\varphi(X)$ with topology $\sigma(L^\varphi(X), L^\varphi(X^*))$, where φ is a Young function satisfying the condition: $\varphi(t)/t \to \infty$ as $t \to \infty$ (so the space $L^1(X)$ is excluded) and φ^* denotes the complementary Young function.

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J. Batt and W. Hiermeyer [BH] characterized the relatively $\sigma(L^p(X), L^q(X^*))$-compact subsets of $L^p(X)$ $(1 \le p < \infty)$ for a general Banach space X. Moreover, G. Schluchtermann and R. Wheeler [SW] found a characterization of sequential $\sigma(L^1(X), L^\infty(X^*))$-completeness of $L^1(X)$.

The purpose of this paper is to extend the above results to the class of Köthe-Bochner spaces $E(X)$ provided with the weak topology $\sigma(E(X), E'(X^*))$, where E is a Banach function space (not necessarily order continuous), E' denotes the Köthe dual of E and X is a real Banach space.

Now we establish notation and terminology (see [KA], [AB]).

From now on we shall assume that (Ω, Σ, μ) is a complete finite measure space, and let $(E, \|\cdot\|_E)$ be a Banach function spaces over (Ω, Σ, μ) such that $L^\infty \subset E \subset L^1$, where the inclusion maps are continuous. For a subset A of Ω let χ_A stand for its characteristic function. Let $\mathbb{N}$ denote the set of all natural numbers.

Let E' stand for the Köthe dual of E. Then the associated norm $\|\cdot\|_{E'}$ on E' can be defined by $\|v\|_{E'} = \sup\{|\int_\Omega u(\omega)\,v(\omega)\,d\mu| : u \in E, \|u\|_E \le 1\}$. Recall that E is said to be *perfect* if $E = E''$.

The following characterization of perfect Banach function spaces will be needed (see [KA, Theorem 6.1.7, Corollary 10.3.2], [N$_3$]).

Theorem 1.1 *For a Banach function space $(E, \|\cdot\|_E)$ the following statements are equivalent:*

(i) E is perfect.

(ii) The norm $\|\cdot\|_E$ satisfies both the σ-Fatou property and the σ-Levy property.

(iii) E is sequentially $\sigma(E, E')$-complete.

It is known that a the norm bounded subset Z of E is conditionally $\sigma(E, E')$-compact iff for each $v \in E'$ the subset $\{uv : u \in Z\}$ of L^1 is uniformly integrable (see [N$_1$, Proposition 2.1]).

Let $(X, \|\cdot\|_X)$ be a real Banach space, and let X^* stand for the Banach dual of X. Let S_X and S_{X^*} denote the unit spheres in X and X^* resp.

By $L^0(X)$ we will denote the set of equivalence classes of strongly Σ-measurable function $f : \Omega \to X$. For $f \in L^0(X)$ let $\widetilde{f}(\omega) = \|f(\omega)\|_X$ for $\omega \in \Omega$. The space $E(X) = \{ f \in L^0(X) : \widetilde{f} \in E \}$ equipped with the norm $\|f\|_{E(X)} := \|\widetilde{f}\|_E$ is called a Köthe-Bochner space.

We shall need the following two lemmas:

Lemma 1.2 *Let $(E, \|\cdot\|_E)$ be a Banach function space with the σ-Fatou property (i.e., $u_n \uparrow u$ in E implies $\|u_n\|_E \uparrow \|u\|_E$). The for $f \in E(X)$ we have*

$$\|f\|_{E(X)} = \sup\{|\textstyle\int_\Omega \langle f(\omega), g(\omega)\rangle\,d\mu| : g \in L^\infty(X^*), \|g\|_{E'(X^*)} \le 1\}.$$

Proof. In view of [Bu, Theorem 1.1 (4)] we have

$$\|f\|_{E(X)} = \sup\{|\textstyle\int_\Omega \big\langle f(\omega), g(\omega)\big\rangle\,d\mu| : g \in E'(X^*), \|g\|_{E'(X^*)} \le 1\}.$$

Let $\varepsilon > 0$ be given. Hence there exists $g \in E'(X^*)$ with $\|g\|_{E'(X^*)} \le 1$ such that $\|f\|_{E(X)} \le |\int_\Omega \big\langle f(\omega), g(\omega)\big\rangle\,d\mu| + \varepsilon$. For $n \in \mathbb{N}$ let us put

$$g_n(\omega) = \begin{cases} g(\omega) & \text{if } \|g(\omega)\|_{X^*} \le n, \\ 0 & \text{elsewhere.} \end{cases}$$

Then $g_n \in L^\infty(X^*)$ and $\|g_n(\omega) - g(\omega)\|_{X^*} \longrightarrow 0$ for $\omega \in \Omega$. Moreover, $\|g_n(\omega)\|_{X^*} \leq \|g(\omega)\|_{X^*}$ for $\omega \in \Omega$, so $g_n \in E'(X^*)$ and $\|g_n\|_{E'(X^*)} \leq \|g\|_{E'(X^*)} \leq 1$. Since $\widetilde{f}\widetilde{g} \in L^1$, by the Lebesgue dominated convergence theorem we get

$$|\int_\Omega \langle f(\omega), g(\omega)\rangle \, d\mu - \int_\Omega \langle f(\omega), g_n(\omega)\rangle \, d\mu| \leq \int_\Omega \|f(\omega)\|_X \, \|g(\omega) - g_n(\omega)\|_{X^*} \, d\mu \ \to 0$$

so there exists $n_o \in \mathbb{N}$ such that

$$|\int_\Omega \langle f(\omega), g(\omega)\rangle \, d\mu| \ \leq \ |\int_\Omega \langle f(\omega), g_{n_o}(\omega)\rangle \, d\mu| \ + \ \tfrac{\varepsilon}{2}.$$

It follows that

$$\|f\|_{E(X)} = \sup\{|\int_\Omega \langle f(\omega), g(\omega)\rangle \, d\mu| : \ g \in L^\infty(X^*), \ \|g\|_{E'(X^*)} \leq 1\},$$

and the proof is complete. ■

Lemma 1.3 *Let $(E, \|\cdot\|_E)$ be a perfect Banach function space. Then for $f \in L^0(X)$ the following statements are equivalent:*

(i) $f \in E(X)$.

(ii) $\sup\{|\int_\Omega \langle f(\omega), g(\omega)\rangle \, d\mu| : \ g \in L^\infty(X^*), \ \|g\|_{E'(X^*)} \leq 1\} < \infty$.

Proof. (i) $\Longrightarrow$ (ii) It follows from Lemma 1.2.

(ii) $\Longrightarrow$ (i) Since $\widetilde{f} \in L^0$ by [KA, Corollary 4.3.1] there exists a sequence (A_n) in Σ such that $A_n \uparrow \Omega$ with $\widetilde{f}\chi_{A_n} \in E$ for $n \in \mathbb{N}$ and $\widetilde{f}\chi_{A_n} \uparrow \widetilde{f}$. Let $f_n = f\chi_{A_n}$ for $n \in \mathbb{N}$. Then $\widetilde{f_n} = \widetilde{f}\chi_{A_n}$ and $f_n \in E(X)$ for $n \in \mathbb{N}$. In view of Lemma 1.1 and (ii) for all $n \in \mathbb{N}$ we have

$$\begin{aligned} \|f_n\|_{E(X)} &= \sup\{|\int_\Omega \langle f_n(\omega), g(\omega)\rangle \, d\mu| : \ g \in L^\infty(X^*), \ \|g\|_{E'(X^*)} \leq 1\} \\ &= \sup\{|\int_\Omega \langle f(\omega)\, \chi_{A_n}(\omega), g(\omega)\rangle \, d\mu| : \ g \in L^\infty(X^*), \ \|g\|_{E'(X^*)} \leq 1\} \qquad (1) \\ &\leq \sup\{|\int_\Omega \langle f(\omega), g(\omega)\rangle \, d\mu| : \ g \in L^\infty(X^*), \ \|g\|_{E'(X^*)} \leq 1\} < \infty. \end{aligned}$$

To show that $f \in E(X)$, let $0 \leq v \in E'$. Then $\widetilde{f_n}v \uparrow \widetilde{f}v$. Hence in view of the Fatou lemma, the Hölder's inequality and (1) we get

$$\int_\Omega \widetilde{f}(\omega)v(\omega)\, d\mu \ \leq \ \sup_n \int_\Omega \widetilde{f_n}(\omega)v(\omega)\, d\mu \ \leq \ \sup_n \|f_n\|_{E(X)} \|v\|_{E'} < \infty.$$

It follows that $\widetilde{f} \in (E')' = E$, i.e., $f \in E(X)$, as desired. ■

2. Sequential $\sigma(E(X), E'(X^*))$-compactness in $E(X)$

We start by recalling some definitions. For a finite Σ-partition π of Ω let E_π stand for the conditional expectation operator for π. Following [BH] for all increasing sequences (π_n) of finite Σ-partition of Ω let us define

$$\begin{aligned}\Lambda_1(c_o,(\pi_n)) := \{ T \in \mathcal{L}(L^1(X),c_o) : \exists (g_n) \subset L^\infty(X^*),\ \|g_n\|_\infty \le 1,\\ g_n = (E_{\pi_{n+1}} - E_{\pi_n}) g_n\ ,\ n\in\mathbb{N},\\ \forall f \in L^1(X),\ T(f) = (\textstyle\int_\Omega \langle f(\omega), g_n(\omega)\rangle\, d\mu)\ \},\end{aligned}$$

and let $\Lambda_1(c_o) := \bigcup \Lambda_1(c_o,(\pi_n))$.

The following criterion for relative $\sigma(L^1(X),L^\infty(X^*))$-compactness in $L^1(X)$ will be of importance.

Theorem 2.1 *[BH, Theorem 2.1]. For a subset H of $L^1(X)$ the following statements are equivalent:*

(i) H is relatively $\sigma(L^1(X),L^\infty(X^*))$-compact.

(ii) H is relatively sequentially $\sigma(L^1(X),L^\infty(X^*))$-compact.

(iii) H is relatively countably $\sigma(L^1(X),L^\infty(X^*))$-compact.

(iv) a) $\sup_{f\in H} \|f\|_{L^1(X)} < \infty$,

b) the set $\{\widetilde{f} : f \in H\}$ in L^1 is uniformly integrable,

c) for each $A\in\Sigma$ the set $\{\int_A f(\omega)d\mu :\ f\in H\}$ is relatively weakly compact in X,

d) $T(H)$ is relatively weakly compact in c_o for all $T\in\Lambda_1(c_o)$.

Now we are ready to state our main result.

Theorem 2.2 *Let (Ω,Σ,μ) be a finite measure space and let $(E,\|\cdot\|_E)$ be a perfect Banach function space. Then for a norm bounded subset H of $E(X)$ the following statements are equivalent:*

(i) H is relatively sequentially $\sigma(E(X),E'(X^*))$-compact.

(ii) (a) the set $\widetilde{H} = \{\widetilde{f} : f\in H\}$ is relatively sequentially $\sigma(E,E')$-compact,

(b) for each $A\in\Sigma$ the set $\{\int_A f(\omega)d\mu :\ f\in H\}$ is relatively weakly compact in X,

(c) $T(H)$ is relatively weakly compact in c_o for all $T\in\Lambda_1(c_o)$.

(iii) (a) the set $\widetilde{H} = \{\widetilde{f} : f\in H\}$ is relatively sequentially $\sigma(E,E')$-compact,

(b) for each $A\in\Sigma$ the set $\{\int_A f(\omega)d\mu :\ f\in H\}$ is relatively weakly compact in X,

(c) any measure $m:\Sigma\to X$ of the form $m(A) = weak-\lim\int_A f_n(\omega)d\mu$ for every $A\in\Sigma$ and some sequence (f_n) in H has the RNP (with respect to μ).

Proof. (i) $\Longrightarrow$ (ii) Assume that H is relatively sequentially $\sigma(E(X),E'(X^*))$-compact. Since H is conditionally compact, by [N_2, Theorem 2.3], the set $\widetilde{H} = \{\widetilde{f} : f\in H\}$ is conditionally $\sigma(E,E')$-compact. In view of Theorem 1.1 $\widetilde{H}$ is relatively sequentially $\sigma(E,E')$-compact. Thus (a) holds.

Since $H\subset E(X)\subset L^1(X)$ and $\sigma(L^1(X),L^\infty(X^*))|_{E(X)} \subset \sigma(E(X),E'(X^*))$ the set H is relatively sequentially $\sigma(L^1(X),L^\infty(X^*))$-compact, so by Theorem 2.1 conditions (b) and (c) are satisfied.

(ii) $\Longrightarrow$ (i) Assume that (ii) holds and let (f_n) be a sequence in H. Since $H\subset E(X)\subset L^1(X)$ and $L^\infty(X^*)\subset E(X^*)$, by Theorem 2.1 the subset H of $L^1(X)$ is relatively sequentially $\sigma(L^1(X), L^\infty(X^*))$-compact. Thus there exist a subsequence (f_{k_n}) of (f_n) and $f_0 \in L^1(X)$ such that $f_{k_n} \to f_0$ for $\sigma(L^1(X), L^\infty(X^*))$. We shall show that $f_0 \in E(X)$ and $f_{k_n} \to f_0$ for $\sigma(E(X), E'(X^*))$. To show that $f_0 \in E(X)$ in view of Lemma 1.3 it is enough to show that

$$\sup \{ | \int_\Omega \langle f_0(\omega), g(\omega) \rangle \, d\mu | : \ g \in L^\infty(X^*), \ \|g\|_{E'(X^*)} \le 1 \} < \infty.$$

Indeed, let $g \in L^\infty(X^*)$ and $\|g\|_{E'(X^*)} \le 1$. Since $f_{k_n} \to f_0$ for $\sigma(L^1(X), L^\infty(X^*))$ we can choose $n_o \in \mathbb{N}$ such that $| \int_\Omega \langle f_0(\omega) - f_{k_{n_o}}(\omega), g(\omega) \rangle \, d\mu | \le 1$.
Hence, since $\sup_{f \in H} \|f\|_{E(X)} = c < \infty$ for some $c > 0$, by the Hölder's inequality we get

$$\begin{aligned} | \int_\Omega \langle f_0(\omega), g(\omega) \rangle \, d\mu | &\le | \int_\Omega \langle f_0(\omega) - f_{k_{n_o}}(\omega), g(\omega) \rangle \, d\mu | + | \int_\Omega \langle f_{k_{n_o}}(\omega), g(\omega) \rangle \, d\mu | \\ &\le 1 + \int_\Omega \|f_{k_{n_o}}(\omega)\|_X \|g(\omega)\|_{X^*} \, d\mu \\ &\le 1 + \|f_{k_{n_o}}\|_{E(X)} \|g\|_{E'(X^*)} \quad \le \quad 1 + c. \end{aligned}$$

We shall now show that $f_{k_n} \to f_0$ for $\sigma(E(X), E'(X^*))$. Indeed, let $g \in E'(X^*)$ and for every $m \in \mathbb{N}$ let us put

$$g_m(\omega) = \begin{cases} g(\omega) & \text{if} \quad \|g(\omega)\|_{X^*} \le m, \\ 0 & \text{elsewhere.} \end{cases}$$

Let $\varepsilon > 0$ be given. Then by (a) there exists $\delta > 0$ such that for every $A \in \Sigma$ with $\mu(A) \le \delta$ we have

$$\sup_n \int_A \|f_{k_n}(\omega)\|_X \|g(\omega)\|_{X^*} \, d\mu \le \tfrac{\varepsilon}{8} \quad \text{and} \quad \int_A \|f_0(\omega)\|_X \|g(\omega)\|_{X^*} \, d\mu \le \tfrac{\varepsilon}{8}.$$

Let $r = \max(c, \int_\Omega \|f_{k_{n_o}}(\omega)\|_X \, d\mu)$. For $\eta = \min(\frac{\varepsilon}{8r}, 1)$ and $m \in \mathbb{N}$ let us put

$$B_m = \{ \omega \in \Omega : \|g(\omega) - g_m(\omega)\|_{X^*} \ge \eta \}.$$

It is seen that $B_m = \{ \omega \in \Omega : \|g(\omega)\|_{X^*} \ge m \}$ and $B_m \downarrow$, $\mu(\bigcap_{m=1}^\infty B_m) = 0$, so $\mu(B_m) \longrightarrow 0$. Choose $m_o \in \mathbb{N}$ such that $\mu(B_{m_o}) \le \delta$. Then we get

$$\sup_n \int_{B_{m_o}} \|f_{k_n}(\omega)\|_X \|g(\omega)\|_{X^*} \, d\mu \le \tfrac{\varepsilon}{8} \quad \text{and} \quad \int_{B_{m_o}} \|f_0(\omega)\|_X \|g(\omega)\|_{X^*} \, d\mu \le \tfrac{\varepsilon}{8}.$$

Hence for all $n \in \mathbb{N}$ we have

$$\begin{aligned} & | \int_\Omega \langle f_{k_n}(\omega) - f_0(\omega), g(\omega) - g_{m_o}(\omega) \rangle \, d\mu | \\ \le \ & \int_\Omega \|f_{k_n}(\omega) - f_0(\omega)\|_X \|g(\omega) - g_{m_o}(\omega)\|_{X^*} \, d\mu \\ \le \ & \int_{B_{m_o}} \|f_{k_n}(\omega)\|_X \|g(\omega) - g_{m_o}(\omega)\|_{X^*} \, d\mu \\ & + \int_{B_{m_o}} \|f_0(\omega)\|_X \|g(\omega) - g_{m_o}(\omega)\|_{X^*} \, d\mu \\ & + \int_{\Omega \setminus B_{m_o}} \|f_{k_n}(\omega)\|_X \|g(\omega) - g_{m_o}(\omega)\|_{X^*} \, d\mu \\ & + \int_{\Omega \setminus B_{m_o}} \|f_0(\omega)\|_X \|g(\omega) - g_{m_o}(\omega)\|_{X^*} \, d\mu \\ \le \ & \tfrac{\varepsilon}{8} + \tfrac{\varepsilon}{8} + \eta r + \eta r \quad \le \quad \tfrac{\varepsilon}{2}. \end{aligned}$$

Choose $n_1 \in \mathbb{N}$ such that for $n \geq n_1$

$$|\int_\Omega \langle f_{k_n}(\omega) - f_0(\omega), g_{m_o}(\omega)\rangle \, d\mu| \quad \leq \quad \tfrac{\varepsilon}{2}.$$

Hence for $n \geq n_1$ we have

$$\begin{aligned} &|\int_\Omega \langle f_{k_n}(\omega) - f_0(\omega), g(\omega)\rangle \, d\mu| \\ &\leq |\int_\Omega \langle f_{k_n}(\omega) - f_0(\omega), g(\omega) - g_{m_o}(\omega)\rangle \, d\mu| \; + \; |\int_\Omega \langle f_{k_n}(\omega) - f_0(\omega), g_{m_o}(\omega)\rangle \, d\mu| \\ &\leq \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} \quad = \quad \varepsilon, \end{aligned}$$

and the proof is complete.

(i) $\Longrightarrow$ (iii) Assume that H is relatively sequentially $\sigma(E(X), E'(X^*))$-compact. As in the proof of (i) $\Longrightarrow$ (ii) we obtain that conditions (a), (b) in (iii) hold. Assume that condition (c) does not hold, i.e., there exists a measure $m : \Sigma \to X$ with $|m|(\Omega) < \infty$, $|m| \ll \mu$ of the form

$$m(A) = \text{ weak} - \lim \int_A f_n(\omega) d\mu \quad \text{for} \quad A \in \Sigma,$$

for some (f_n) in H such that m does not have a Bochner derivative with respect to μ. Then there exist a subsequence (f_{k_n}) of (f_n) and $f_0 \in E(X)$ such that $f_{k_n} \to f_0$ for $\sigma(E(X), E'(X^*))$. Hence for each $A \in \Sigma$ and $x^* \in X^*$ we get $x^*(\int_A f_{k_n}(\omega) d\mu) \to x^*(\int_A f_0(\omega) d\mu)$. It easily follows that for each $x^* \in X^*$ $x^*(m(A)) = x^*(\int_A f_0(\omega) d\mu)$ for each $A \in \Sigma$, so $m(A) = \int_A f_0(\omega) d\mu$. This means that m has the RNP (with respect to μ), and we get a contradiction.

(iii) $\Longrightarrow$ (i) Assume that (iii) holds. Then in view of [N_2, Theorem 2.3] the set H is conditionally $\sigma(E(X), E'(X^*))$-compact. Let (f_n) be a sequence in H. Then there is a $\sigma(E(X), E'(X^*))$-Cauchy subsequence (f_{k_n}) of (f_n). Since $H \subset E(X) \subset L^1(X)$ and $L^\infty(X^*) \subset E'(X^*)$, (f_{k_n}) is also a $\sigma(L^1(X), L^\infty(X^*))$-Cauchy sequence. But for each $A \in \Sigma$, and $x^* \in X^*$, $x^*(\int_A f_{k_n}(\omega) d\mu) = \int_\Omega \langle f_{k_n}(\omega), \chi_A(\omega) x^*\rangle d\mu$ and $\chi_A x^* \in L^\infty(X^*)$, so for each $A \in \Sigma$, $(\int_A f_{k_n}(\omega) d\mu)$ is a weak Cauchy sequence in X. By the (b) there is a subsequence $(f_{l_{k_n}})$ of (f_{k_n}) such that the sequence $(\int_A f_{l_{k_n}}(\omega) d\mu)$ is weakly convergent in X, i.e., there is $m(A) \in X$ such that $m(A) = \text{ weak} - \lim \int_A f_{l_{k_n}}(\omega) d\mu$. But for each $x^* \in X^*$,

$$\begin{aligned} &|x^*(m(A) - \int_A f_{k_n}(\omega) d\mu)| \\ &\leq |x^*(m(A) - \int_A f_{l_{k_n}}(\omega) d\mu)| + |x^*(\int_A f_{l_{k_n}}(\omega) d\mu - \int_A f_{k_n}(\omega) d\mu)|, \end{aligned}$$

so

$$m(A) = \text{ weak} - \lim \int_A f_{k_n}(\omega) d\mu.$$

One can show that the set function $m : \Sigma \to X$ is a countably additive measure with $|m|(\Omega) < \infty$ and $|m| \ll \mu$ (see [BD, p. 178]). Hence by (c) there exists $f_0 \in L^1(X)$ such that $m(A) = \int_A f_0(\omega) d\mu$ for each $A \in \Sigma$. Arguing as in [BD, p. 178–179] we obtain that $f_{k_n} \longrightarrow f_0$ for $\sigma(L^1(X), L^\infty(X^*))$. Similarly as in the proof of implication (ii) $\Longrightarrow$ (i) we obtain that $f_0 \in E(X)$ and $f_{k_n} \longrightarrow f_0$ for $\sigma(E(X), E'(X^*))$. ■

As a consequence of Theorem 2.2 we obtain a characterization of Banach spaces having the RNP in terms of relatively sequentially $\sigma(E(X), E'(X^*))$-compact sets in $E(X)$.

Theorem 2.3 *Let (Ω, Σ, μ) be a finite measure space, and let $(E, \|\cdot\|_E)$ be a perfect Banach function space. Then following statements are equivalent:*

(i) X has the RNP (with respect to μ).

(ii) Every norm bounded subset H of $E(X)$ satisfying the following conditions:

(a) the set $\widetilde{H} = \{\widetilde{f} : f \in H\}$ *is relatively sequentially* $\sigma(E, E')$*-compact,*

(b) for each $A \in \Sigma$ *the set* $\{\int_A f(\omega) d\mu : f \in H\}$ *is relatively weakly compact in X,*

is relatively sequentially $\sigma(E(X), E'(X^*))$*-compact.*

Proof. (i) $\Longrightarrow$ (ii) If X has the RNP (with respect to μ), then the condition (c) in (iii) of Theorem 2.2 is superfluous.

(ii) $\Longrightarrow$ (i) Assume that X lacks the RNP with respect to μ. Then there is a μ-continuous vector measure $G : \Sigma \longrightarrow X$ of bounded variation that does not have a Bochner integrable Radon-Nikodym derivative with respect to μ. Moreover, by the discussion of [DU, Chap. III] we may assume that $\|G(A)\|_X \leq \mu(A)$ for all $A \in \Sigma$. Let S stand for the set of all finite Σ-partitions π of Ω partially ordered by refinement. For $\pi \in S$ let us set $f_\pi = \sum_{A\in\pi} \chi_A \frac{G(A)}{\mu(A)}$. Let $H = \{f_\pi : \pi \in S\}$. Arguing as in the proof of Theorem 3 of [DU, Chap. III, § 2] one can show that H obeys (b).

To show that (a) holds, it is enough to show that for each $v \in E'$ the set $\widetilde{f_\pi} v : \pi \in S\}$ in L^1 is uniformly integrable. Indeed, let $v \in E'$. Then for each $B \in \Sigma$ and $\pi \in S$ we get $\int_B \|f_\pi(\omega)\|_X |v(\omega)| \, d\mu \leq \int_B |v(\omega)| \, d\mu$.
Since $v \in E' \subset (L^\infty)' = L^1$, $\sup_{\pi\in S} \int_B \|f_\pi(\omega)\|_X |v(\omega)| \, d\mu \longrightarrow 0$ as $\mu(B) \longrightarrow 0$, as desired.

We shall now show that H is not relatively sequentially $\sigma(E(X), E'(X^*))$-compact subset of $E(X)$. Assume on the contrary that H is relatively sequentially $\sigma(E(X), E'(X^*))$-compact. Then H is relatively sequentially $\sigma(L^1(X), L^\infty(X^*))$-compact, so by Theorem 2.1 H is relatively $\sigma(L^1(X), L^\infty(X^*))$-compact in $L^1(X)$. Since (f_π) is a net in $L^1(X)$, there is a subnet $(f_{\pi'})$ of (f_π) such that $f_{\pi'} \underset{\pi'}{\longrightarrow} f_0$ for $\sigma(L^1(X), L^\infty(X^*))$ and some $f_0 \in L^1(X)$. Hence, for each $B \in \Sigma$ and $x^* \in X^*$, we have

$$\int_\Omega \left\langle f_{\pi'}(\omega), \chi_B(\omega) x^* \right\rangle d\mu \underset{\pi'}{\longrightarrow} \int_\Omega \left\langle f_0(\omega), \chi_B(\omega) x^* \right\rangle d\mu.$$

But $\int_\Omega \left\langle f_0(\omega), \chi_B(\omega) x^* \right\rangle d\mu = x^*(\int_B f_0(\omega) \, d\mu)$.
On the other hand, we easily get $\int_\Omega \left\langle f_{\pi'}(\omega), \chi_B(\omega) x^* \right\rangle d\mu = x^*(\sum_{A\in\pi'} \frac{\mu(B\cap A)}{\mu(A)} G(A))$. One can calculate that

$$\sum_{A\in\pi'} \frac{\mu(B\cap A)}{\mu(A)} G(A) \underset{\pi'}{\longrightarrow} G(B).$$

Thus $x^*(G(B)) = x^*(\int_B f_0(\omega) \, d\mu)$, so $G(B) = \int_B f_0(\omega) \, d\mu$, and this means that G has the RNP (with respect to μ). This contradicts the choice of G. It follows that H is not relatively sequentially $\sigma(E(X), E'(X^*))$-compact, as desired. ■

In particular, we obtain:

Corollary 2.4 *For a Banach space X the following statements are equivalent:*

(i) X has the RNP.

(ii) For every finite measure space (Ω, Σ, μ) any norm bounded subset H of $L^1(X)$ satisfying the conditions:

(a) the set $\{\widetilde{f} : f \in H\}$ in L^1 is uniformly integrable,

(b) for each $A \in \Sigma$ the set $\{\int_A f(\omega) d\mu : f \in H\}$ is relatively weakly compact in X,

is relatively sequentially $\sigma(L^1(X), L^\infty(X^*))$-compact.

3. Sequential $\sigma(E(X), E'(X^*))$-completeness of $E(X)$

We start by defining for $A \in \Sigma$ two linear mappings:

$$\Phi_A : X \longrightarrow E(X) \quad \text{and} \quad \Psi_A : E(X) \longrightarrow X$$

by

$$\Phi_A(x) = \chi_A x \quad \text{and} \quad \Psi_A(f) = \int_A f(\omega) d\mu.$$

It is easy to observe that Φ_A is sequentially $(\sigma(X, X^*)), \sigma(E(X), E'(X^*)))$-continuous and Ψ_A is $(\sigma(E(X), E'(X^*)), \sigma(X, X^*))$-continuous.

Now, given $x_o \in S_X$ choose $x_o^* \in S_{X^*}$ such that $x_o^*(x_o) = 1$. Define two linear mappings:

$$P_{x_o^*} : E(X) \longrightarrow E \quad \text{and} \quad Q_{x_o} : E \longrightarrow E(X)$$

by

$$P_{x_o^*}(f) = x_o^* f \quad \text{and} \quad Q_{x_o}(u) = u\, x_o\ .$$

It is seen that $P_{x_o^*}$ is $(\sigma(E(X), E'(X^*)), \sigma(E, E'))$-continuous and Q_{x_o} is $(\sigma(E, E'), \sigma(E(X), E'(X^*)))$-continuous.

As a consequence we obtain:

Lemma 3.1 *The sets $\Phi_\Omega(X)$ $(= \{\chi_\Omega x : x \in X\})$ and $Q_{x_o}(E)$ $(= \{u x_o : u \in E\})$ are sequentially $\sigma(E(X), E'(X^*))$-closed in $E(X)$.*

Now we are ready to present a characterization of sequential completeness of the space $(E(X), \sigma(E(X), E'(X^*)))$.

Theorem 3.2 *Assume that (Ω, Σ, μ) is a finite measure space, and let $(E, \|\cdot\|_E)$ be a Banach function space. Then the following statements are equivalent:*

(i) $E(X)$ is sequentially $\sigma(E(X), E'(X^*))$-complete.

(ii) a) E is sequentially $\sigma(E, E')$-complete (i.e., E is perfect).

b) X has the RNP (with respect to μ) and X is sequentially weakly complete.

Proof. (i) $\Longrightarrow$ (ii) By Lemma 3.1 $(X, \sigma(X, X^*))$ and $(E, \sigma(E, E'))$ embed as sequentially closed subspaces in $(E(X), \sigma(E(X), E'(X^*)))$. It follows that X is sequentially weakly complete, and E is sequentially $\sigma(E, E')$-complete.

Moreover, assume that H is a subset of $E(X)$ that satisfies conditions (a), (b) of (ii) in Theorem 2.2. By [N$_2$, Theorem 2.3] H is conditionally $\sigma(E(X), E'(X^*))$-compact. Making use of (i) we conclude that H is relatively sequentially $\sigma(E(X), E'(X^*))$-compact. Hence in view of Theorem 2.3 X has the RNP (with respect to μ).

(ii) $\Longrightarrow$ (i) Let (f_n) be a $\sigma(E(X), E'(X^*))$-Cauchy sequence in $E(X)$. Then the set $\{f_n : n \in \mathbb{N}\}$ is conditionally $\sigma(E(X), E'(X^*))$-compact. Hence is view of [N$_2$, Theorem 2.3] the set $\widetilde{H} = \{\widetilde{f}_n : n \in \mathbb{N}\}$ is conditionally $\sigma(E, E')$-compact and for each $A \in \Sigma$ the set $\{\int_A f_n(\omega) d\mu : n \in \mathbb{N}\}$ is conditionally weakly compact in X. By (a) the set $\{\widetilde{f}_n : n \in \mathbb{N}\}$ is relatively sequentially $\sigma(E, E')$-compact. In view of [KA, Theorem 10.4.7] for $f \in E(X)$ we have

$$\|\widetilde{f}\|_E = \sup\{|\int_\Omega \widetilde{f}(\omega) v(\omega) d\mu| : v \in E', \|v\|_{E'} \leq 1\}.$$

It easily follows (see [L, Lemma 1.3.1]) that $\sup_n \|f_n\|_{E(X)} < \infty$. Hence by (b) and Theorem 2.2 the set $\{f_n : n \in \mathbb{N}\}$ is relatively sequentially $\sigma(E(X), E'(X^*))$-compact. Thus there is a subsequence (f_{k_n}) of (f_n) and $f_0 \in E(X)$ such that $f_{k_n} \longrightarrow f_0$ for $\sigma(E(X), E'(X^*))$. It easily follows that $f_n \longrightarrow f_0$ for $\sigma(E(X), E'(X^*))$, and this means that $E(X)$ is sequentially $\sigma(E(X), E'(X^*))$-complete, as desired. ■

Remark An analogical characterization of sequential completeness of $(L^p(X), \sigma(L^p(X), L^q(X^*)))$ $(1 < p < \infty)$ and $(L^\varphi(X), \sigma(L^\varphi(X), L^{\varphi^*}(X^*)))$ was found by F. Bombal (see [B$_1$], [B$_2$]) and of $(L^1(X), \sigma(L^1(X), L^\infty(X^*)))$ by G. Schlüchtermann and R. Wheeler (see [SH], Lemma 3.3]).

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