UNCOUNTABLE LEVEL SETS OF LIPSCHITZ FUNCTIONS AND ANALYTIC SETS

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ABSTRACT. We show that a subset of the interval $[0,1]$ is an analytic set with Lebesgue measure zero if and only if it coincides with the set of values of uncountable order of some Lipschitz function from $[0,1]$ into $[0,1]$.

1. INTRODUCTION

Let $f : [0,1] \to \mathbb{R}$. A value y of the function f is said to be of *uncountable order* if the set $f^{-1}(\{y\})$ is uncountable.
The characterization of the set of points where level sets of continuous functions are uncountable is a very old result of S. Mazurkiewicz and W. Sierpinski ([5], [4]). Their characterization is as follows.

Theorem 1.1. *A subset of the interval* $[0,1]$ *is analytic if and only if it coincides with the set of values of uncountable order of some continuous function from* $[0,1]$ *into* $[0,1]$.

Recently, in a joint paper with U. B. Darji, we have characterized the set of points where level sets of C^1 functions are uncountable [3]. Our result is as follows.

Theorem 1.2. *A subset of the interval* $[0,1]$ *coincides with the set of values of uncountable order of some* C^1 *function* $f : [0,1] \to [0,1]$ *if and only if it is analytic and its closure has Lebesgue measure zero.*

In this paper we characterize such sets for Lipschitz functions. Our characterization is as follows.

Theorem 1.3. *Let* M *be a subset of* $[0,1]$. *Then* M *is equal to* $\{y : f^{-1}(\{y\})$ *is uncountable*$\}$ *for some Lipschitz function* $f : [0,1] \to [0,1]$ *if and only if* M *is an analytic set with Lebesgue measure zero.*

2. UNCOUNTABLE LEVEL SETS

We proceed towards the goal of this paper.
We first need few definitions and terminology. Throughout, λ denotes the Lebesgue measure, and π_1 and π_2 denote coordinate projections.

Definition 2.1. *Let* f *be a Lipschitz function on a closed interval* I. *By* U_f, D_f *and* $\tilde{Z}_{(f,1)}$ *we denote the sets*

$$\{y : f^{-1}(\{y\}) \text{ is uncountable}\},$$

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$$\{x : f \textit{ is differentiable at } x\}$$

and

$$\{x \in D_f : f^{(1)}(x) = 0\},$$

respectively.

Theorem 2.2. *([2]: Lemma 1) If f is a continuous function of bounded variation on $[0,1]$, there exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is a Lipschitz function.*

Definition 2.3. *A* box *is a set of the form $I \times J$ where I, J are compact intervals.*

Definition 2.4. *Let I be a closed interval. Then, we use I_L, I_M, I_R to denote the left third, middle third and the right third intervals of I, respectively. If $B = I \times J$ is a box, then $B_L = I_L \times J$, $B_M = I_M \times J$, and $B_R = I_R \times J$. We call B_L, B_M, B_R* the vertical splitting *of B.*

Definition 2.5. *Let B be a box. A continuous function f is* diagonal to *B if the restriction of f to B is a linear function which passes through the diagonal corners of B.*

Definition 2.6. *A continuous function f is said to be* jagged inside box *B if f is diagonal to each of B_L, B_M, B_R.*

Henceforth, we shall denote by CBV continuous functions of bounded variation, and, given a CBV function f, we shall denote by $V(f)$ its variation.

Lemma 2.7. *Let $I = [a,b]$, $J = [c,d]$, $B = I \times J$ and $\epsilon > 0$. Let $\{C_i\}_{i \in \mathbb{N}}$ be a sequence of closed subsets of J, let A be a subset of J with $\lambda(A) = 0$ and $A \cap C_i \neq \emptyset$, for every i. Then, there is a CBV function f from I onto J and there is a sequence $\{\mathcal{G}_i\}_{i \in \mathbb{N}}$ of countable collections of boxes contained in B such that*

1. *the variation of f on I is less than $\lambda(J) + \epsilon$,*
2. *$f^{-1}(\{y\})$ is countable for all $y \in J$,*
3. *$f(a) = c$, $f(b) = d$,*
4. *f is linearly jagged in each $B' \in \cup_{i=1}^{\infty} \mathcal{G}_i$,*
5. *if $i \neq j$, then $\mathcal{G}_i \cap \mathcal{G}_j = \emptyset$ and $\cup_{i=1}^{\infty} \mathcal{G}_i$ is a pairwise disjoint collection,*
6. *$A \cap C_i \subseteq \pi_2(\cup \mathcal{G}_i)$ and $\pi_2(B') \cap A \cap C_i \neq \emptyset$ for all $B' \in \mathcal{G}_i$, and*
7. *$diam(B') < \epsilon$ for all $B' \in \mathcal{G}_i$.*

Proof. We will construct a sequence $\{f_k\}_{k \in \mathbb{N}}$ of CBV functions whose uniform limit is the desired function.

Let $f_0 : I \to J$ be a linear function which satisfies Condition 3 of the Lemma. Let $J_1^1, J_2^1, \ldots, J_n^1, \ldots$ be a sequence of non-overlapping closed intervals contained in J with the following properties:

a. $A \cap C_1 \subseteq \cup_{i=1}^{\infty} J_i^1$, $A \cap C_1 \cap J_i^1 \neq \emptyset$,
b. $\sum_{i=1}^{\infty} \lambda(J_i^1) < \frac{\epsilon}{5 \cdot 2}$ and $\lambda(f_0^{-1}(J_i^1)) < \frac{\epsilon}{2}$.

Let $I_i^1 = f_0^{-1}(J_i^1)$. In each of I_i^1, replace f_0 by an appropriate continuous function which is jagged in $(I_i^1 \times J_i^1)_L$, diagonal to $(I_i^1 \times J_i^1)_M$ and diagonal to $(I_i^1 \times J_i^1)_R$. Let f_1 be the resulting continuous piecewise linear function and let $\mathcal{G}_1 = \{(I_i^1 \times J_i^1)_L : i \in \mathbb{N}\}$. Then, at the end of stage 1, the following properties are satisfied:

(i) f_1 is a continuous function linearly jagged inside each $B' \in \mathcal{G}_1$ with $f_1(a) = c$ and $f_1(b) = d$,
(ii) $|f_1^{-1}(\{y\})| \leq 5$ for all $y \in J$,

(iii) f_1 is a CBV function and

$$\begin{aligned} V(f_1) &\leq V(f_0) + 5 \cdot \sum_{i=1}^{\infty} \lambda(J_i^1) \\ &< V(f_0) + 5 \cdot \frac{\epsilon}{5 \cdot 2} \\ &= \lambda(J) + \frac{\epsilon}{2}. \end{aligned}$$

(iv) $A \cap C_1 \subseteq \pi_2(\cup \mathcal{G}_1)$ and $\pi_2(B') \cap A \cap C_1 \neq \emptyset$ for all $B' \in \mathcal{G}_1$,
(v) $\|f_1 - f_0\|_0 \leq \sum_{i=1}^{\infty} \lambda(J_i^1) < \frac{\epsilon}{5 \cdot 2}$.

Let us now construct f_2. Let $J_1^2, J_2^2, \ldots, J_n^2, \ldots$ be a sequence of non-overlapping closed intervals contained in J such that

a. either there is i' such that $J_i^2 \subseteq J_{i'}^1$ or J_i^2 does not overlap with any J_s^1, $s \in \mathbb{N}$,
b. $A \cap C_2 \subseteq \cup_{i=1}^{\infty} J_i^2$, $A \cap C_2 \cap J_i^2 \neq \emptyset$,
c. $\sum_{i=1}^{\infty} \lambda(J_i^2) < \frac{\epsilon}{5 \cdot 2^2}$ and $\lambda(f_1^{-1}(J_i^2)) < \frac{\epsilon}{2}$.

Now, if there is i' such that $J_i^2 \subseteq J_{i'}^1$, we let $I_i^2 = f_1^{-1}(J_i^2) \cap (I_{i'}^1)_R$, otherwise we let $I_i^2 = f_1^{-1}(J_i^2)$. Let f_2 be the modification of f_1 on $\cup_i I_i^2 \times J_i^2$ as earlier and $\mathcal{G}_2 = \{(I_i^2 \times J_i^2)_L : i \in \mathbb{N}\}$. Then, at the end of stage 2, the following properties are satisfied:

(i) f_2 is a continuous function linearly jagged inside each $B' \in \mathcal{G}_2$, with $f_2(a) = c$ and $f_2(b) = d$,
(ii) $|f_2^{-1}(\{y\})| \leq 5 + (2-1) \cdot 4$ for all $y \in J$,
(iii) f_2 is a CBV function and

$$\begin{aligned} V(f_2) &\leq V(f_1) + 5 \cdot \sum_{i=1}^{\infty} \lambda(J_i^2) \\ &< V(f_1) + 5 \cdot \frac{\epsilon}{5 \cdot 2^2}, \\ &= V(f_1) + \frac{\epsilon}{2^2}. \end{aligned}$$

(iv) $A \cap C_2 \subseteq \pi_2(\cup \mathcal{G}_2)$ and $\pi_2(B') \cap A \cap C_2 \neq \emptyset$ for all $B' \in \mathcal{G}_2$,
(v) $\|f_2 - f_1\|_0 \leq \sum_i \lambda(J_i^2) < \frac{\epsilon}{5 \cdot 2^2}$.

Now let us assume that we are at stage $k > 1$, f_k and $\mathcal{G}_k$ have been constructed so that the following properties are satisfied:

(i) f_k is a continuous function linearly jagged inside each $B' \in \mathcal{G}_k$, with $f_1(a) = c$ and $f_1(b) = d$, ,
(ii) $|f_k^{-1}(\{y\})| \leq 5 + (k-1) \cdot 4$ for all $y \in J$,
(iii) f_k is a CBV function and

$$V(f_k) < V(f_{k-1}) + \frac{\epsilon}{2^k},$$

(iv) $A \cap C_k \subseteq \pi_2(\cup \mathcal{G}_k)$ and $\pi_2(B') \cap A \cap C_k \neq \emptyset$ for all $B' \in \mathcal{G}_k$,
(v) $\|f_k - f_{k-1}\|_0 \leq \sum_i \lambda(J_i^k) < \frac{\epsilon}{5 \cdot 2^k}$.

Let us now construct f_{k+1}. Let $J_1^{k+1}, J_2^{k+1}, \ldots, J_n^{k+1}, \ldots$ be a sequence of non-overlapping closed intervals contained in J such that

a. either there is i' such that $J_i^{k+1} \subseteq J_{i'}^k$ or J_i^{k+1} does not overlap with any J_s^k, $s \in \mathbb{N}$,
b. $A \cap C_{k+1} \subseteq \cup_{i=1}^{\infty} J_i^{k+1}$, $A \cap C_{k+1} \cap J_i^{k+1} \neq \emptyset$,
c. $\sum_{i=1}^{\infty} \lambda(J_i^{k+1})) < \frac{\epsilon}{5 \cdot 2^{k+1}}$ and $\lambda(f_k^{-1}(J_i^{k+1})) < \frac{\epsilon}{2}$.

Now, if there is i' such that $J_i^{k+1} \subseteq J_{i'}^k$, we let $I_i^{k+1} = f_k^{-1}(J_i^{k+1}) \cap (I_{i'}^k)_R$, otherwise we let $I_i^{k+1} = f_k^{-1}(J_i^{k+1})$. Let f_{k+1} be the modification of f_k on $\cup_i I_i^{k+1} \times J_i^{k+1}$ as earlier and $\mathcal{G}_{k+1} = \{(I_i^{k+1} \times J_i^{k+1})_L : i \in \mathbb{N}\}$.
Now, it is easy to verify that f_{k+1} satisfies all induction hypotheses of stage $k+1$ except (iii). In order to prove (iii) we notice that

$$\begin{aligned} V(f_{k+1}) &\leq V(f_k) + 5 \cdot \sum_i \lambda(J_i^k) \\ &\leq V(f_k) + 5 \cdot \frac{\epsilon}{5 \cdot 2^{k+1}} \\ &= V(f_k) + \frac{\epsilon}{2^{k+1}}. \end{aligned}$$

By (v), we have that $\{f_k\}$ converges uniformly to some continuous function f. By (iii) we have that

$$V(f) \leq \lambda(J) + \epsilon.$$

Hence, f is of bounded variation. Clearly, f satisfies the required conditions. $\square$

Definition 2.8. *We will use τ, σ etc. to denote an element of $\mathbb{N}^{<N}$ (= finite sequences of elements of $\mathbb{N}$) or $\mathbb{N}^N$. We use $|\sigma|$ to denote the length of σ and, if $|\sigma| > k$, then $\sigma|k$ to denote the restriction of σ to the first k coordinates, and $\sigma(k)$ to denote the $k-$th coordinate of σ. If σ is a finite string and k is a positive integer, then σk denotes the concatenation of σ followed by k.*

Proposition 2.9. *Let $A \subseteq [0,1]$ be an analytic set with $\lambda(A) = 0$. Then, there is a CBV function $f : [0,1] \to [0,1]$ such that*

(i) *$f^{-1}(\{y\})$ is uncountable for all $y \in A$, and*
(ii) *$f^{-1}(\{y\})$ is countable for all $y \notin A$.*

Proof. As A is an analytic subset of $[0,1]$, we may obtain a Suslin scheme [1] $\{C_\tau\}_{\tau \in \mathbb{N}^{<N}}$ such that

(a) each of C_τ is a closed subset of $[0,1]$,
(b) for each $\sigma \in \mathbb{N}^N$, $C_{\sigma|k+1} \subseteq C_{\sigma|k}$, and $diam(C_{\sigma|k}) < \frac{1}{2^k}$,
(c) $A = \cup_{\sigma \in \mathbb{N}^N} \cap_{k=1}^\infty C_{\sigma|k}$.

We will construct the desired f as the uniform limit of a sequence of continuous functions $\{f_k\}_{k \in \mathbb{N}}$. Let $f_0 : [0,1] \to [0,1]$ be the identity map and let $\mathcal{G}_0 = \{[0,1] \times [0,1]\}$. Applying Lemma 2.7 to $B = [0,1] \times [0,1]$, $\{C_{\sigma|1}\}_{\sigma \in \mathbb{N}^N}$, A and $\epsilon = \frac{1}{2}$, we obtain a function f_1 and a sequence of countable collections of boxes $\{\mathcal{H}_i\}_{i \in \mathbb{N}}$ which satisfy the conclusion of Lemma 2.7. Let $\mathcal{G}_1 = \{B'_L, B'_R : B' \in \mathcal{H}_i \text{ for some } i\}$. For each $B' \in \mathcal{H}_i$, define $\phi_1(B'_L) = \phi_1(B'_R) = i$. Note that ϕ_1 is well-defined as $\mathcal{H}_i \cap \mathcal{H}_j = \emptyset$ for $i \neq j$. Then, f_1, ϕ_1 and $\mathcal{G}_1$ satisfy the following conditions:

1. f_1 is a CBV function with $V(f_1) < 1 + \frac{1}{2}$,
2. $f_1 = f_0$ outside $\pi_1(\cup \mathcal{G}_1)$,
3. $f_1^{-1}(\{y\})$ is countable for all y,
4. $\mathcal{G}_1$ is a pairwise disjoint collection,
5. f_1 is diagonal to each $B' \in \mathcal{G}_1$,
6. for each $\sigma \in \mathbb{N}^1$, $\phi_1^{-1}(\{\sigma\})$ is a countable collection of boxes and $A \cap C_\sigma \subseteq \pi_2(\cup \phi_1^{-1}(\{\sigma\}))$, and if $B' \in \phi_1^{-1}(\{\sigma\})$, then $\pi_2(B') \cap A \cap C_\sigma \neq \emptyset$, and
7. for each $B' \in \mathcal{G}_1$, $diam(B') < \frac{1}{2}$.

Now, suppose that we are at stage $k > 1$ and a function f_k, a countable collection of boxes $\mathcal{G}_k$ contained in the unit square and a function $\phi_k : \mathcal{G}_k \to \mathbb{N}^k$ have been constructed such that:

1. f_k is a CBV function with $V(f_k) < 1 + \sum_{i=1}^{k} \frac{1}{2^i}$,
2. $f_k^{-1}(\{y\})$ is countable for all y,
3. $f_k = f_{k-1}$ outside $\pi_1(\cup \mathcal{G}_k)$,
4. $\mathcal{G}_k$ is a pairwise disjoint collection,
5. f_k is diagonal to each $B' \in \mathcal{G}_k$,
6. for each $\sigma \in \mathbb{N}^k$, $\phi_k^{-1}(\{\sigma\})$ is a countable collection of boxes and $A \cap C_\sigma \subseteq \pi_2(\cup \phi_k^{-1}(\{\sigma\}))$, and if $B' \in \phi_k^{-1}(\{\sigma\})$, then $\pi_2(B') \cap A \cap C_\sigma \neq \emptyset$,
7. for each $\sigma \in \mathbb{N}^k$, $\cup \phi_k^{-1}(\{\sigma\}) \subseteq \cup \phi_{k-1}^{-1}(\{\sigma|(k-1)\})$,
8. for each $B' \in \mathcal{G}_k$, $diam(B') < \frac{1}{2^k}$, and
9. for each $B' \in \mathcal{G}_{k-1}$ with $\tau = \phi_{k-1}(B')$ and for positive integer m, we have that if $\pi_2(B') \cap C_{\tau m} \neq \emptyset$, then for each $y \in A \cap C_{\tau m} \cap \pi_2(B')$, there are two disjoint boxes $B_1, B_2 \in \mathcal{G}_k$ with $B_1 \cup B_2 \subseteq B'$ such that $y \in \pi_2(B_1) \cap \pi_2(B_2)$ and $\phi_k(B_1) = \phi_k(B_2) = \tau m$.

Let us now define f_{k+1}. Enumerate $\mathcal{G}_k$ as $B_1, B_2, \ldots$. Let $l \geq 1$ and $\sigma = \phi_k(B_l)$. If there is no i so that $\pi_2(B_l) \cap A \cap C_{\sigma i} \neq \emptyset$, then we let $g_l = f_k$ on $\pi_1(B_l)$. Otherwise, we apply Lemma 2.7 to B_l, $\pi_2(B_l) \cap C_{\sigma i}$, $i = 1, 2, \ldots$ (listing only the non-empty ones), $A \cap \pi_2(B_l)$ and $\epsilon = \frac{1}{2^{k+1+l}}$, and obtain a function g_l and a sequence of countable collections of boxes $\{\mathcal{H}_i^l\}_{i \in \mathbb{N}}$ which satisfy the conclusion of Lemma 2.7. For each $B' \in \mathcal{H}_i^l$, define $\phi_{k+1}^l(B'_L) = \phi_{k+1}^l(B'_R) = \sigma i$. We do this for each l and let $f_{k+1} = f_k$ outside $\cup_{l=1}^{\infty} \pi_1(B_l)$ and $f_{k+1} = g_l$ on $\pi_1(B_l)$. We let $\mathcal{G}_{k+1} = \{B'_L, B'_R : B' \in \mathcal{H}_i^l \text{ for some } i, l\}$ and let ϕ_{k+1} be the union of all the partial ϕ_{k+1}^l. These $f_{k+1}, \mathcal{G}_{k+1}, \phi_{k+1}$ satisfy the induction hypotheses. As f_{k+1} is continuous and modified only inside boxes of stage k and these boxes have diameters less than $\frac{1}{2^k}$, we have that $\{f_k\}_{k \in \mathbb{N}}$ converges uniformly to some continuous function f. By induction hypothesis 1, we have that f also is a CBV function.

Let us now show that $f^{-1}(\{y\})$ is uncountable for $y \in A$ and countable otherwise. We shall prove that $y \in A$ if and only if $f^{-1}(\{y\})$ is uncountable.

($\Rightarrow$) Let $y \in A$. Let $\sigma \in \mathbb{N}^N$ be such that $y \in \cap_{k=1}^{\infty} C_{\sigma|k}$. Applying induction hypothesis 9 at stage $k = 1$ with $B = [0,1] \times [0,1]$, we may obtain two disjoint boxes B_0^y and B_1^y in $\mathcal{G}_1$ such that $y \in \pi_2(B_0^y) \cap \pi_2(B_1^y)$ and that $\phi_1(B_0^y) = \phi_1(B_1^y) = \sigma|1$. Now suppose that $k \geq 1$ and we have 2^k many pairwise disjoint boxes B_α^y, $\alpha \in \{0,1\}^k$ with each $B_\alpha^y \in \mathcal{G}_k, y \in \cap_{\alpha \in \{0,1\}^k} \pi_2(B_\alpha^y)$ and $\phi_k(B_\alpha^y) = \sigma|k$ for all α. Applying induction hypothesis 9 at stage $k+1$ to each B_α^y, for $\alpha \in \{0,1\}^k$ and $m = \sigma(k+1)$, we obtain an analogous appropriate collection of boxes at stage $k+1$. Now, it is easy to verify that the Cantor set $\cup_{\alpha \in \{0,1\}^{\mathbb{N}}} \cap_{k=1}^{\infty} \pi_1(B_{\alpha|k}^y)$ maps to y under f.

($\Leftarrow$) Let $f^{-1}(\{y\})$ be uncountable. As $f = f_1$ outside $\pi_1(\cup \mathcal{G}_1)$ and $f_1^{-1}(\{y\})$ is countable, we have that there is $B_1 \in \mathcal{G}_1$ such that B_1 contains uncountably many points of the graph of f whose second coordinate is y. Let $l_1 = \phi_1(B_1)$. By a similar argument, there has to be $B_2 \in \mathcal{G}_2$ such that B_2 contains uncountably many points of the graph of f whose second coordinate is y and $B_2 \subseteq B_1$. By induction hypotheses 4 and 7, we have that $\phi_2(B_2) = (l_1, l_2)$ for some l_2. Continuing in this fashion, we obtain a sequence of boxes $\{B_k\}_{k \in \mathbb{N}}$ and a sequence of integers $\{l_k\}_{k \in \mathbb{N}}$ such that $y \in \pi_2(B_k)$, $B_k \in \mathcal{G}_k$, $B_{k+1} \subseteq B_k$ and $\phi_k(B_k) = \sigma|k$ where $\sigma = (l_1, l_2, \ldots)$. From Condition 6 we have that $\pi_2(B_k) \cap A \cap C_{\sigma|k} \neq \emptyset$, and from Condition 8 that $diam(B_k) \to 0$ as $k \to \infty$. Hence, $y \in \cap_{k=1}^{\infty} C_{\sigma|k}$. Therefore, $y \in A$. □

Proposition 2.10. *Let A be an analytic subset of $[0,1]$ with $\lambda(A)=0$. Then, there is a Lipschitz function $f:[0,1]\to[0,1]$ such that*

(i) *$f^{-1}(\{y\})$ is uncountable for all $y\in A$, and*
(ii) *$f^{-1}(\{y\})$ is countable for all $y\notin A$.*

Proof. By Proposition 2.9 there exists a CBV function $g:[0,1]\to[0,1]$ such that $g^{-1}(\{y\})$ is uncountable for all $y\in A$ and countable otherwise. Applying Theorem 2.2 we obtain a homeomorphism h from $[0,1]$ onto $[0,1]$ such that $g\circ h$ is a Lipschitz function. Now, $f=g\circ h$ is the desired function. □

Proposition 2.11. *Suppose that $f:[0,1]\to\mathbb{R}$ is a Lipschitz function. Then, the set of points where level sets are uncountable is an analytic set with Lebesgue measure zero.*

Proof. By a very old result of S. Mazurkiewicz and W. Sierpinski [5] U_f is an analytic set. Let ${U_f}^1=\{y\in U_f:\exists x_y\in D_f \text{ with } f(x_y)=y\}$. As f is Lipschitz, it follows that $\lambda(U_f\setminus {U_f}^1)=0$. As for every $y\in U_f$ $f^{-1}(\{y\})$ is uncountable, it contains a perfect set and hence it is clear that for every $y\in {U_f}^1$ it is $x_y\in\tilde{Z}_{(f,1)}$. Let ${U_f}^2$ be the set of all points in ${U_f}^1$ which are not a local extremum of f. As ${U_f}^1\setminus{U_f}^2$ is at most countable, it has Lebesgue measure equal to zero. Let $\epsilon>0$. For every $y\in{U_f}^2$ choose a sequence $\{p_{y,k}\}_{k\in\mathbb{N}}$ converging to x_y such that, for every k,

(i) the image under f of the semi-open interval $J_{y,k}$ containing x_y and having as end-points x_y and $p_{y,k}$ is a non-degenerate interval, and
(ii) $|f(x_y)-f(p_{y,k})|<\epsilon\cdot|x_y-p_{y,k}|$.

Let, for every $y\in{U_f}^2$ and for every k, $f(\overline{J_{y,k}})=[f(a_{y,k}),f(b_{y,k})]$. Now, $V_y=\{[f(a_{y,k}),f(b_{y,k})]\}_{k\in\mathbb{N}}$ is a family of non-degenerate intervals containing y and with diameters going to zero. Let $V=\cup_{y\in{U_f}^2}V_y$. Clearly, V is a Vitali covering of ${U_f}^2$. By the Vitali covering theorem ([1]), there exists a countable sub-collection of pair-wise disjoint intervals $\{[f(a_{y_i,k_i}),f(b_{y_i,k_i})]\}_{i\in\mathbb{N}}$ such that $\lambda({U_f}^2\setminus\cup_{i=1}^{\infty}[f(a_{y_i,k_i}),f(b_{y_i,k_i})])=0$. The collection $\{I_i\}_{i\in\mathbb{N}}$ of closed intervals having as end-points a_{y_i,k_i} and b_{y_i,k_i} is pair-wise disjoint since so is $\{[f(a_{y_i,k_i}),f(b_{y_i,k_i})]\}_{i\in\mathbb{N}}$. Moreover, we have that

$$\begin{aligned} & |f(a_{y_i,k_i})-f(b_{y_i,k_i})| \\ \le\ & |f(a_{y_i,k_i})-f(x_{y_i})|+|f(x_{y_i})-f(b_{y_i,k_i})| \\ <\ & \epsilon\cdot(|a_{y_i,k_i}-x_{y_i}|+|x_{y_i}-b_{y_i,k_i}|). \end{aligned}$$

Therefore,

$$\begin{aligned} & \sum_{i=1}^{\infty}|f(a_{y_i,k_i})-f(b_{y_i,k_i})| \\ \le\ & \sum_{i=1}^{\infty}(|f(a_{y_i,k_i})-f(x_{y_i})|+|f(x_{y_i})-f(b_{y_i,k_i})| \\ \le\ & \epsilon\cdot\sum_{i=1}^{\infty}(|a_{y_i,k_i}-x_{y_i}|+|x_{y_i}-b_{y_i,k_i}|) \\ \le\ & 2\cdot\epsilon. \end{aligned}$$

Hence, $\lambda(U_f)=\lambda({U_f}^2)\le 2\cdot\epsilon$, for every ϵ.

□

Theorem 2.12. *Let $M \subseteq [0,1]$. Then the following are equivalent:*

1. *M is an analytic set with $\lambda(M) = 0$,*
2. *there is a Lipschitz function f from $[0,1]$ into $[0,1]$ such that $f^{-1}(\{y\})$ is uncountable for every $y \in M$ and countable otherwise.*

Proof. (1) $\Rightarrow$ (2) This is Proposition 2.10.
(2) $\Rightarrow$ (1) This is Proposition 2.11. □

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