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ON FUZZY QUOTIENT BCI-ALGEBRAS INDUCED BY FUZZY IDEALS

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ABSTRACT. We define fuzzy quotient BCI-algebras induced by fuzzy ideals and study the relation between fuzzy quotient BCI-algebras and fuzzy ideals. We establish isomorphism theorem.

1. INTRODUCTION

For the general development of BCI-algebras, the (fuzzy) ideal theory plays an important role. Of course, the quotient structure by (fuzzy) ideal plays an important role also. In general, the relation "$\sim$" on a BCI-algebra X defined by $x \sim y$ if and only if $x * y \in A$ and $y * x \in A$ is used, where $x, y \in X$ and A is an ideal of X, to constructing quotient structure of BCI-algebra induced by an ideal. F. L. Zhang [8] gave an equivalent relation on a BCI-algebra by using a different method, and constructed the corresponding quotient structures. S. M. Hong and Y. B. Jun [1] fuzzified the equivalence relation obtained by Zhang's way, and established a quotient BCI-algebra which is induced by a fuzzy ideal. In this paper, we consider another fuzzification of the equivalence relation given by F. L. Zhang, and construct fuzzy quotient BCI-algebras induced by fuzzy ideals. We establish an isomorphism theorem, and give a characterization for a quotient BCI-algebra induced by a fuzzy ideal to be commutative (positive implicative).

2. PRELIMINARIES

In this section we include some elementary aspects that are necessary for this paper.

Recall that a *BCI-algebra* is an algebra $(X, *, 0)$ of type (2, 0) satisfying the following axioms for every $x, y, z \in X$,

(a1) $((x * y) * (x * z)) * (z * y) = 0$,
(a2) $(x * (x * y)) * y = 0$,
(a3) $x * x = 0$,
(a4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

A partial ordering $\leq$ on X can be defined by $x \leq y$ if and only if $x * y = 0$. In a BCI-algebra X, the following hold:

(b1) $x * 0 = x$.
(b2) $(x * y) * z = (x * z) * y$.
(b3) $0 * (x * y) = (0 * x) * (0 * y)$.
(b4) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.

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A mapping $f: X \to Y$ of BCI-algebras is called a *homomorphism* if $f(x*y) = f(x)*f(y)$ for all $x, y \in X$. An *ideal* of a BCI-algebra X is defined to be a subset A of X containing 0 such that if $x * y \in A$ and $y \in A$ then $x \in A$. If x is an element of an ideal A of a BCI-algebra X and $y \leq x$, then $y \in A$. For any elements x and y of a BCI-algebra X and $n \in \mathbb{N}$, let us write $x * y^n$ instead of $\big(((x * y) * y) * \cdots\big) * y$ in which y occurs n times.

Proposition 2.1. (Huang [2]) *For any elements x and y of a BCI-algebra X and $n \in \mathbb{N}$, we have $0 * (x * y)^n = (0 * x^n) * (0 * y^n)$.*

We now review some fuzzy logic concepts. Let X be a set. A *fuzzy set* in X is a mapping from X to $[0, 1]$. In the sequel, we place a bar over a symbol to denote a fuzzy set so $\bar{A}$, $\bar{B}$, $\bar{G}$, $\cdots$ all represent fuzzy sets in X. A *fuzzy ideal* of a BCI-algebra X is defined to be a fuzzy set $\bar{A}$ in X such that

(F1) $\bar{A}(0) \geq \bar{A}(x)$ for all $x \in X$,

(F2) $\bar{A}(x) \geq \min\{\bar{A}(x * y),\ \bar{A}(y)\}$ for all $x, y \in X$.

Note that every fuzzy ideal $\bar{A}$ of a BCI-algebra X is order reversing, i.e., if $x \leq y$ then $\bar{A}(x) \geq \bar{A}(y)$. A fuzzy ideal $\bar{A}$ of a BCI-algebra X is said to be *closed* if $\bar{A}(0 * x) \geq \bar{A}(x)$ for all $x \in X$. A fuzzy set $\bar{A}$ in a BCI-algebra X is called a *fuzzy commutative ideal* if it satisfies (F1) and

(F3) $\bar{A}\big(x * ((y * (y * x)) * (0 * (0 * (x * y))))\big) \geq \min\big\{\bar{A}\big((x * y) * z\big),\ \bar{A}(z)\big\}$ for all $x, y, z \in X$.

A fuzzy set $\bar{A}$ in a BCI-algebra X is called a *fuzzy positive implicative ideal* if it satisfies (F1) and

(F4) $\bar{A}(x * z) \geq \min\big\{\bar{A}\big(((x * z) * x) * (y * z)\big),\ \bar{A}(y)\big\}$ for all $x, y, z \in X$.

3. Quotient Structures

Let A be an ideal of a BCI-algebra X and let $n \in \mathbb{N}$. We define a relation "$\sim$" on X as follows:

$$x \sim y(A) \text{ if and only if } 0 * (x * y)^n \in A \text{ and } 0 * (y * x)^n \in A.$$

Then "$\sim$" is a congruence relation on X (see [8] and [1]).

Let X be a BCI-algebra and denote by A_x the equivalence class containing $x \in X$, and by X/A the set of all equivalence classes of X with respect to "$\sim$", that is,

$$A_x := \{y \in X \mid x \sim y(A)\} \text{ and } X/A := \{A_x \mid x \in X\}.$$

Define a binary operation "$\diamond$" on X/A by $A_x \diamond A_y = A_{x*y}$ for all $A_x, A_y \in X/A$. Then $(X/A; \diamond, A_0)$ is a BCI-algebra (see [8]).

Theorem 3.1. *If A is an ideal of a BCI-algebra X, then the mapping $\phi : X \to X/A$ given by $\phi(x) = A_x$ is an epimorphism with kernel A.*

Proof. The map $\phi : X \to X/A$ is clearly surjective and since

$$\phi(x * y) = A_{x*y} = A_x \diamond A_y = \phi(x) \diamond \phi(y),$$

ϕ is an epimorphism. Now

$$\mathrm{Ker}\phi = \{x \in X \mid \phi(x) = A_x = A_0\} = \{x \in X \mid x \in A\} = A.$$

This completes the proof. □

Theorem 3.2. *Let $f : X \to Y$ be an epimorphism of BCI-algebras. If Y satisfies the implication $0 * x^n = 0 * y^n \Rightarrow x = y$ for every $x, y \in Y$ and $n \in \mathbb{N}$, then the quotient algebra $X/\mathrm{Ker}f$ is isomorphic to Y.*

Proof. Obviously, Ker$f$ is an ideal of X. Let $x, y \in X$ be such that $f(x) = f(y)$. Then

$$f\big(0*(x*y)^n\big) = f(0)*f\big((x*y)^n\big) = 0*f(x*y)^n = 0*\big(f(x)*f(y)\big)^n = 0.$$

Similarly, $f\big(0*(y*x)^n\big) = 0$, and so $0*(x*y)^n \in \mathrm{Ker}f$ and $0*(y*x)^n \in \mathrm{Ker}f$. Hence $x \sim y(\mathrm{Ker}f)$. This means that x and y belong to a class of $X/\mathrm{Ker}f$. Conversely if $x \sim y(\mathrm{Ker}f)$, then $0*(x*y)^n \in \mathrm{Ker}f$ and $0*(y*x)^n \in \mathrm{Ker}f$, which imply that

$$0 = f\big(0*(x*y)^n\big) = f\big((0*x^n)*(0*y^n)\big) = f(0*x^n)*f(0*y^n) = \big(0*f(x)^n\big)*\big(0*f(y)^n\big)$$

and $\big(0*f(y)^n\big)*\big(0*f(x)^n\big) = 0$ by the similar way. It follows from (a4) that $0*f(x)^n = 0*f(y)^n$ so from the hypothesis that $f(x) = f(y)$. Therefore $X/\mathrm{Ker}f \ni (\mathrm{Ker}f)_x \mapsto f(x) \in Y$ is a one-to-one correspondence between $X/\mathrm{Ker}f$ and Y. Moreover $(\mathrm{Ker}f)_x \diamond (\mathrm{Ker}f)_y = (\mathrm{Ker}f)_{x*y}$ implies $f(x)*f(y) = f(x*y)$. Hence the above correspondence gives the required isomorphism. □

Let $\bar{A}$ be a fuzzy ideal of a BCI-algebra X. Define a binary relation "$\approx$" on X as follows:

$$x \approx y(\bar{A}) \text{ if and only if } \bar{A}\big(0*(x*y)^n\big) = \bar{A}(0) = \bar{A}\big(0*(y*x)^n\big)$$

for all $x, y \in X$ and $n \in \mathbb{N}$.

Lemma 3.3. *The binary relation "$\approx$" is an equivalence relation on a BCI-algebra X.*

Proof. Obviously, "$\approx$" is reflexive and symmetric. Let $x, y, z \in X$ be such that $x \approx y(\bar{A})$ and $y \approx z(\bar{A})$. Then

$$\bar{A}\big(0*(x*y)^n\big) = \bar{A}(0) = \bar{A}\big(0*(y*x)^n\big) \text{ and } \bar{A}\big(0*(y*z)^n\big) = \bar{A}(0) = \bar{A}\big(0*(z*y)^n\big)$$

for every $n \in \mathbb{N}$. On the other hand,

$$\begin{aligned}\big(0*(x*z)^n\big)*\big(0*(x*y)^n\big) &= \big((0*x^n)*(0*z^n)\big)*\big((0*x^n)*(0*y^n)\big)\\ &\le (0*y^n)*(0*z^n) = 0*(y*z)^n.\end{aligned}$$

Since $\bar{A}$ is order reversing, it follows that

$$\bar{A}\big((0*(x*z)^n)*(0*(x*y)^n)\big) \ge \bar{A}\big(0*(y*z)^n\big)$$

so from (F2) that

$$\begin{aligned}\bar{A}\big(0*(x*z)^n\big) &\ge \min\big\{\bar{A}\big((0*(x*z)^n)*(0*(x*y)^n)\big),\ \bar{A}\big(0*(x*y)^n\big)\big\}\\ &\ge \min\big\{\bar{A}\big(0*(y*z)^n\big),\ \bar{A}\big(0*(x*y)^n\big)\big\}\\ &= \bar{A}(0).\end{aligned}$$

Clearly $\bar{A}\big(0*(x*z)^n\big) \le \bar{A}(0)$ by (F1), and so $\bar{A}\big(0*(x*z)^n\big) = \bar{A}(0)$. Similarly, we obtain $\bar{A}\big(0*(z*x)^n\big) = \bar{A}(0)$. Hence $x \approx z(\bar{A})$, which proves the transitivity of $\approx$. This completes the proof. □

Lemma 3.4. *For any elements x, y and z of a BCI-algebra X, $x \approx y(\bar{A})$ implies $x*z \approx y*z(\bar{A})$ and $z*x \approx z*y(\bar{A})$.*

Proof. If $x \approx y(\bar{A})$, then $\bar{A}\big(0*(x*y)^n\big) = \bar{A}(0) = \bar{A}\big(0*(y*x)^n\big)$ for every $n \in \mathbb{N}$. Note that

$$\begin{aligned}&\big(0*((x*z)*(y*z))^n\big)*\big(0*(x*y)^n\big)\\ =\ &\big((0*(x*z)^n)*(0*(y*z)^n)\big)*\big(0*(x*y)^n\big)\\ =\ &\big(((0*x^n)*(0*z^n))*((0*y^n)*(0*z^n))\big)*\big(0*(x*y)^n\big)\\ \le\ &\big((0*x^n)*(0*y^n)\big)*\big(0*(x*y)^n\big)\\ =\ &\big(0*(x*y)^n\big)*\big(0*(x*y)^n\big)\\ =\ &0.\end{aligned}$$

Since $\bar{A}$ is order reversing, it follows that

$$\bar{A}\big((0*((x*z)*(y*z))^n)*(0*(x*y)^n)\big) \ge \bar{A}(0)$$

so from (F2) that

$$\begin{aligned} & \bar{A}\big(0 * ((x * z) * (y * z))^n\big) \\ \geq\ & \min\{\bar{A}\big((0 * ((x * z) * (y * z))^n) * \big(0 * (x * y)^n\big),\ \bar{A}\big(0 * (x * y)^n\big)\} \\ \geq\ & \bar{A}(0). \end{aligned}$$

Obviously, $\bar{A}\big(0 * ((x * z) * (y * z))^n\big) \leq \bar{A}(0)$ by (F1). Hence

$$\bar{A}\big(0 * ((x * z) * (y * z))^n\big) = \bar{A}(0).$$

Similarly, we get $\bar{A}\big(0 * ((y * z) * (x * z))^n\big) = \bar{A}(0)$, and therefore $x * z \approx y * z(\bar{A})$. Similar argument induces $z * x \approx z * y(\bar{A})$. This completes the proof. □

Using Lemma 3.4 and the transitivity of $\approx$, we have the following lemma.

Lemma 3.5. *If $x \approx u(\bar{A})$ and $y \approx v(\bar{A})$ in a BCI-algebra X, then $x * y \approx u * v(\bar{A})$.*

Let X be a BCI-algebra and denote by $\bar{A}_x$ the equivalence class containing $x \in X$, and by $X/\bar{A}$ the set of all equivalence classes of X with respect to "$\approx$", that is,

$$\bar{A}_x := \{y \in X \mid x \approx y(\bar{A})\} \text{ and } X/\bar{A} := \{\bar{A}_x \mid x \in X\}.$$

Define a binary operation "$\oslash$" on $X/\bar{A}$ by $\bar{A}_x \oslash \bar{A}_y = \bar{A}_{x*y}$ for all $\bar{A}_x,\ \bar{A}_y \in X/\bar{A}$. We first verify that the operation "$\oslash$" is well defined. Let $x, y, u, v \in X$ be such that $\bar{A}_x = \bar{A}_u$ and $\bar{A}_y = \bar{A}_v$. Then $x \approx u(\bar{A})$ and $y \approx v(\bar{A})$, which imply that $x * y \approx u * v(\bar{A})$ by Lemma 3.5. Let $w \in \bar{A}_x \oslash \bar{A}_y$. Then $w \approx x * y \approx u * v(\bar{A})$, and so $w \in \bar{A}_{u*v} = \bar{A}_u \oslash \bar{A}_v$. Now if $z \in \bar{A}_u \oslash \bar{A}_v$, then $z \approx u * v \approx x * y(\bar{A})$, and thus $z \in \bar{A}_{x*y} = \bar{A}_x \oslash \bar{A}_y$. Therefore $\bar{A}_x \oslash \bar{A}_y = \bar{A}_u \oslash \bar{A}_v$, that is, "$\oslash$" is well defined. Next we shall show that $(X/\bar{A}; \oslash, \bar{A}_0)$ is a BCI-algebra. Let $\bar{A}_x, \bar{A}_y, \bar{A}_z \in A/\bar{A}$. Then

$$\begin{aligned} & \big((\bar{A}_x \oslash \bar{A}_y) \oslash (\bar{A}_x \oslash \bar{A}_z)\big) \oslash (\bar{A}_z \oslash \bar{A}_y) \\ =\ & (\bar{A}_{x*y} \oslash \bar{A}_{x*z}) \oslash \bar{A}_{z*y} \\ =\ & \bar{A}_{(x*y)*(x*z)} \oslash \bar{A}_{z*y} \\ =\ & \bar{A}_{((x*y)*(x*z))*(z*y)} \\ =\ & \bar{A}_0, \end{aligned}$$

which shows that $X/\bar{A}$ satisfies the condition (a1). Similarly, we can deduce the conditions (a2) and (a3). Let $x, y \in X$ be such that $\bar{A}_x \oslash \bar{A}_y = \bar{A}_0$ and $\bar{A}_y \oslash \bar{A}_x = \bar{A}_0$. Then $\bar{A}_{x*y} = \bar{A}_0 = \bar{A}_{y*x}$, and so $x * y \approx 0 \approx y * x(\bar{A})$. It follows from (b1) that

$$\bar{A}\big(0 * (x * y)^n\big) = \bar{A}\big(0 * ((x * y) * 0)^n\big) = \bar{A}(0)$$

and

$$\bar{A}\big(0 * (y * x)^n\big) = \bar{A}\big(0 * ((y * x) * 0)^n\big) = \bar{A}(0)$$

so that $x \approx y(\bar{A})$. Hence $\bar{A}_x = \bar{A}_y$. We shall state this as a theorem.

Theorem 3.6. *If $\bar{A}$ is a fuzzy ideal of a BCI-algebra X, then $(X/\bar{A}; \oslash, \bar{A}_0)$ is a BCI-algebra.*

We then call $X/\bar{A}$ *fuzzy quotient BCI-algebra* of X induced by the fuzzy ideal $\bar{A}$.

Lemma 3.7. (Xi [7]) *Let $f : X \to Y$ be an epimorphism of BCI-algebras. If $\bar{B}$ is a fuzzy ideal of Y, then the homomorphic preimage of $\bar{B}$ under f, denoted by $f^{-1}(\bar{B})$, is a fuzzy ideal of X.*

Theorem 3.8. (Isomorphism theorem) *Let $f : X \to Y$ be an epimorphism of BCI-algebras and let $\bar{B}$ be a fuzzy ideal of Y. Then $X/\bar{A}$ is isomorphic to $Y/\bar{B}$, where $\bar{A} = f^{-1}(\bar{B})$.*

Proof. Note that $X/\bar{A}$ and $Y/\bar{B}$ are BCI-algebras (see Theorem 3.6 and Lemma 3.7). Let $\Phi : X/\bar{A} \to Y/\bar{B}$ be a mapping defined by $\Phi(\bar{A}_x) = \bar{B}_{f(x)}$, where $x \in X$. Let $x, y \in X$ be such that $\bar{A}_x = \bar{A}_y$. Then

$$\begin{aligned} \bar{B}(0) &= \bar{B}\big(f(0)\big) = f^{-1}(\bar{B})(0) = \bar{A}(0) = \bar{A}\big(0 * (x * y)^n\big) \\ &= f^{-1}(\bar{B})\big(0 * (x * y)^n\big) = \bar{B}\big(f(0 * (x * y)^n)\big) = \bar{B}\big(0 * (f(x) * f(y))^n\big). \end{aligned}$$

Similarly $\bar{B}\big(0 * (f(y) * f(x))^n\big) = \bar{B}(0)$. Hence $f(x) \approx f(y)(\bar{B})$, that is, $\bar{B}_{f(x)} = \bar{B}_{f(y)}$. Therefore Φ is well defined. For any $\bar{A}_x, \bar{A}_y \in X/\bar{A}$, we have

$$\Phi\big(\bar{A}_x \oslash \bar{A}_y\big) = \Phi(\bar{A}_{x*y}) = \bar{B}_{f(x*y)} = \bar{B}_{f(x)*f(y)} = \bar{B}_{f(x)} \oslash \bar{B}_{f(y)} = \Phi(\bar{A}_x) \oslash \Phi(\bar{A}_y).$$

Hence Φ is a homomorphism. Now let $x, y \in X$ be such that $\bar{B}_{f(x)} = \bar{B}_{f(y)}$. Then $f(x) \approx f(y)(\bar{B})$, and so

$$\begin{aligned} \bar{A}(0) &= f^{-1}(\bar{B})(0) = \bar{B}\big(f(0)\big) = \bar{B}(0) = \bar{B}\big(0 * (f(x) * f(y))^n\big) \\ &= \bar{B}\big(f(0) * f((x * y)^n)\big) = \bar{B}\big(f(0 * (x * y)^n)\big) \\ &= f^{-1}(\bar{B})\big(0 * (x * y)^n\big) = \bar{A}\big(0 * (x * y)^n\big), \end{aligned}$$

and $\bar{A}\big(0 * (y * x)^n\big) = \bar{A}(0)$ by the same way. Thus $x \approx y(\bar{A})$, that is, $\bar{A}_x = \bar{A}_y$. This shows that Φ is injective. Clearly Φ is surjective, and the proof is complete. □

Lemma 3.9. (Meng and Xin [6]) *A BCI-algebra X is positive implicative if and only if it satisfies $x * y = \big((x * y) * y\big) * (0 * y)$ for all $x, y \in X$.*

Lemma 3.10. (Liu and Meng [4]) *A fuzzy ideal $\bar{A}$ of a BCI-algebra X is fuzzy positive implicative if and only if it satisfies $\bar{A}(x * y) = \bar{A}\big(((x * y) * y) * (0 * y)\big)$ for all $x, y \in X$.*

Theorem 3.11. *Let $\bar{A}$ be a fuzzy ideal of a BCI-algebra X. Then the fuzzy quotient BCI-algebra $X/\bar{A}$ of X induced by $\bar{A}$ is positive implicative if and only if $\bar{A}$ is a fuzzy positive implicative ideal of X.*

Proof. Assume that the quotient algebra $X/\bar{A}$ is positive implicative. Then

$$\bar{A}_{x*y} = \bar{A}_x \oslash \bar{A}_y = \big((\bar{A}_x \oslash \bar{A}_y) \oslash \bar{A}_y\big) \oslash (\bar{A}_0 \oslash \bar{A}_y) = \bar{A}_{((x*y)*y)*(0*y)},$$

that is, $x * y \approx \big((x * y) * y\big) * (0 * y)(\bar{A})$. It follows from (F1) and (F2) that

$$\bar{A}(x*y) \geq \min\big\{\bar{A}\big((x*y)*(((x*y)*y)*(0*y))\big),\, \bar{A}\big(((x*y)*y)*(0*y)\big)\big\} = \bar{A}\big(((x*y)*y)*(0*y)\big).$$

Obviously $\bar{A}(x * y) \leq \bar{A}\big(((x * y) * y) * (0 * y)\big)$ because $((x * y) * y) * (0 * y) \leq x * y$ by (a1), (b1) and (b2) and $\bar{A}$ is order reversing. Hence $\bar{A}(x * y) = \bar{A}\big(((x * y) * y) * (0 * y)\big)$, and thus $\bar{A}$ is a fuzzy positive implicative ideal of X. Conversely suppose that $\bar{A}$ is a fuzzy positive implicative ideal of X. Using (b2) and Lemma 3.10, we have

$$\begin{aligned} & \bar{A}\big((x * y) * (((x * y) * y) * (0 * y))\big) \\ =\ & \bar{A}\big((x * (((x * y) * y) * (0 * y))) * y\big) \\ =\ & \bar{A}\big((((x * (((x * y) * y) * (0 * y))) * y) * y) * (0 * y)\big) \\ =\ & \bar{A}(0). \end{aligned}$$

Since $(((x * y) * y) * (0 * y)) * (x * y) = 0$, it follows that

$$\bar{A}\big((((x * y) * y) * (0 * y)) * (x * y)\big) = \bar{A}(0).$$

Hence $x * y \approx ((x * y) * y) * (0 * y)(\bar{A})$, and so

$$\bar{A}_x \oslash \bar{A}_y = \bar{A}_{x*y} = \bar{A}_{((x*y)*y)*(0*y)} = ((\bar{A}_x \oslash \bar{A}_y) \oslash \bar{A}_y) \oslash (\bar{A}_0 \oslash \bar{A}_y).$$

It follows from Lemma 3.9 that $X/\bar{A}$ is a positive implicative BCI-algebra. □

Lemma 3.12. (Meng and Xin [5]) *A BCI-algebra X is commutative if and only if it satisfies $x * (x * y) = y * (y * (x * (x * y)))$ for all $x, y \in X$.*

Lemma 3.13. (Jun and Meng [3]) *Let $\bar{A}$ be a closed fuzzy ideal of a BCI-algebra X. Then $\bar{A}$ is fuzzy commutative if and only if it satisfies $\bar{A}(x*(y*(y*x))) \geq \bar{A}(x*y)$ for all $x, y \in X$.*

Theorem 3.14. *Let $\bar{A}$ be a closed fuzzy ideal of a BCI-algebra X. Then the fuzzy quotient BCI-algebra $X/\bar{A}$ of X induced by $\bar{A}$ is commutative if and only if $\bar{A}$ is fuzzy commutative.*

Proof. Assume that $\bar{A}$ is a closed fuzzy commutative ideal of X. Then, by Lemma 3.13, (b2) and (a3), we have

$$\bar{A}\big((x*(x*y))*(y*(y*(x*(x*y))))\big) \geq \bar{A}\big((x*(x*y))*y\big) = \bar{A}\big((x*y)*(x*y)\big) = \bar{A}(0).$$

On the other hand, note that

$$\bar{A}\big((y*(y*(x*(x*y))))*(x*(x*y))\big) = \bar{A}\big((y*(x*(x*y)))*(y*(x*(x*y)))\big) = \bar{A}(0)$$

by (b2) and (a3). Hence $x*(x*y) \approx y*(y*(x*(x*y)))(\bar{A})$, which implies that

$$\bar{A}_x \oslash (\bar{A}_x \oslash \bar{A}_y) = \bar{A}_{x*(x*y)} = \bar{A}_{y*(y*(x*(x*y)))} = \bar{A}_y \oslash (\bar{A}_y \oslash (\bar{A}_x \oslash (\bar{A}_x \oslash \bar{A}_y))).$$

It follows from Lemma 3.12 that $X/\bar{A}$ is commutative. Conversely let $\bar{A}$ be a closed fuzzy ideal of X such that $X/\bar{A}$ is commutative. Then

$$\bar{A}_{x*(x*y)} = \bar{A}_x \oslash (\bar{A}_x \oslash \bar{A}_y) = \bar{A}_y \oslash (\bar{A}_y \oslash (\bar{A}_x \oslash (\bar{A}_x \oslash \bar{A}_y))) = \bar{A}_{y*(y*(x*(x*y)))},$$

and hence $x*(x*y) \approx y*(y*(x*(x*y)))(\bar{A})$. It follows from (b2) and (F1) that

$$\bar{A}\big((x*(y*(y*(x*(x*y)))))*(x*y)\big) = \bar{A}\big((x*(x*y))*(y*(y*(x*(x*y))))\big) = \bar{A}(0) \geq \bar{A}(x*y),$$

so from (F2) that

$$\begin{aligned} & \bar{A}\big(x*(y*(y*(x*(x*y))))\big) \\ \geq\ & \min\big\{\bar{A}\big((x*(y*(y*(x*(x*y)))))*(x*y)\big),\ \bar{A}(x*y)\big\} \\ =\ & \bar{A}(x*y). \end{aligned}$$

Using (a1), (b2) and (a3), we get

$$\big(x*(y*(y*x))\big)*\big(x*(y*(y*(x*(x*y))))\big) \leq 0*(x*y).$$

Since $\bar{A}$ is order reversing, it follows from (F2) and its closedness that

$$\begin{aligned} & \bar{A}\big(x*(y*(y*x))\big) \\ \geq\ & \min\big\{\bar{A}\big((x*(y*(y*x)))*(x*(y*(y*(x*(x*y)))))\big), \\ & \qquad \bar{A}\big(x*(y*(y*(x*(x*y))))\big)\big\} \\ \geq\ & \min\big\{\bar{A}\big(0*(x*y)\big),\ \bar{A}(x*y)\big\} \\ =\ & \bar{A}(x*y). \end{aligned}$$

Hence, by Lemma 3.13, $\bar{A}$ is fuzzy commutative. □

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