BETTER-THAN-OPPONENT——A STOPPING GAME FOR POISSON-ARRIVING OFFERS

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Abstract. Two players observe a Poisson stream of offers. The offers are i.i.d.r.v.s from $U_{[0,1]}$ distribution. Each player wants to accept one offer in the interval $[0, T]$ and aims to select an offer larger than the opponent's one. Offers arrive sequentially and decisions to accept or reject must be made immediately after the offers arrive. Players have equal weights, so if both players want to accept a same r.v., a lottery is used to the effect that each player can get it with probability 1/2. If one player accepts a r.v. and the other doesn't the latter player waits for a larger r.v. appearing before time T. Call this event his win. Each player wants to maximize his probability of winning. The normal form of the game is formulated and the explicit solution is given with Nash values and equilibrium strategies. The bilateral-move version of the game is also analysed and the explicit solution is found. It is shown that the second mover stands unfavorable, on the contrary to the case in multi-round poker.

1 Better-than-opponent—Simultanious-move game. Players I and II must make a decision to accept (A) or reject (R) an offered job at each offer presentation. The offers arrive during time interval $[0, T]$ as a Poisson process with rate λ. The offered jobs have random sizes being i.i.d. random variables from uniform distribution on $[0, 1]$. Whenever an offer with size x arrives it is presented to both players simultaniously, and players must choose either A or R. If players' choice-pair is A-R or R-A then the player choosing A gets x dropping out from the game thereafter, and the other player continues his (or her) one-person game. If the choice-pair is A-A then a lottery is used to the effect that A-R or R-A is enforced to the players with equal probability 1/2. If the choice-pair is R-R, then the current sample x is rejected and the game passes on to the time when a new job arrives next. A player wins if he accepts a r.v. that is larger than the opponent's one, or his opponent fails to accept any r.v. before the deadline T comes. The aim of each player is to find his strategy by following which he maximizes probability of his winning.

Define state (x, t) to mean that (1) both players remain in the game, and (2) an offer with size x has just arrived at time $T - t$(*i.e* the remaining time before deadline is t). Let $\varphi(x, t)(\psi(x, t))$ be the probability of choosing A by player I(II) in state (x, t). Also let $V_i(t, \varphi, \psi)$ be the winning probability for player i at time t left to go, if players employ strategies φ and ψ, respectively. Then the game is described by the following differential equations (if one considers the possible events when the residual time decreases from t to $t - \triangle t$ and takes the limit as $\triangle t \to 0$).

$$\lambda^{-1}(V_1'(t), V_2'(t)) = -(V_1(t), V_2(t)) + \int_0^1 (\overline{\varphi}, \varphi) M(x, t)(\overline{\psi}, \psi)^{\mathbf{T}} dx \tag{1.1}$$

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with the initial conditions $V_1(0) = V_2(0) = 0$ and

$$
(1.2) \qquad M(x,t) = (I) \left\{ \begin{array}{c} \text{R} \\ \text{A} \end{array} \right. \overbrace{\begin{array}{|c|c|} \multicolumn{1}{c}{\text{R}} & \multicolumn{1}{c}{\text{A}} \\ \hline V_1(t), V_2(t) & \overline{g}(x,t), g(x,t) \\ \hline g(x,t), \overline{g}(x,t) & 1/2, 1/2 \\ \hline \end{array}}^{(\text{II})}
$$

Here $V_i(t,\varphi,\psi), \varphi(x,t)$ and $\psi(x,t)$ are abreviated by $V_i(t), \varphi$ and ψ, respectively, and

$$
\begin{aligned}
g(x,t) &= \text{Pr.}\{\text{he wins} \mid \text{one player only chooses } A \text{ in state } (x,t)\} \\
&= e^{-\lambda t} \sum_{j=0}^{\infty} x^j (\lambda t)^j / j! = e^{-\lambda t \overline{x}}. \quad (\overline{x} \text{ means } 1-x)
\end{aligned}
$$

Rewriting (1.1)-(1.2) we get the equations

$$
(1.3) \qquad \left\{ \begin{array}{l} \lambda^{-1} V_1'(t) = -V_1(t) + \int_0^1 [\psi \overline{g} + \overline{\psi} V_1 + \{\psi(\frac{1}{2} - \overline{g}) + \overline{\psi}(g - V_1)\}\varphi] dx, \\ \\ \lambda^{-1} V_2'(t) = -V_2(t) + \int_0^1 [\varphi \overline{g} + \overline{\varphi} V_2 + \{\varphi(\frac{1}{2} - \overline{g}) + \overline{\varphi}(g - V_2)\}\psi] dx, \end{array} \right.
$$

which reflects the symmetry of the players' roles in the game. The problem here is to find the solution to $(V_1(T,\varphi,\psi), V_2(T,\varphi,\psi)) \underset{(\varphi,\psi)}{\to} \text{Nash eq.}$

The explicit solution to the simultanious-move version of the game (1.1) $\sim$ (1.3) is given in Section 1, and that to the bilateral-move version is given in Section 2.

The optimal stopping games have various phases according to the various essentials of (1) zero-sum or nonzero-sum, (2) information pattern under which players choose their decision, and (3) the objectives that players attempt. The problem we consider in this paper belong to a class of best-choice problems combined with continuous-time sequential games. In the recent works by the present author [9, 10], the game under ENV (= expected-net-value)-maximization are investigated, whereas [11] and the present article discusses the games under WP (= winning- probability)-maximization. Other recent works related to this area of problems are [1 $\sim$ 5, 6, 7, 9 and 12]. Also a recent work for the optimal stopping games in various phases can be found in [8]. Moreover, some open problems in this area are mentioned in Section 3 of the present paper.

The following lemma plays a fundamental role to derive the main results in this article.

Lemma 1.1 (Garnaev[3]) *Let $y' = a(t) + b(t)y$, with $y(0) = 0$. Then*

$$
y(t) \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} 0, \forall t \in (0,T), \qquad \text{if } a(t) \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} 0, \forall t \in (0,T]
$$

This result follows from the fact that

$$y(t) = \int_0^t a(\tau)\exp(\int_\tau^t b(s)ds)d\tau.$$

Also the next lemma helps to derive the key equations (1.11) and (1.12) which will appear in Theorem 1.

Lemma 1.2 *Let* $0 < V(t) < \frac{1}{2}$, $\forall t \in [0, T]$. *For each* $x \in [0, 1]$ *and* $t \in (0, T]$, *the bimatrix game*

$M(x,t) =$

	R	A
R	$V(t), V(t)$	$\overline{g}(x,t), g(x,t)$
A	$g(x,t), \overline{g}(x,t)$	$1/2, 1/2$

has the equilibria:

If	Common eq. Value	Common eq. Strategy
(i) $0 < x < [1 + (\lambda t)^{-1} \log V(t)]^+$	$V(t)$	choose R
(ii) $[1 - (\lambda t)^{-1} \log 2]^+ < x < 1$	$1/2$	choose A
(iii) otherwise	$\frac{1}{2} - \frac{(1/2 - g(x,t))^2}{1/2 - V(t)}$	randomize R and A with prob. $\varphi = \frac{g(x,t) - V(t)}{1/2 - V(t)}$, for A

(c.f. If (iii) applies, the choice-pairs R-A and A-R give pure-strategy eq. also.)

Proof. First note that

$$(1.4) \qquad \begin{cases} V(t) > g(x,t) \Longleftrightarrow x < [1 + (\lambda t)^{-1} \log V]^+ \\ \bar{g}(x,t) > 1/2 \Longleftrightarrow x < [1 - (\lambda t)^{-1} \log 2]^+ \end{cases}$$

and $V(t) < 1/2$, Then evidently R-R [A-A] gives a unique equilibrium if (i) [(ii)] apllies. If (iii) applies a mixed-strategy equilibrium exists which is given by solving

$$\overline{\varphi} V(t) + \varphi \overline{g}(x,t) = \overline{\varphi} g(x,t) + \frac{1}{2}\varphi = \text{Eq. payoff to II},$$

$$\overline{\psi} V(t) + \psi \overline{g}(x,t) = \overline{\psi} g(x,t) + \frac{1}{2}\psi = \text{Eq. payoff to I}.$$

This gives the statements of the lemma. □

Let denote, by $\varphi_*(x,t)$-$\psi_*(x,t)$ and $V_{1*}(t)$-$V_{2*}(t)$ the strategy-pair and game values in Nash equilibrium, respectively, for the game described by (1.1)-(1.2). From the symmetry of the players' role in the game, we can take $V_{1*}(t) = V_{2*}(t) = V_*(t)$, say.

Therefore $0 < V_*(t) < \frac{1}{2}$, $\forall t > 0$, is evident.

We consider the following three cases

Case 1° $0 < \lambda T < \log 2$.

The condition (ii) in Lemma 1.2 applies, and the choice-pair A-A is selected. Since the time left is too short, any offer, however small, are accepted by both players.

Case 2° $\log 2 < \lambda T < t_0$, where $t_0 \in (\log 2, \infty)$, is a unique root of the equation $-\log V_*(\lambda^{-1}t) = t$.

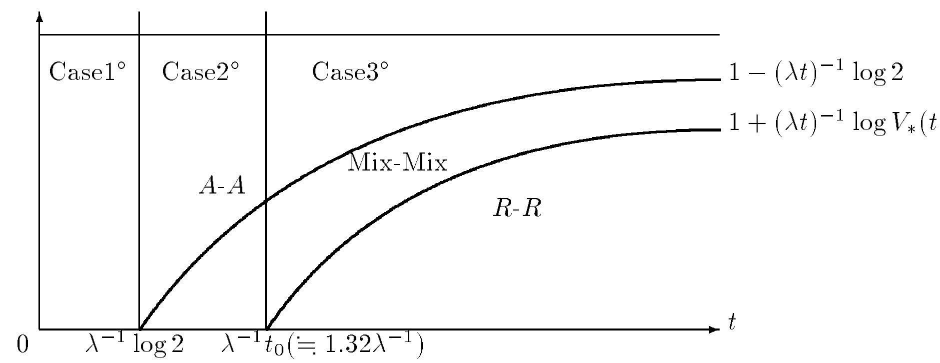
The conditions (ii)-(iii) in Lemma 1.2 apply. The choice-pair Mix-Mix (A-A) is selected if $x < (>) [1 - (\lambda t)^{-1} \log 2]^+$. Here Mix means to randomize A and R with respective probabilities $\varphi_*(x,t)(\psi_*(x,t))$ and $\overline{\varphi}_*(x,t)(\overline{\psi}_*(x,t))$ for player I(II).

Case 3° $\lambda T > t_0$.

The conditions (i)$\sim$(iii) in Lemma 1.2 apply. Choice-pairs R-R, Mix-Mix, and A-A are selected in this order as $x \in [0,1]$ increases. Since the time left is long, small offers are rejected by both players expecting to face a larger r.v. in the future.

Figure 1 shows devision of (x,t) region by the equilibrium strategy-pair (1.5) into the three subregions. The appropriateness of the figure is assured later in Corollaries 1.1 and 1.2.

Figure 1: Equibrium strategy-pairs in Theorem 1



We prove

Theorem 1. *The strategy-pair $\varphi_*(x,t)$-$\psi_*(x,t)$ with*

$$\varphi_*(x,t) = \psi_*(x,t) = \begin{cases} 0, & \text{if} \quad 0 \le x < [1 + (\lambda t)^{-1} \log V_*(t)]^+ \\ 1, & \text{if} \quad [1 - (\lambda t)^{-1} \log 2]^+ < x \le 1 \\ (1/2 - V_*(t))^{-1}(g(x,t) - V_*(t)), & \text{if otherwise} \end{cases} \tag{1.5}$$

is in Nash equilibrium and the game has common eq. value $V_*(T)$, which is given by the solution of the differential equation :

In Case 1°,

$$\lambda^{-1} V_*'(t) = \frac{1}{2} - V_*(t), \quad \text{with} \quad V_*(0) = 0; \tag{1.6}$$

In Case 2°,

$$\lambda^{-1}V_*'(t) = \frac{1}{2} - V_*(t) - \left(\frac{1}{2} - V_*(t)\right)^{-1} a(t), \quad \text{with } V_*(\lambda^{-1}\log 2) = 1/4, \tag{1.7}$$

where $a(t) \equiv \frac{1}{4} - (\lambda T)^{-1}\left(\frac{1}{4}\log 2 + \frac{3}{8} - e^{-\lambda t} + \frac{1}{2}e^{-2\lambda t}\right)$;

In Case 3°,

$$tV_*'(t) = \frac{1}{2} + \left(\frac{1}{2} - V_*(t)\right)\left(\frac{1}{2} - \log V_*(t)\right) + \frac{1}{4}\left(\frac{1}{2} - V_*(t)\right)^{-1}\log(2V_*(t)), \quad \text{with } V_*(\lambda^{-1}t_0) = e^{-t_0}. \tag{1.8}$$

Proof. We want to prove that $\varphi_* - \psi_*$ given by (1.5) satisfies

$$V(t,\varphi,\psi_*) \le V(t,\varphi_*,\psi_*), \quad \forall \text{ strategy } \varphi \text{ for I}, \tag{1.9}$$

and

$$V(t,\varphi_*,\psi) \le V(t,\varphi_*,\psi_*), \quad \forall \text{ strategy } \psi \text{ for II}. \tag{1.10}$$

We temporally use the abreviated notations $V(t,\varphi,\psi_*) = V(t)$ and $V(t,\varphi_*,\psi_*) = V_*(t)$ for the proof of (1.9). Then from (1.3) we have

$$\lambda^{-1}V'(t) = -V + \int_0^1 \left[\psi_*\overline{g} + \overline{\psi_*}V + \left\{\psi_*\left(\frac{1}{2} - \overline{g}\right) + \overline{\psi_*}(g - V)\right\}\varphi\right]dx,$$

and

$$\lambda^{-1}V_*'(t) = -V_* + \int_0^1 \left[\psi_*\overline{g} + \overline{\psi_*}V_* + \left\{\psi_*\left(\frac{1}{2} - \overline{g}\right) + \overline{\psi_*}(g - V_*)\right\}\varphi_*\right]dx. \tag{1.11}$$

(In the r.h.s. of these equations, the arguments t and (x,t) are ommitted. This way of the simplified descriptions are often used in this paper)

By subtracting side-by-side and doing some rearrangement, we obtain

$$\lambda^{-1}(V(t)-V_*(t))' = -(V-V_*)\int_0^1 (1-\overline{\varphi}\overline{\psi_*})dx + \int_0^1 \left\{\overline{\psi_*}(g - V_*) + \psi_*\left(\frac{1}{2} - \overline{g}\right)\right\}(\varphi-\varphi_*)dx \tag{1.12}$$

Let $B(t)$ denote the second term in the r.h.s. of (1.12), that is,

$$B(t) \equiv \int_0^1 \left\{\overline{\psi_*}(g - V_*) + \psi_*\left(\frac{1}{2} - \overline{g}\right)\right\}(\varphi - \varphi_*)dx \tag{1.13}$$

Then in Case 1° we have, since $\varphi_* = \psi_* = 1$, and $\overline{g} < 1/2$,

$$B(t) = -\int_0^1 \left(\frac{1}{2} - \overline{g}\right)\overline{\varphi}dx \le 0, \quad \forall \text{ strategy } \varphi \text{ for } I.$$

In Case 2° we have

$$B(t) = \int_0^{1-(\lambda t)^{-1}\log 2} \left\{\overline{\psi_*}(g - V_*) + \psi_*\left(\frac{1}{2} - \overline{g}\right)\right\}(\varphi-\varphi_*)dx + \int_{1-(\lambda t)^{-1}\log 2}^1 \left(\frac{1}{2} - \overline{g}\right)(-\overline{\varphi})dx$$

and

$$\overline{\psi_*}(g - V_*) + \psi_* \left(\frac{1}{2} - \overline{g}\right) = \left(\frac{1}{2} - V_*\right)^{-1} \left\{\left(\frac{1}{2} - g\right)(g - V_*) + (g - V_*)\left(\frac{1}{2} - \overline{g}\right)\right\} = 0.$$

Therefore $B(t) \leq 0$, for any strategy φ for I.

In Case 3°, Eq.(1.12) becomes

$$B(t) = \int_0^{1+(\lambda t)^{-1}\log V_*} (g - V_*)\varphi dx + \int_{1+(\lambda t)^{-1}\log V_*}^{1-(\lambda t)^{-1}\log 2} 0 dx$$
$$+ \int_{1-(\lambda t)^{-1}\log 2}^{1} \left(\frac{1}{2} - \overline{g}\right)(-\overline{\varphi})dx \leq 0 + 0 + 0 = 0, \qquad \forall\varphi.$$

So in all cases, " the constant term" $B(t)$ in the linear differential equation (1.11) of $V(t) - V_*(t)$ is non-positive. This implies, from Lemma 1.1, that (1.9) is valid.

Symmetry of the game leads to the similar fact (1.10) in the analogous way. This completes the proof of the first half of the theorem.

Now it remains to show the second half. The above arguments combined with (1.11) with $\varphi_* = \psi_*$ on th r.h.s. give the differential equation

$$\lambda^{-1}V_*'(t) = \left(\frac{1}{2} - V_*\right)\int_0^1 (2\varphi_* - \varphi_*^2)dx \left[\equiv \left(\frac{1}{2} - V_*\right) B_*(t), \quad \text{say.}\right] \tag{1.14}$$

Computation shows that

$B_*(t) = 1,$ in Case 1°;

$$\begin{aligned} B_*(t) &= \int_0^{1-(\lambda t)^{-1}\log 2} \left\{1 - \left(\frac{1}{2} - V_*\right)^{-2}\left(\frac{1}{2} - g\right)^2\right\} dx + \int_{1-(\lambda t)^{-1}\log 2}^{1} dx \\ &= 1 - \left(\frac{1}{2} - V_*\right)^{-2} a(t), \end{aligned}$$

where

$$\begin{aligned} a(t) &= \int_0^{1-(\lambda t)^{-1}\log 2} \left(\frac{1}{2} - g\right)^2 dx \\ &= \frac{1}{4} - (\lambda t)^{-1}\left\{\frac{1}{4}\log 2 + \frac{3}{8} - e^{-\lambda t} + \frac{1}{2}e^{-2\lambda t}\right\}, \quad \text{in Case } 2° \end{aligned} \tag{1.15}$$

and finally in Case 3°,

$$\begin{aligned} B_*(t) &= \int_{1+(\lambda t)^{-1}\log V_*}^{1-(\lambda t)^{-1}\log 2} \left\{1 - \left(\frac{1/2 - g}{1/2 - V_*}\right)^2\right\} dx + \int_{1-(\lambda t)^{-1}\log 2}^{1} dx \\ &= -(\lambda t)^{-1}\log V_* - \left(\frac{1}{2} - V_*\right)^{-2} b(t), \end{aligned}$$

where

$$\begin{aligned} b(t) &= \int_{1+(\lambda t)^{-1}\log V_*}^{1-(\lambda t)^{-1}\log 2} \left(\frac{1}{2}-g\right)^2 dx \\ &= -(\lambda t)^{-1}\left\{\frac{1}{4}\log(2V_*)+\frac{1}{2}\left(\frac{1}{2}-V_*\right)+\frac{1}{2}\left(\frac{1}{2}-V_*\right)^2\right\}, \end{aligned}$$

and hence

$$B_*(t) = (\lambda t)^{-1}\left[\frac{1}{2}-\log V_* + \frac{1}{4}\left(\frac{1}{2}-V_*\right)^{-2}\log(2V_*)+\frac{1}{2}\left(\frac{1}{2}-V_*\right)^{-1}\right].$$

These expressions of $B_*(t)$, combined with (1.14), give the equations (1.6) $\sim$ (1.8). □

An equivalent empression of (1.8) is

$$(1.8') \qquad \int_{e^{-t_0}}^{V_*(t)} \left[\frac{3}{4}-\frac{1}{2}y+\left(\frac{1}{2}-y\right)^{-1}\log\left(2^{1/4}y^{y\overline{y}}\right)\right]^{-1} dy = \log\left(\frac{\lambda t}{t_0}\right)$$

for $t > \lambda^{-1}t_0$, where to will be given in Corollary 1.1, $e^{-t_0} \fallingdotseq e^{-1.32} \fallingdotseq 0.267$, and the integrand in the l.h.s. is positive for $e^{-t_0} < y < 1/2$.

Corollary 1.1 *Differential equations* (1.6) *and* (1.7) *give after integration,*

$$(1.16) \qquad V_*(t) = \frac{1}{2}(1-e^{-\lambda t}), \quad \text{for} \quad 0 \le \lambda t \le \log 2,$$

$$(1.17) \quad V_*(t) = \frac{1}{2} - \left[\frac{5}{4}e^{-2\lambda t}-\frac{1}{4}+2e^{-2\lambda t}\int_{\log 2}^{\lambda t}\tau^{-1}\left\{\left(\frac{1}{4}\log 2+\frac{3}{8}\right)e^{2\tau}-e^{\tau}+\frac{1}{2}\right\}d\tau\right]^{1/2},$$

$$\text{for} \quad \log 2 \le \lambda t \le t_0.$$

The value t_0 is determined by $t_0 = \log u_0$ where $u_0(\fallingdotseq 3.74$ and hence $t_0 = 1.32)$ is a unique root of the equation

$$(1.18) \qquad \int_{\log 2}^{\log u} \tau^{-1}\left\{\left(\frac{1}{4}\log 2+\frac{3}{8}\right)e^{2\tau}-e^{\tau}+\frac{1}{2}\right\}d\tau = \frac{1}{4}u^2-\frac{1}{2}u-\frac{1}{8}.$$

Proof. (1.16) is evident from (1.6). We derive (1.17) as follows: By considering $z(t) = \left(\frac{1}{2}-V_*(t)\right)^2$. Eq.(1.7) becomes a linear differential equation

$$z'(t)+2\lambda z(t) = -2\lambda a(t), \qquad \text{with} \quad z(\lambda^{-1}\log 2) = 1/16.$$

and integration gives

$$\begin{aligned} z(t) &= e^{-2\lambda t}\left[\frac{1}{4}-2\lambda\int_{\lambda^{-1}\log 2}^{t} a(\tau)e^{2\lambda\tau}d\tau\right] \\ &= \frac{5}{4}e^{-2\lambda t}-\frac{1}{4}+2e^{-2\lambda t}\int_{\log 2}^{\lambda t}\tau^{-1}\left\{\left(\frac{1}{4}\log 2+\frac{3}{8}\right)e^{2\tau}-e^{\tau}+\frac{1}{2}\right\}d\tau. \end{aligned}$$

Since $V_*(t) = \frac{1}{2} - \sqrt{z(t)}$, Eq.(1.7) follows. Substituting $V_*(\lambda^{-1}t_0) = e^{-t_0}$ into (1.17) we have

$$\int_{\log 2}^{t_0} \tau^{-1}\left\{\left(\frac{1}{4}\log 2 + \frac{3}{8}\right)e^{2\tau} - e^{\tau} + \frac{1}{2}\right\}d\tau = \frac{1}{4}e^{2t_0} - \frac{1}{2}e^{t_0} - \frac{1}{8},$$

or equivalently, $u_0 = e^{t_0}$ satisfies (1.18).

Finally we show that (1.18) has a unique root in $u \in (2, \infty)$. Let

$$f(u) \equiv (\text{1.h.s. minus r.h.s.}) \text{ of } (1.18).$$

Then $f(2) = 1/8$ and $f'(u) = (u\log u)^{-1}\left\{\left(\frac{1}{4}\log 2 + \frac{3}{8}\right)u^2 - u + \frac{1}{2} - \frac{1}{2}(u-1)u\log u\right\} = (u\log u)^{-1}g(u)$, say. Besides $g(2) = 0$, $g'(2) = -\frac{1}{2}\log 2 < 0$, $g'(u) = \left(\frac{1}{2}\log 2 + \frac{1}{4}\right)u - \left(u - \frac{1}{2}\right)\log u - \frac{1}{2}$, $g''(u) = \frac{1}{2u} - \log u - \left(\frac{3}{4} - \frac{1}{2}\log 2\right) < 0$, for $u > 2$, and therefore $g(u) \leq g(2) = 0$. This implies $f'(u) < 0$ for $u > 2$, and hence the equation $f(u) = 0$ has a unique root in $u \in (2, \infty)$. This completes the proof of Corollary 1.1. □

The next corollary gives the approproateness of Figure 1.

Corollary 1.2 *Both of $V_*(t)$ and $t^{-1}\log V_*(t)$ are increasing and concave in $t > 0$.*

Proof. *Increasingness of $V_*(t)$* is intuitively evident, but it is also clear in our approach, since $V_*'(t) > 0$, $\forall t > 0$, from (1.14).

Concavity of $V_*(t)$: In Case 1°, $V_*''(t) < 0$ by (1.6). In Case 2°,

$$\lambda^{-1}V_*''(t) = V_*' - \left(\frac{1}{2} - V_*\right)^{-1} a'(t) - \left(\frac{1}{2} - V_*\right)^{-2} a(t)$$

from(1.7). All three terms in the r.h.s. are negative since, by (1.15), $a(t)$ positive and both of $1 - (\lambda t)^{-1}\log 2$ and $\left(\frac{1}{2} - g\right)^2$ are increasing. In Case 3°, by differentiating $tV_*'(t)$ in (1.8) and doing some rearrangement, we get

$$tV_*''(t) = \frac{1}{4}V_*'\left(\frac{1}{2} - V_*\right)^{-2} k(V_*),$$

where

$$k(V_*) = \frac{1}{2} + \log 2 + 2(1 - 2V_* + 2V_*^2)\log V_* - 2V_*^2.$$

This function $k(V)$ has the property that $k(0+) = -\infty$, $k\left(\frac{1}{2} - 0\right) = 0$, and

$$\frac{1}{2}k'(V) = (V^{-1} - 2) - 4\left(\frac{1}{2} - V\right)\log V > 0, \quad \text{for } 0 < V < \frac{1}{2}.$$

Thus $tV_*''(t)$ and hence $V_*''(t)$ is negative.

Increasingness of $u(t) = t^{-1}\log V_*(t)$:

$$u'(t) = V_*'/(tV_*) - t^{-2}\log V_* > 0 + 0 = 0.$$

Concavity of $u(t)$: Differentiate both sides of $tu = \log V_*$, twice. Then we get

$$\begin{aligned} tu''(t) &= V_*^{-2}\{V_*''V_* - (V_*')^2\} - 2u' \\ &= \frac{V_*''}{V_*} - \left(\frac{V_*'}{V_*}\right)^2 - 2u' < 0 + 0 + 0 = 0. \end{aligned}$$

□

Remark 1. An immediate consequence of Theorem 1 and Corollary 1.2 is as follows: Evidently $\alpha \equiv \lim_{t\to\infty} V_*(t)$ exists and is equal to $1/2$, provided that $\lim_{t\to\infty} t^{-1}V_*(t) = 0$. Because, from (1.8), α should satisfy the equation

$$-\frac{1}{4}\left(\frac{1}{2}-\alpha\right)^{-1}\log(2\alpha) - \alpha\log\alpha = \frac{3}{4} - \frac{1}{2}(\alpha + \log\alpha), \quad \text{for } \ 0 < \alpha \le \frac{1}{2}.$$

It is easy to show that the above equation has a unique root $\alpha = 1/2$.

Corollary 1.3 *The time s until the first acceptance of an offer is made has the defective p.d.f.*

$$g(s) = \lambda p(s) \exp\left[-\lambda \int_0^s p(w)dw\right], \tag{1.19}$$

where
$$p(s) \equiv \int_0^1 \left\{2\varphi_*(x, T-s) - \varphi_*^2(x, T-s)\right\} dx.$$

The probability that both players lose is $1 - \int_0^T g(s)ds$.

Proof. Note that $p(s)$ is the probability of acceptance of an offer in state $(x, T-s)$. Let $0 < s_1 < s_2 < \cdots < s_{n-1}$ be the ellapsed times from T of the succesive offers arrived and were rejected. Then we have

$$g(s) = \lambda e^{-\lambda s} \sum_{n=1}^{\infty} \int_{0<s_1<\cdots} \int_{<s_{n-1}<s} p(s) \prod_{j=1}^{n-1} (1 - p(s_j))(\lambda ds_j),$$

which gives (1.19), if we apply the identity

$$\int_{0<s_1<\cdots} \int_{<s_{n-1}<s} \prod_{j=1}^{n-1} f(s_j)ds_j = \frac{1}{(n-1)!}\left[\int_0^s f(w)dw\right]^{n-1}, \quad \forall f(\cdot).$$ □

2 Bilateral move Game We shall discuss in this section about the bilateral-move version of the game. In each state (x, t) player's moves are split into two steps. Player I first decides to choose either R or A, and then player II, after being informed of the choice made by I, decides to choose either R or A. The rest of the game rule is the same as in the simultanious-move version. So the game in state (x, t) is described by.

Players	1 st step	2nd step	Payoffs
I: (x,t)	R	R	$(V_1(t-\triangle t), V_2(t-\triangle t))$
II: (x,t)	R	A	$(\overline{g}(x,t), g(x,t))$
	A	R	$(g(x,t), \overline{g}(x,t))$
	A	A	$(1/2, 1/2)$

Let $\psi_R(x,t)(\psi_A(x,t))$ be the probability that II chooses A after he is informed of the fact that I has chosen $R(A)$. Also let $\varphi(x,t)$ be the probability that I chooses A. Denote $V_1(t,\varphi,\psi_{R_*},\psi_{A_*})$ and $V_i(t,\varphi_*,\psi_{R_*},\psi_{A_*})$ simoly by $V_1(t)$ and $V_{i_*}(t)$ respectively. Then player II's behavior after being informed of I's $\left\{\begin{array}{c} R \\ A \end{array}\right\}$ is evidently to choose $A(R)$ if $g(x,t) > (<) \left\{\begin{array}{c} V_{2_*}(t) \\ 1/2 \end{array}\right\}$. So we have

$$\psi_{R_*}(x,t) = I(g(x,t) > V_{2_*}(t) \qquad \text{and} \qquad \psi_{A_*}(x,t) = I\left(g(x,t) > \frac{1}{2}\right) \tag{2.1}$$

Also we obtain

$$\lambda^{-1}V_1'(t) + V_1(t) = \int_0^1 \left\{\overline{\varphi}(\overline{\psi}_{R_*}V_1 + \psi_{R_*}\overline{g}) + \varphi\left(\overline{\psi}_{A_*}g + \frac{1}{2}\psi_{A_*}\right)\right\}dx \tag{2.2}$$

$$\lambda^{-1}V_{1_*}'(t) + V_{1_*}(t) = \int_0^1 \left\{\overline{\varphi_*}(\overline{\psi}_{R_*}V_{1_*} + \psi_{R_*}\overline{g}) + \varphi_*\left(\overline{\psi}_{A_*}g + \frac{1}{2}\psi_{A_*}\right)\right\}dx \tag{2.3}$$

$$\lambda^{-1}V_{2_*}'(t) + V_{2_*}(t) = \int_0^1 \left\{\overline{\varphi_*}(\overline{\psi}_{R_*}V_{2_*} + \psi_{R_*}g) + \varphi_*\left(\overline{\psi}_{A_*}\overline{g} + \frac{1}{2}\psi_{A_*}\right)\right\}dx \tag{2.3}$$

with the initial conditions $V_1(0) = V_{i_*}(0) = 0,\ i = 1,2,\cdots$. In the r.h.s. of (2.2) $\sim$ (2.4) simplified notations for $\varphi(x,t), V_1(t)$ ect. are used.

Now we obtain after some algebra,

$$\begin{aligned}\lambda^{-1}\left(V_1(t) - V_{1_*}(t)\right)' &= -\left(V_1 - V_{1_*}\right)\int_0^1 \left(1 - \overline{\varphi}\overline{\psi}_{R_*}\right)dx \\ &+ \int_0^1 \left\{g - V_{1_*} + (V_{1_*} - \overline{g})\,\psi_{R_*} + \left(\frac{1}{2} - g\right)\psi_{A_*}\right\}(\varphi - \varphi_*)dx\end{aligned} \tag{2.5}$$

from (2.2) and (2.3);

$$\begin{aligned}\lambda^{-1}\left(V_{1_*}(t) - V_{2_*}(t)\right)' &= -\left(V_{1_*} - V_{2_*}\right)\int_0^1 \left(1 - \overline{\varphi}\overline{\psi}_{R_*}\right)dx \\ &+ \int_0^1 (\overline{g} - g)(\overline{\varphi_*}\psi_{R_*} - \varphi_*\overline{\psi}_{A_*})dx\end{aligned} \tag{2.6}$$

from (2.3) and (2.4).

As in the previuos section we consider the following three cases:

Case 1^+ $\quad 0 < \lambda T < \log 2$.
This case is the same as Case $1°$.

Case 2^+ $\quad \log 2 < \lambda T < t_1$, where $t_1 \in (\log 2, \infty)$ is a unique root of the equation $-\log V_{2_*}(\lambda^{-1}t) = t$.

Case 3^+ $\quad \lambda T > t_1$.

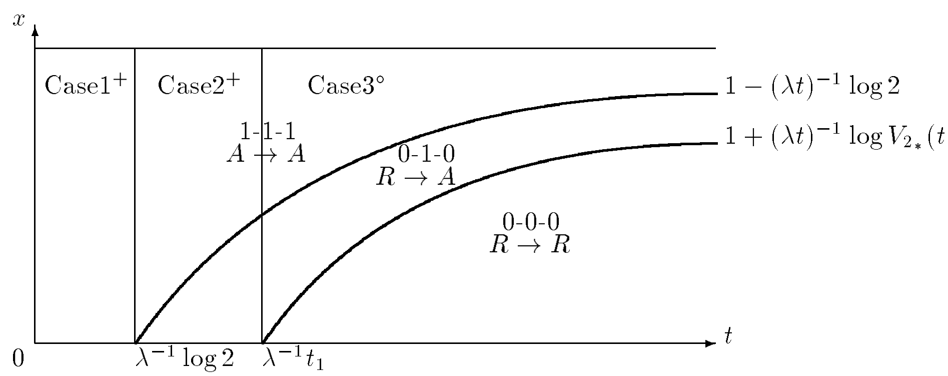
We consider the strategy-triple

$$\varphi_*(x,t) = \psi_{A_*}(x,t) = I(g(x,t) > 1/2) = I\left(x > \left[1-(\lambda t)^{-1}\log 2\right]^+\right), \tag{2.7}$$

$$\psi_{R_*}(x,t) = I(g(x,t) > V_{2_*}(t)) = I\left(x > \left[1+(\lambda t)^{-1}\log V_{2_*}(t)\right]^+\right)$$

which plays a central role hereafter. The (x,t) region is devided into the "upper" "middle" and "lower" subregions as is shown by Figure 2. In the figure the triple values $\varphi'_* - \psi_{R*} - \psi_{A*}$ and the bilateral choices made by players in each subregions are mentioned. The $R \to A$, for example, means the bilateral choice first I' s R, then followed by II's A. The appropriateness of Figure 2 will be made clear later. We find later that t_1 is approximately 1.11.

Figure 2: Division of (x,t) region by the strategy-triple(2.7)



Lemma 2.1 *For the strategy-triple* (2.7), *we have*

$$V_{2_*}(t) = (<)V_{1_*}(t). \qquad \text{in Case}^+(\text{Case}^+\text{and } 3^+),$$

and therefore $0 < V_{2_*}(t) < 1/2, \ \forall t > 0.$

Proof. Denote by $D_*(t)$ the second integral in the r.h.s.of (2.6). Then the lemma is proven from Lemma 1.1, if we show that $D_*(t) > 0, \ \forall t > 0$. Eq. (2.7) together with Figure 2 yield

$$\begin{aligned} D_*(t) &\equiv \int_0^1 (1-2g)(\overline{\varphi}_*\psi_{R_*} - \varphi_*\overline{\psi}_{A_*})dx = 0, && \text{in Case1}^+; \\ &= \int_0^{1-(\lambda t)^{-1}\log 2} (1-2g)dx > 0, && \text{in Case2}^+; \\ &= \int_{1+(\lambda t)^{-1}V_{2*}}^{1-(\lambda t)^{-1}\log 2} (1-2g)dx > 0, && \text{in Case3}^+; \end{aligned} \tag{2.8}$$

This implies that $V_{2_*}(t) = V_{1_*}(t)$ in Case 1^+ and $V_{2_*}(t) < V_{1_*}(t)$ in Case 2^+ and 3^+ □

Lemma 2.2 *For the strategy-triple* (2.7), *we find that*

$$V_{2_*}(t) = \frac{1}{2}(1-e^{-\lambda t}), \qquad \text{in Case1}^+; \tag{2.9}$$

$$V_{2_*}(t) = e^{-\lambda t}\left[\frac{1}{2} + \int_{\log 2}^{\lambda t} \tau^{-1}\left\{\frac{1}{2}(1+\log 2)e^{\tau} - 1\right\}d\tau\right], \qquad \text{in Case2}^+, \tag{2.10}$$

and in Case 3^+, $V_{2_*}(t)$ satisfies the equation

$$V'_{2_*}(t) = t^{-1}\left\{\frac{1}{2}(1+\log 2) - V_{2_*}(1-\log V_{2_*})\right\}, \quad \text{with } V_{2_*}(t_1) = e^{-t_1} \fallingdotseq 0.33, \tag{2.11}$$

or equivalently

$$\int_{V_{2_*}(\lambda^{-1}t_1)}^{V_{2_*}(t)} \frac{dy}{\log\{k(y/e)^y\}} = \log\left(\frac{\lambda t}{t_1}\right) \tag{2.11$'$}$$

where $k \equiv \sqrt{2}e^{1/2} \fallingdotseq 2.33164$. Here $t_1 = 1.11$ is a unique root in $t \in (\log 2, \infty)$ of the equation

$$\int_{\log 2}^{\lambda t} \tau^{-1}\left\{\frac{1}{2}(1+\log 2)e^{\tau} - 1\right\} d\tau = \frac{1}{2}. \tag{2.12}$$

The integrand in the l.h.s. of (2.11$'$) is positive and convexly increasing from $2/(1+\log 2)$ at $y = 0$ t_0 ∞ at $y = 1/2$.

Proof. Eq. (2.4) becomes

$$\lambda^{-1}V'_{2_*}(t) = -V_{2_*}\int_0^1 (1-\overline{\varphi}_*\overline{\psi}_{R_*})dx + \int_0^1 \left\{\overline{\varphi}_*\psi_{R_*}g + \varphi_*\left(\overline{\psi}_{A_*}\overline{g} + \frac{1}{2}\psi_{A_*}\right)\right\} dx \tag{2.13}$$

and let $E_*(t)$ and $F_*(t)$ be the first and second integral in the r.h.s. Then

$$\begin{aligned}
(2.14)\, E_*(t) &\equiv \int_0^1 (1-\overline{\varphi}_*\overline{\psi}_{R_*})dx \\
&= 1, \quad \text{in Cases 1 and 2;} \\
&= \int_{1+(\lambda t)^{-1}\log V_{2_*}}^{1-(\lambda t)^{-1}\log 2} dx + \int_{1-(\lambda t)^{-1}\log 2}^{1} dx = -(\lambda t)^{-1}\log V_{2_*}, \qquad \text{in Case } 3^+,
\end{aligned}$$

and

$$\begin{aligned}
F_*(t) &\equiv \int_0^1 \left\{\overline{\varphi}_*\psi_{R_*}g + \varphi_*\left(\overline{\psi}_{A_*}\overline{g} + \frac{1}{2}\psi_{A_*}\right)\right\} dx, \\
&= \frac{1}{2}, \quad \text{in Case } 1^+; \\
&= \int_0^{1-(\lambda t)^{-1}\log 2} g\,dx + \int_{1-(\lambda t)^{-1}\log 2}^{1} 1/2\,dx, \\
&= (\lambda t)^{-1}\left\{\frac{1}{2}(1+\log 2) - e^{-\lambda t}\right\}, \quad \text{in Case } 2^+;
\end{aligned}$$

and

$$\begin{aligned}
&= \int_{1+(\lambda t)^{-1}\log V_{2_*}}^{1-(\lambda t)^{-1}\log 2} g\,dx + \int_{1-(\lambda t)^{-1}\log 2}^{1} 1/2\,dx \\
&= (\lambda t)^{-1}\left\{\frac{1}{2}(1+\log 2) - V_{2_*}\right\}, \quad \text{in Case } 3^+.
\end{aligned}$$

Hence by substituting this expression into (2.13), we get the differential equation

$$\lambda^{-1}V_{2_*}'(t) = -V_{2_*} + 1/2 \qquad \text{in Case } 1^+; \tag{2.15}$$

$$= -V_{2*} + (\lambda t)^{-1}\left\{\frac{1}{2}(1+\log 2) - e^{-\lambda t}\right\}, \quad \text{in Case } 2^+; \tag{2.16}$$

$$= (\lambda t)^{-1}\left\{\frac{1}{2}(1+\log 2) - V_{2_*}'(1-\log V_{2_*})\right\}, \quad \text{in Case } 3^+; \tag{2.17}$$

Note that the r.h.s. of the above expression is continuously connected in case-to-case. Integration of (2.15)-(2.16) give (2.9)-(2.10). The bordering value t_1 deviding Case 2^+ and 3^+ is obtained from $V_{2_*}(\lambda^{-1}t_1) = e^{-t_1} \fallingdotseq 0.3296$ in (2.10) with $t = \lambda^{-1}t_1$ and the result is (2.12). The fact that (2.12) has a unique root is easy to prove. This completes the proof of Lemma 2.2. □

Now we proceed to stating the main result in Section 2.

Theorem 2. *The strategy-triple* (2.7) *is in equilibrium, that is*

$$V_1(t,\varphi,\psi_{R_*},\psi_{A_*}) \le V_1(t,\varphi_*,\psi_{R_*},\psi_{A_*}). \qquad \forall\varphi \tag{2.18}$$

$$V_2(t,\varphi_*,\psi_R,\psi_A) \le V_2(t,\varphi_*,\psi_{R_*},\psi_{A_*}). \qquad \forall\psi_R{-}\psi_A, \tag{2.19}$$

and gives the eq. values $(V_{1_*}(t), V_{2_*}(t))$.

Proof. (2.18) is, by our abreviated notation, (see the begining of Section 2)

$$V_1(t) \le V_{1_*}(t), \quad \forall\psi. \tag{2.18$'$}$$

Let $D(t)$ be the second term in the r.h.s. of (2.5) (Don't confuse this with $D_*(t)$ defined by (2.8)), that is,

$$D(t) \equiv \int_0^1 \left\{g - V_{1_*} + (V_{1_*} - \overline{g})\psi_{R_*} + \left(\frac{1}{2} - g\right)\psi_{A_*}\right\}(\varphi - \varphi_*)dx. \tag{2.20}$$

Then, by referring to Figure 2, we find that

$$\begin{aligned} D(t) &= \int_0^1 \left(\frac{1}{2} - \overline{g}\right)(-\overline{\varphi})dx < 0, \quad \text{in Case } 1^+; \\ &= \int_0^{1-(\lambda t)^{-1}\log 2} (g-\overline{g})\varphi dx + \int_{1-(\lambda t)^{-1}\log 2} \left(\frac{1}{2} - \overline{g}\right)(-\overline{\varphi})dx \\ &< 0 + 0 = 0, \quad \text{in Case } 2^+; \end{aligned}$$

and in Case 3^+,

$$\begin{aligned} D(t) &= \int_0^{1+(\lambda t)^{-1}\log V2_*} (g - V_{1_*})\varphi dx + \int_{1+(\lambda t)^{-1}\log V2_*}^{1-(\lambda t)^{-1}\log 2} (g-\overline{g})\varphi dx \\ &+ \int_{1-(\lambda t)^{-1}\log 2}^1 \left(\frac{1}{2} - \overline{g}\right)(-\overline{\varphi})dx < 0+0+0 = 0, \end{aligned}$$

since $g - V_{1_*} < g - V_{2_*}$ (by Lemma 2.1) < 0 in the first term. Hence $D(t) < 0,\ \forall t > 0$, and (2.18′) is proven by applying Lemma 1.1.

Next we have to prove (2.19). Let us use, terporally, the simplified notation $V_2(t)$, instead of $V_2(t, \varphi_*, \psi_R, \psi_A)$. Then (2.19) becomes

$$V_2(t) \le V_{2_*}(t), \qquad \forall \psi_R\text{–}\psi_A. \tag{2.19′}$$

Since we have, similarly as in (2.2),

$$\lambda^{-1} V_2'(t) = -V_2(t) + \int_0^1 \left\{ \varphi_* \left(\frac{1}{2}\psi_A + \overline{\psi}_A \overline{g} \right) + \overline{\varphi}_* (\psi_R g + \overline{\psi}_R V_2) \right\} dx. \tag{2.21}$$

This equation and (2.4) yield, after some rearrangement,

$$\lambda^{-1}(V_2(t) - V_{2_*}(t))' = -(V_2 - V_{2_*}) \int_0^1 (1 - \overline{\varphi}_* \overline{\psi}_R) dx + H(t), \tag{2.22}$$

$$H(t) \equiv \int_0^1 \left\{ \varphi_* \left(\frac{1}{2} - \overline{g} \right) (\psi_A - \psi_{A_*}) + \overline{\varphi}_* (g - V_{2_*})(\psi_R - \psi_{R_*}) \right\} dx. \tag{2.23}$$

We find by referring to Figure 2, that

$$\begin{aligned} H(t) &= \int_0^1 \left(\frac{1}{2} - \overline{g} \right) (-\overline{\psi}_A) dx < 0, \quad \text{in Case } 1^+; \\ &= \int_0^{1-(\lambda t)^{-1}\log 2} (g - V_{2_*})(-\overline{\psi}_R) dx + \int_{1-(\lambda t)^{-1}\log 2} \left(\frac{1}{2} - \overline{g} \right)(-\overline{\psi}_A) dx \\ &< 0 + 0 = 0, \quad \text{in Case } 2^+; \end{aligned}$$

and in Case 3^+,

$$\begin{aligned} H(t) &= \int_0^{1+(\lambda t)^{-1}\log V2_*} (g - V_{2_*}) \psi_R dx + \int_{1+(\lambda t)^{-1}\log V2_*}^{1-(\lambda t)^{-1}\log 2} (g - V_{2_*})(-\psi_R) dx \\ &+ \int_{1-(\lambda t)^{-1}\log 2}^1 \left(\frac{1}{2} - \overline{g} \right)(-\overline{\psi}_A) dx < 0 + 0 + 0 = 0, \end{aligned}$$

Thus $H(t) < 0,\ \forall t > 0$, and (2.19) is proven by applying Lemma 1.1. This completes the proof of Theorem 2. □

Theorem 2 and Lemma 2.2 give

Corollary 2.1 *The eq. strategy-triple is given by* (2.7), *and satisfies* (2.9) $\sim$ (2.12).

Corollary 2.2 *Both of* $V_{2_*}(t)$ *are increasing and concave in* $t > 0$.

Proof. Increasingness of $V_{2_*}(t)$ is intuitively evident, so its proof is ommited. In Case $1^+, V_{2_*}(t) = \frac{1}{2}(1 - e^{-\lambda t})$ is concave. From (2.4), we have

$$\lambda^{-1} V_{2_*}'(t) = \left(\frac{1}{2} - V_{2_*}(t) \right) E_*(t) - \frac{1}{2} D_*(t). \tag{2.24}$$

(c.f. $E_*(t)$ and $D_*(t)$ are given by (2.14) and (2.8), respectively. This equation correspond to (1.14) in Section 1.)

In Case 2^+, since $E_*(t) = 1$ and $D_*(t) = \int_0^{1-(\lambda t)^{-1}\log 2}(1-2g)dx$, we find that

$$\lambda^{-1}V''_{2_*}(t) = -V'_{2_*} - \frac{1}{2}D'_*(t) < 0,$$

and hence $V_{2_*}(t)$ is concave. In Case 3^+ we use (2.17) i.e. $tV'_{2_*}(t) = \frac{1}{2}(1+\log 2) - V_{2_*}(1-\log V_{2_*})$. Differentitation gives

$$V''_{2_*} = t^{-1}V'_{2_*}(\log V_{2_*} - 1) < 0.$$

The fact that if $V_{2_*}(t)$ is concave, then so is $t^{-1}\log V_{2_*}(t)$ has been proven in Corollary 1.2. □

Since $V_{2_*}(t)$ is increasing and less than $1/2$, the limit as $t \to \infty$ exists. From (2.17) we obtain

Remark 2. If $\lim_{t\to\infty} tV'_{2_*}(t) = 0$, then $\lim_{t\to\infty} V_{2_*}(t) = \frac{1}{2}$, because the equation based on (2.17)

$$-v\log v = \frac{1}{2}(1+\log 2) - v, \quad \text{in } v \in [0,1]$$

has a unique root $1/2$.

Therefore $\lim_{t\to\infty} V_{1_*}(t) = 1/2$ also. Thus the advantage of the first mover is lost when $T \to \infty$. This is simply because players have qual weights. (Also, see the next Remark 3.)

Corollary 2.3

$$V_{1_*}(t) + V_{2_*}(t) = 1 - e^{-\lambda t}, \quad \textit{if} \quad 0 \le t \le \lambda^{-1}t_1 \tag{2.25}$$

$$= 1 - \exp.\left[\int_0^t \tau^{-1}\log V_{2_*}(\tau)d\tau\right], \quad \textit{if} \quad t > \lambda^{-1}t_1$$

Proof. Eq. (2.3) gives just in the same way as for Eq.(2.24),

$$\lambda^{-1}V'_{1_*}(t) = \left(\frac{1}{2} - V_{1_*}(t)\right)E_*(t) + \frac{1}{2}D_*(t). \tag{2.24$'$}$$

Let $s(t) \equiv V_{1_*}(t) + V_{2_*}(t)$. Then (2.24)-(2.24$'$)gives

$$\lambda^{-1}s'(t) = (1-s(t))E_*(t), \quad \text{with} \quad s(0) = 0,$$

and hence

$$\log(1-s(t)) = -\lambda\int_0^t E_*(\tau)d\tau.$$

We obtain (2.25), if $E_*(\tau)$ in (2.24) is used. □

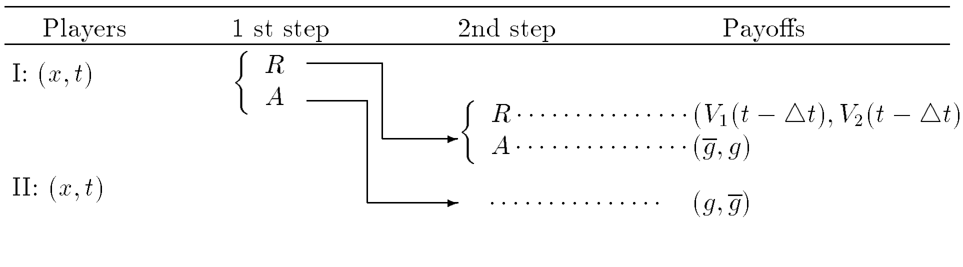
Corollary 2.4. *The time s until the first acceptance of an offer is made has the defective p.d.f.*

$$g(s) = \lambda q(s)\text{exp.}\left[-\lambda\int_0^s q(w)dw\right] \tag{2.26}$$

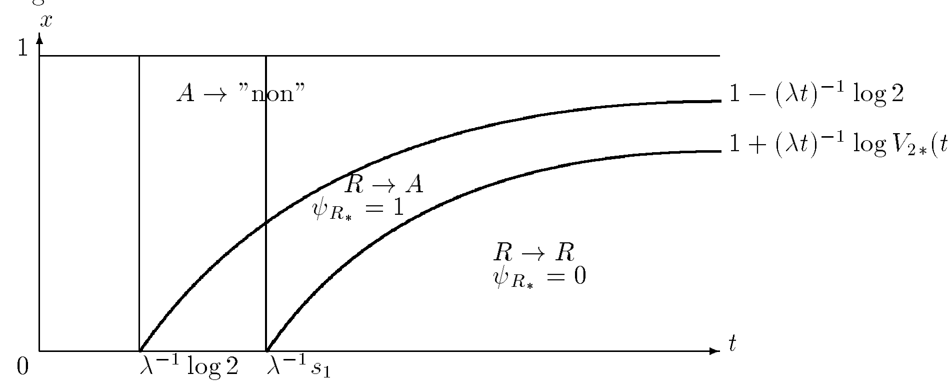
where $q(s) = \begin{cases} -\lambda^{-1}(T-s)^{-1}\log V_{2_*}(T-s), & \text{if } T-s > \lambda^{-1}t_1 \\ 1, & \text{if otherwise.} \end{cases}$ *The probability that both players lose is* $1 - \int_0^T g(s)ds$.

Proof. Note that $q(s)$ is the probability that acceptance is made at time $T - s$. Proof is the same as in Corollary 1.3. □

Remark 3. The bilateral-move game version by unequal-weight players are analysed in Enns and Ferenstein [2]. Players' weight is 1 for I, and 0 for II. So lottery need not be used for the choice-pair A-A. The game diagram is:



in contrast with one for the game by equal-weight players. The authors show in [2] that the solution to the game is :



where s_1 is appoximately 1.53. Compare the figure with our Figure 2. Their $V_{2_*}(t)$ is of course different from one given by our Lemma 2.2. Moreover, they prove that $\lim_{t\to\infty} V_{2_*}(t) \fallingdotseq 0.32756$, which is the unique root of the equation $u \log u = \log 2 - u$, and hence the advantage of the first mover remains even when $T \to \infty$ (c.f. Remark 2.)

3. Final Remarks.

Remark 4. A remarkable feature contained in the present work is that in the simultanious move version of game, the equilibrium strategy uses some randomization between R and A, whereas in the bilateral-move version of the game, they employ non-randomized strategies only. See Figures 1 and 2.

Remark 5. In the bilateral-move game, the first mover stands at advantage to the second mover. See (2.8) in Lemmma 2.1. This is different from earlier works on single and multiround poker, where the first mover stands at disadvantage, because player's hand x for I and y for II are private informations, and I leaks some information to his rival about his private x by moving first. (See Garnaev [3] and Sakaguchi [7])

Remark 6. It is interesting to investigate some open problems mentioned in the following.

(a) Solve the three-person game version. See Sakaguchi [9, 10, 11] and Ramsey and Szajowski [12])

(b) Solve the game where a player wins if he gets the largest offer among those arrived and will arrive thereafter before time T. Each player aims to maximize his probability of winning.

(c) Solve no-information version of the game. Players do not know the size distribution of Poisson-arriving offers, but can only observe the *ralative rank* among those arrived so far of the offer. Each player wants to accept the offer with smaller *absolute rank* than the opponent's. The best (worst) has the absolute rank 1 (n, if the n-th is the last offer arrived beforre time T).

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