FOLDNESS OF QUASI-ASSOCIATIVE IDEALS IN BCI-ALGEBRAS

Young Bae Jun, Seok Zun Song and Celestin Lele

Received January 30, 2002

Abstract. The notion of an n-fold quasi-associative ideal is introduced. Characterizations of n-fold quasi-associative ideals are given, and conditions for an ideal to be an n-fold quasi-associative ideal is provided. The extension property for an n-fold quasi-associative ideal is established.

1. Introduction

The study of BCK/BCI-algebras was initiated by Iséki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Since then a great deal of literature has been produced on the theory of BCK/BCI-algebras, in particular, emphasis seems to have been put on the ideal theory of BCK/BCI-algebras. In [1], Y. Huang and Z. Chen introduced the foldness of some ideals in BCK-algebras. In 1992, Z. C. Yue and X. H. Zhang [2] introduced the notion of a quasi-associative ideal in a BCI-algebra, and investigated a few properties. In this paper, using Yue and Chen's idea, we discuss the foldness of quasi-associative ideals in a BCI-algebra. Firstly, we introduce the notion of n-fold quasi-associative ideals, and then we give characterizations of an n-fold quasi-associative ideal. We provide conditions for an ideal to be an n-fold quasi-associative ideal. Finally, we establish the extension property for an n-fold quasi-associative ideal.

2. Preliminaries

A nonempty set X with a constant 0 and a binary operation $*$ is called a *BCI-algebra* if the following conditions hold:

(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(VI) $x * y = 0$ and $y * x = 0$ imply $x = y$,

for all $x, y, z \in X$. In a BCI-algebra X, we can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$. The following statements are true in any BCI-algebra X.

(2.1) $(x * y) * z = (x * z) * y$.
(2.2) $x * 0 = x$.
(2.3) $(x * z) * (y * z) \leq x * y$.
(2.4) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.
(2.5) $0 * (x * y) = (0 * x) * (0 * y)$.
(2.6) $x * (x * (x * y)) = x * y$.

An *ideal* of a BCI-algebra X is a subset I of X containing 0 such that if $x * y \in I$ and $y \in I$ then $x \in I$. Note that every ideal I of a BCI-algebra X has the following property.

$$x \leq y \text{ and } y \in I \text{ imply } x \in I.$$

2000 *Mathematics Subject Classification.* Primary 06F35, 03G25.
Key words and phrases. Ideal, quasi-associative ideal, (n-fold) quasi-associative ideal.
The first author was supported by Korea Research Foundation Grant (KRF-2001-005-D00002).

3. n-fold quasi-associative ideals

In what follows, let X and n denote a BCI-algebra and a positive integer, respectively, unless specified otherwise. Z. C. Yue and X. H. Zhang [2] introduced the notion of a quasi-associative ideal in a BCI-algebra as follows.

Definition 3.1. ([2, Definition 3]) A nonempty subset I of X is called a *quasi-associative ideal* of X if it satisfies

(I1) $0 \in I$,

(I2) $x * (y * z) \in I$ and $y \in I$ imply $x * z \in I$ for all $x, y, z \in X$.

We begin with the definition of n-fold quasi-associative ideals. For any elements x and y of X, let $x * y^n$ denote $(\cdots((x * y) * y) * \cdots) * y$ in which y occurs n-times.

Definition 3.2. A nonempty subset I of X is called an *n-fold quasi-associative ideal* of X if it satisfies

(I1) $0 \in I$,

(I3) $x * (y * z^n) \in I$ and $y \in I$ imply $x * z^n \in I$ for all $x, y, z \in X$.

Example 3.3. (1) Let $X = \{0, a, b\}$ be a BCI-algebra with the Cayley table as follows.

$*$	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

It is easily checked that $I = \{0, a\}$ is an n-fold quasi-associative ideal of X.

(2) Let $X = \{0, a, b, c\}$ be a BCI-algebra with the Cayley table as follows.

$*$	0	a	b	c
0	0	0	b	b
a	a	0	c	b
b	b	b	0	0
c	c	b	a	0

It is easily checked that $I = \{0, a\}$ is an even-fold quasi-associative ideal of X.

Theorem 3.4. *Every n-fold quasi-associative ideal is an ideal.*

Proof. The proof is by taking $z = 0$ in (I3) and using (2.2). □

We state characterizations of an n-fold quasi-associative ideal.

Theorem 3.5. *Let I be an ideal of X. Then the following are equivalent.*

(a) *I is an n-fold quasi-associative ideal of X.*

(b) *$x * (0 * y^n) \in I$ implies $x * y^n \in I$ for all $x, y \in X$.*

(c) *$x * (y * z^n) \in I$ implies $(x * y) * z^n \in I$ for all $x, y, z \in X$.*

Proof. Assume that I is an n-fold quasi-associative ideal of X and let $x, y \in X$ be such that $x * (0 * y^n) \in I$. Since $0 \in I$, it follows from (I3) that $x * y^n \in I$. Suppose that (b) is true and let $x, y, z \in X$ be such that $x * (y * z^n) \in I$. Using (2.1), (2.4), (I) and (III), we have

$$\begin{aligned} & \Big((x * y) * (0 * z^n)\Big) * \Big(x * (y * z^n)\Big) \\ = & \Big((x * y) * (x * (y * z^n))\Big) * \Big(0 * z^n\Big) \\ \leq & \Big((y * z^n) * y\Big) * \Big(0 * z^n\Big) \\ = & \Big((y * y) * z^n\Big) * \Big(0 * z^n\Big) \\ = & (0 * z^n) * (0 * z^n) = 0 \in I. \end{aligned}$$

Since I is an ideal, we get $(x*y)*(0*z^n) \in I$ and so $(x*y)*z^n \in I$ by (b). Finally suppose that (c) holds and let $x, y, z \in X$ be such that $x * (y * z^n) \in I$ and $y \in I$. Using (2.1) and (c) we have $(x * z^n) * y = (x * y) * z^n \in I$ and so $x * z^n \in I$ because I is an ideal. This completes the proof. □

Theorem 3.6. *Let I be an ideal of X and let k be a positive integer. Then the following are equivalent.*

(a) *I is an n-fold quasi-associative ideal of X.*
(b) *$(x * z^k) * (0 * y^n) \in I$ implies $(x * z^k) * y^n \in I$ for all $x, y, z \in X$.*
(c) *$(x * z^k) * (0 * y^n) \in I$ and $z \in I$ imply $x * y^n \in I$ for all $x, y, z \in X$.*

Proof. Suppose that I is an n-fold quasi-associative ideal of X. Then (b) follows directly from Theorem 3.5(b). Assume that (b) holds and let $x, y, z \in X$ be such that $(x * z^k) * (0 * y^n) \in I$ and $z \in I$. Then $(x * y^n) * z^k = (x * z^k) * y^n \in I$ by (b). Since I is an ideal of X, it follows that $x * y^n \in I$. Finally suppose that (c) is true. Let $x, y \in X$ be such that $x * (0 * y^n) \in I$. Using (2.2) continuously, we have

$$(x * 0^k) * (0 * y^n) = x * (0 * y^n) \in I.$$

Since $0 \in I$, it follows from (c) that $x * y^n \in I$. Hence, by Theorem 3.5, I is an n-fold quasi-associative ideal of X. □

The following example shows that there is a positive integer n such that an ideal may not be an n-fold quasi-associative ideal.

Example 3.7. Consider a BCI-algebra $X = \{0, a, b, c\}$ with the following Cayley table.

$*$	0	a	b	c
0	0	c	b	a
a	a	0	c	b
b	b	a	0	c
c	c	b	a	0

Then the zero ideal $\{0\}$ is not an 1-fold quasi-associative ideal because $c * (0 * a) = c * c = 0 \in \{0\}$ and $c * a = b \notin \{0\}$.

We provide a condition for an ideal to be an n-fold quasi-associative ideal.

Theorem 3.8. *Let I be an ideal of X such that*

$$x \in I \quad \text{and} \quad y \in X \quad \text{imply} \quad x * y^n \in I. \tag{3.1}$$

Then I is an n-fold quasi-associative ideal of X.

Proof. Let $x, y \in X$ be such that $x * (y * z^n) \in I$ and $y \in I$. Using (2.1) and (3.1) yields $(x * z^n) * (y * z^n) = \Big(x * (y * z^n)\Big) * z^n \in I$ and $y * z^n \in I$. Since I is an ideal, it follows that $x * z^n \in I$. This completes the proof. □

We finally establish the extension property for n-fold quasi-associative ideals

Theorem 3.9. (Extension property for n-fold quasi-associative ideals) *Let I and J be ideals of X such that $I \subseteq J$. If I is an n-fold quasi-associative ideal of X, then J is too.*

Proof. Assume that I is an n-fold quasi-associative ideal of X and let $x, y \in X$ be such that $x * (0 * y^n) \in J$. Then

$$\Big(x * (x * (0 * y^n))\Big) * \Big(0 * y^n\Big) = \Big(x * (0 * y^n)\Big) * \Big(x * (0 * y^n)\Big) = 0 \in I.$$

It follows from (2.1) and Theorem 3.5(b) that

$$\Big(x * y^n\Big) * \Big(x * (0 * y^n)\Big) = \Big(x * (x * (0 * y^n))\Big) * y^n \in I \subseteq J.$$

Since J is an ideal, we get $x * y^n \in J$. Using Theorem 3.5, we know that J is an n-fold quasi-associative ideal of X. □

Corollary 3.10. *If the zero ideal $\{0\}$ of X is an n-fold quasi-associative ideal of X, then every ideal is an n-fold quasi-associative ideal.*

4. Conclusions and future works

We introduced the notion of n-fold quasi-associative ideals in BCI-algebras, and gave characterizations of n-fold quasi-associative ideals. We also provided conditions for an ideal to be an n-fold quasi-associative ideal, and established the extension property for n-fold quasi-associative ideals. This ideas could be able to discuss the foldness of a p-ideal, which is introduced by X. H. Zhang, J. Hao and S. A. Bhatti [3], in BCI-algebras. There are deep relations between n-fold quasi-associative ideals and n-fold p-ideals. We first will study n-fold p-ideals, and then we will investigate relations between n-fold quasi-associative ideals and n-fold p-ideals. For the general development of BCI/BCK-algebras, fuzzy theory also plays an important role as well as the ideal theory. So we will discuss the fuzzifications of n-fold quasi-associative ideals and n-fold p-ideals in BCI-algebras.

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Y. B. Jun, Department of Mathematics Education, Gyeongsang National University, Chinju (Jinju) 660-701, Korea
E-mail: ybjun@nongae.gsnu.ac.kr

S. Z. Song, Department of Mathematics, Cheju National University, Cheju 690-756, Korea
E-mail: szsong@cheju.cheju.ac.kr

C. Lele, Faculty of sciences, Department of Mathematics, University of Yaounde I, Box 812, Cameroon
E-mail: lele_clele@yahoo.com