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ON THE LATTICE OF IDEALS OF AN MV -ALGEBRA

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ABSTRACT. For an MV-algebra $(A, +,^*, 0)$ we denote by $I(A)$ the set of all ideals of A. For $I_1, I_2 \in I(A)$ we define $I_1 \wedge I_2 = I_1 \cap I_2$, $I_1 \vee I_2 =$ the ideal generated by $I_1 \cup I_2$, and for $I \in I(A)$, $I^* = \{a \in A : a \wedge x = 0$ for every $x \in I\}$.

The aim of this paper is to prove that $(I(A), \vee, \wedge,^*, \{0\}, A)$ is a Boolean lattice iff A is a finite Boolean lattice relative to the natural order on A (Theorem 2.8.)

1 Definitions and preliminaries

Definition 1.1 [2,3]. *An MV-algebra is an algebra $(A, +,^*, 0)$ of type $(2, 1, 0)$ satisfying the following equations:*

$MV_1)$ $x + (y + z) = (x + y) + z$

$MV_2)$ $x + y = y + x$

$MV_3)$ $x + 0 = x$

$MV_4)$ $x^{**} = x$

$MV_5)$ $x + 0^* = 0^*$

$MV_6)$ $(x^* + y)^* + y = (y^* + x)^* + x.$

MV-algebras were originally introduced by Chang in [2] in order to give an algebraic counterpart of the Lukasiewicz many valued logic (MV=many valued). Note that axioms $MV_1)$-$MV_3)$ state that $(A, +, 0)$ is an abelian monoid; following tradition, we denote an MV-algebra $(A, +,^*, 0)$ by its universe A.

Remark 1 *If in $MV_6)$ we put $y = 0$ we obtain $x^{**} = 0^{**} + x$, so, if $0^{**} = 0$ then $x^{**} = x$ for every $x \in A$. Hence, the axiom $MV_4)$ is equivalent with $MV_4')$ $0^{**} = 0$.*

Examples:

$E_1)$ A singleton $\{0\}$ is a trivial example of an MV-algebra; an MV-algebra is said *nontrivial* provided its universe has more that one element.

$E_2)$ Let $(G, \oplus, -, 0, \leq)$ an l-group. For each $u \in G$, $u > 0$, let

$$[0, u] = \{x \in G : 0 \leq x \leq u\}$$

and for each $x, y \in [0, u]$, let $x+y = u \wedge (x \oplus y)$ and $x^* = u - x$. Then $([0, u], +,^*, 0)$ is an MV-algebra. In particular, if consider the real unit interval $[0, 1]$ and for all $x, y \in [0, 1]$ we define $x \oplus y = \min\{1, x + y\}$ and $x^* = 1 - x$, then $([0, 1], \oplus,^*, 0)$ is an MV-algebra.

$E_3)$ If $(A, \vee, \wedge,^*, 0, 1)$ is a Boolean lattice, then $(A, \vee,^*, 0)$ is an MV-algebra.

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E_4) The rational numbers in $[0,1]$, and, for each integer $n \geq 2$, the n-element set $L_n = \left\{0, \frac{1}{(n-1)}, ..., \frac{(n-2)}{(n-1)}, 1\right\}$ yield examples of subalgebras of $[0,1]$.

E_5) Given an MV-algebra A and a set X, the set A^X of all functions $f : X \longrightarrow A$ becomes an MV-algebra if the operations $+$, and * and the element 0 are defined pointwise. The continous functions from $[0,1]$ into $[0,1]$ form a subalgebra of the MV-algebra $[0,1]^{[0,1]}$.

In the rest of this paper, by A we denote an MV-algebra.

On A we define the constant 1 and the operations ,,$\cdot$" and ,,$-$" as follows: $1 = 0^*$, $x \cdot y = (x^* + y^*)^*$ and $x - y = x \cdot y^* = (x^* + y)^*$ (we consider the * operation more binding that any other operation, and the ,,$\cdot$" more binding that + and -).

Lemma 1.2 [3,4] *For $x, y \in A$, the following conditions are equivalent:*

(i) $x^* + y = 1$

(ii) $x \cdot y^* = 0$

(iii) $y = x + (y - x)$

(iv) *There is an element $z \in A$ such that $x + z = y$.*

For any two elements $x, y \in A$ let us agree to write $x \leq y$ iff x and y satisfy the equivalent conditions (i)-(iv) in the above lemma. So, $\leq$ is a partial order relation on A (which is called the *natural order* on A).

Theorem 1.3 [3,4] *If $x, y, z \in A$ then the following hold:*

c$_1$) $1^* = 0$

c$_2$) $x + y = (x^* \cdot y^*)^*$

c$_3$) $x + 1 = 1$

c$_4$) $(x - y) + y = (y - x) + x$

c$_5$) $x + x^* = 1$

c$_6$) $x - 0 = x$, $0 - x = 0$, $x - x = 0$, $1 - x = x^*$, $x - 1 = 0$

c$_7$) $x + x = x$ *iff* $x \cdot x = x$

c$_8$) $x \leq y$ *iff* $y^* \leq x^*$

c$_9$) *If $x \leq y$, then $x + z \leq y + z$ and $x \cdot z \leq y \cdot z$*

c$_{10}$) *If $x \leq y$, then $x - z \leq y - z$ and $z - y \leq z - x$*

c$_{11}$) $x - y \leq x, x - y \leq y^*$

c$_{12}$) $(x + y) - x \leq y$

c$_{13}$) $x \cdot z \leq y$ *iff* $z \leq x^* + y$

c$_{14}$) $x + y + x \cdot y = x + y$

Remark 2 [3,4] *On A, the natural order determines a bounded distributive lattice structure. Specifically, the join $x \vee y$ and the meet $x \wedge y$ of the elements x and y are given by:*
$x \vee y = (x - y) + y = (y - x) + x$
$x \wedge y = (x^* \vee y^*)^*$
Clearly, $x \cdot y \leq x \wedge y \leq x \leq x \vee y \leq x + y$.

For each $x \in A$, we let $0 \cdot x = 0$, and for each integer $n \geq 0$, $(n+1)x = nx + x$.

Theorem 1.4 [3,4] *If $x, y, z, (x_i)_{i \in I}$ are elements of A, then the following hold:*

c$_{15}$) $x + y = (x \vee y) + (x \wedge y)$

c$_{16}$) $x \cdot y = (x \vee y) \cdot (x \wedge y)$

c$_{17}$) $x + \left(\bigvee_{i \in I} x_i\right) = \bigvee_{i \in I} (x + x_i)$

c$_{18}$) $x + \left(\bigwedge_{i \in I} x_i\right) = \bigwedge_{i \in I} (x + x_i)$

c$_{19}$) $x \cdot \left(\bigvee_{i \in I} x_i\right) = \bigvee_{i \in I} (x \cdot x_i)$

c$_{20}$) $x \cdot \left(\bigwedge_{i \in I} x_i\right) = \bigwedge_{i \in I} (x \cdot x_i)$

c$_{21}$) $x \wedge \left(\bigvee_{i \in I} x_i\right) = \bigvee_{i \in I} (x \wedge x_i)$

c$_{22}$) $x \vee \left(\bigwedge_{i \in I} x_i\right) = \bigwedge_{i \in I} (x \vee x_i)$ *(if all the suprema and infima exist).*

Lemma 1.5 *For every $x, y, z \in A$ we have*

c$_{23}$) $(x + y) - z \leq (x - z) + (y - z)$.

Proof. We have $((x+y)-z)^* + (x-z) + (y-z) = (x+y)^* + z + (x-z) + (y-z) =$
$(x+y)^* + (z + (x-z)) + (y-z) = (x+y)^* + (x \vee z) + (y-z) =$
$(x+y)^* + ((x \vee z) + (y-z)) \overset{by\ c_{17}}{=} (x+y)^* + ((x + (y-z)) \vee (z + (y-z))) \overset{by\ c_{17}}{=}$
$(x+y)^* + ((x + (y-z) \vee y \vee z) = (x+y)^* + (((x + (y-z)) \vee y) \vee z) \overset{by\ c_{17}}{=}$
$(x+y)^* + (((x \vee y) + ((y-z) \vee y)) \vee z) = (x+y)^* + (((x \vee y) + y) \vee z)$.
So, to prove c_{23} it suffices to prove $x + y \leq ((x \vee y) + y) \vee z$ which result from c_9 (since $x \leq x \vee y$, hence $x + y \leq (x \vee y) + y \leq ((x-y) + y) \vee z$).■

2 The lattice of ideals of an MV-algebra

Definition 2.1 *A ideal of an MV-algebra A is a non-void subset I of A satisfying the following conditions:*

I$_1$) *If $x \in I$, $y \in A$ and $y \leq x$, then $y \in I$*

I$_2$) *If $x, y \in I$ then $x + y \in I$.*

We denote by $I(A)$ the set of all ideals of A. For $M \subseteq A$ we denote by $(M]$ the *ideal of A generated by M* (that is $(M] = \cap\{I \in I(A) | M \subseteq I\}$). If $M = \{a\}$ with $a \in A$, we denote by $(a]$ the ideal generated by $\{a\}$($(a]$ is called *principal*)

Proposition 2.2 [3,4]
(i) If $M \subseteq A$, then $(M] = \{x \in A : x \leq x_1 + ... + x_n$ for some $x_1, ..., x_n \in M\}$.
In particular, for $a \in A$, $(a] = \{x \in A : x \leq na$ for some integer $n \geq 0\}$.
(ii) If $I_1, I_2 \in I(A)$, then
$I_1 \vee I_2 \overset{def}{=} (I_1 \cup I_2] = \{a \in A : a \leq x_1 + x_2$ *for some* $x_1 \in I_1$ *and* $x_2 \in I_2\}$
(iii) If $x, y \in A$, then $(x] \cap (y] = (x \wedge y]$ (see[4, p.112]).

For $I \in I(A)$ and $a \in A \setminus I$ we denote by $I(a) = (a] \vee I = (I \cup \{a\}]$.

Remark 3 [3,4] *For $I(a)$ we have the next characterization:*

$I(a) = \{x \in A : x \leq y + (na)$ for some $y \in I$ and integer $n \geq 0\}$.

Proposition 2.3 *For $a \in A \setminus I$, $I(a) = \{x \in A : x - a \in I\}$*

Proof. Let $I_a = \{x \in A : x - a \in I\}$. Since $a - a = 0 \in I$ we deduce that $a \in I_a$ .Since for $x \in I$, $x - a \leq x$ (by c_{11}) we deduce that $x - a \in I$, hence $I \subseteq I_a$. To prove $I_a \in I(A)$ we observe that $0 - a = 0 \in I$, hence $0 \in I_a$. If $x \leq y$ and $y \in I_a$, then from $x - a \leq y - a$ (c_{10}) and $y - a \in I$ we deduce $x - a \in I$, hence $x \in I_a$. Let $x, y \in I_a$, that is $x - a, y - a \in I$. From Lemma 1.5. we have $(x + y) - a \leq (x - a) + (y - a)$, hence $(x + y) - a \in I$ that is $x + y \in I_a$.From $a \in I_a$, $I \subseteq I_a$ and $I_a \in I(A)$ we deduce $I(a) \subseteq I_a$. Let now $J \in I(A)$ such that $a \in J$ and $I \subseteq J$. If $x \in I_a$, then $x - a \in I \subseteq J$, hence $x \vee a = (x - a) + a \in J$. Since $x \leq x \vee a$ we deduce $x \in J$ that is $I_a \subseteq J$, hence $I_a \subseteq \cap J = I(a)$. ¿From $I(a) \subseteq I_a$ and $I_a \subseteq I(a)$ we deduce $I_a = I(a)$.■

Corollary 2.4 *If $x, y \in A$ then $(x] \vee (y] = (x + y]$.*

Proof 1: By Proposition 2.3. we have

$$(x] \vee (y] = (y](x) = \{a \in A : a - x \in (y]\}.$$

Since by c_{12} $(x + y) - x \leq y$, we deduce $x + y \in (x] \vee (y]$, hence $(x + y] \subseteq (x] \vee (y]$. Since the inclusion $(x] \vee (y] \subseteq (x + y]$ is obviously, we obtain the equality $(x] \vee (y] = (x + y]$.

Proof 2: It is suffices to show the inclusion $(x + y] \subseteq (x] \vee (y]$. If $z \in (x + y]$ then $z \leq n(x + y)$ for some integer $n \geq 0$. But $n(x + y) = (nx) + (ny)$ and so $z \leq (nx) + (ny)$. Since $nx \in (x]$ and $ny \in (y]$ we deduce that $z \in (x] \vee (y]$, that is $(x + y] \subseteq (x] \vee (y]$.■

For $I_1, I_2 \in I(A)$, we put $I_1 \wedge I_2 = I_1 \cap I_2$, $I_1 \vee I_2 = (I_1 \cup I_2]$,$I_1 \longrightarrow I_2 = \{a \in A : (a] \cap I_1 \subseteq I_2\}$.

Then $(I(A), \vee, \wedge, \{0\}, A)$ is a complete *Brouwerian* lattice ([4, p.114]); we recall that a complete lattice is Brouwerian if it satisfies the identity $a \wedge \left(\bigvee_{i \in I} b_i\right) = \bigvee_{i \in I} (a \wedge b_i)$.

Lemma 2.5 *If $I_1, I_2 \in I(A)$, then*
(i) $I_1 \longrightarrow I_2 \in I(A)$
(ii) If $I \in I(A)$, then $I_1 \cap I \subseteq I_2$ iff $I \subseteq I_1 \longrightarrow I_2$ (that is, $I_1 \longrightarrow I_2 = \sup\{I \in I(A) : I_1 \cap I \subseteq I_2\}$).

Proof (i) Since $(0] \cap I_1 \subseteq I_2$ we deduce that $0 \in I_1 \longrightarrow I_2$. If $x, y \in A$, $x \leq y$ and $y \in I_1 \longrightarrow I_2$, then $(y] \cap I_1 \subseteq I_2$. Since $(x] \subseteq (y]$ we deduce that $(x] \cap I_1 \subseteq (y] \cap I_1 \subseteq I_2$, hence $x \in I_1 \longrightarrow I_2$. Let now $x, y \in I_1 \longrightarrow I_2$; then $(x] \cap I_1 \subseteq I_2$ and $(y] \cap I_1 \subseteq I_2$. We deduce $((x] \cap I_1) \vee ((y] \cap I_1) \subseteq I_2$ hence $((x] \vee (y]) \cap I_1 \subseteq I_2$, so $(x+y] \cap I_1 \subseteq I_2$ (by Corollary 2.4.), that is $x + y \in I_1 \longrightarrow I_2$.

(ii) ($\Longrightarrow$) Let $I \in I(A)$; then $I_1 \cap I \subseteq I_2$. If $x \in I$ then $(x] \cap I_1 \subseteq I \cap I_1 \subseteq I_2$ hence $x \in I_1 \longrightarrow I_2$, that is $I \subseteq I_1 \longrightarrow I_2$.

($\Longleftarrow$) We suppose $I \subseteq I_1 \longrightarrow I_2$ and let $x \in I_1 \cap I$; then $x \in I$, hence $x \in I_1 \longrightarrow I_2$, that is $(x] \cap I_1 \subseteq I_2$. Since $x \in (x] \cap I_1$, then $x \in I_2$, that is $I_1 \cap I \subseteq I_2$.■

Remark 4 *¿From Lemma 2.5. we deduce that* $(I(A), \vee, \wedge, \longrightarrow, \{0\}, A)$ *is a Heyting algebra; for* $I \in I(A)$, $I^* = I \longrightarrow \{0\} = \{x \in A : (x] \cap I = \{0\}\}$.

Corollary 2.6 *(i) For every* $I \in I(A)$, $I^* = \{x \in A : x \wedge y = 0$ *for every* $y \in I\}$ *(see[4, p.114])*

(ii) For any $x \in A$, $(x]^* = \{y \in A : (y] \cap (x] = \{0\} = \{y \in A : x \wedge y = 0\}$ *(by Proposition 2.2., (iii)).*

We recall that for a bounded distributive lattice L, following tradition , by $B(L)$ we denoted the *Boolean lattice* of complemented elements in L.

For an MV-algebra $(A, +, ^*, 0, 1)$ we shall denote by $B(A)$ the Boolean lattice associated with the bounded distributive lattice $(A, \vee, \wedge, 0, 1)$.

Proposition 2.7 [4, p. 127] *For every* $x \in A$, *the following conditions are equivalent:*

(i) $x \in B(A)$

(ii) $x + x = x$

(iii) $x \cdot x = x$

(iv) $x \wedge x^* = 0$

(v) $x \vee x^* = 1$.

Theorem 2.8 *If* A *is an* MV*-algebra, then the following conditions are equivalent:*

(i) $(I(A), \vee, \wedge, ^*, \{0\}, A)$ *is a Boolean lattice*

(ii) $(A, \vee, \wedge, ^*, 0, 1)$ *is a finite Boolean lattice.*

Proof (i)$\Longrightarrow$(ii). Let $x \in A$; since $I(A)$ is a Boolean lattice then $(x] \vee (x]^* = A$. By Proposition 2.3. and Corollary 2.6. (ii), we have $(x] \vee (x]^* = (x]^*(x) = \{y \in A : y - x \in (x]^*\} = \{y \in A : (y - x) \wedge x = 0\}$. Since $(x] \vee (x]^* = A$, then $1 \in (x] \vee (x]^*$, hence $(1 - x) \wedge x = 0$. We obtain that $x^* \wedge x = 0$, hence $x \in B(A)$ (by Proposition 2.7.(iii)), that is $(A, \vee, \wedge, ^*, 0, 1)$ is a Boolean lattice.To show that A is finite it suffices to prove that every ideal of A is principal ([5, p.77]).If $I \in I(A)$, because $I(A)$ is supposed Boolean lattice then $I \vee I^* = A$, hence $1 \in I \vee I^*$. By Proposition 2.2. (ii),$1 = a + b$ with $a \in I$ and $b \in I^*$. By Corrollary 2.6.(i), $x \wedge b = 0$ for every $x \in I$. So $(x^* \vee b^*)^* = 0 \Longleftrightarrow x^* \vee b^* = 1 \Longleftrightarrow (x + b^*)^* + b^* = 1 \Longleftrightarrow x + b^* \leq b^* \Longleftrightarrow x + b^* = b^*$ for every $x \in I$. Since $a + b = 1$ we obtain $b^* \leq a$ hence $x + b^* = b^* \leq a$ for every $x \in I$. Finally, we obtain $x \leq x + b^* \leq a$, hence $x \leq a$ for every $x \in I$, that is $I = (a]$.

(ii)$\Longrightarrow$(i). Suppose $(A, \vee, \wedge, ^*, 0, 1)$ is a finite Boolean lattice. By Remark 4, $I(A)$ is a Heyting algebra. To prove $I(A)$ is a Boolean lattice we must show $I^* = \{0\}$ only for $I = A$ ([1, p.175]). Since in finite Boolean lattice every ideal is principal, then $I = (a]$ for some $a \in A$. By Corollary 2.6. (ii), $I^* = (a]^* = \{x \in A : x \wedge a = 0\}$. Since $I^* = \{0\}$ and $a^* \wedge a = 0$, then $a^* = 0$, hence $a = 1$ so $I = (1] = A$.■

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