A COMPOSITE MAXIMIZING PROBLEM FOR THE OPEN-LOOP NASH STRATEGIES

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Abstract. The n-sector ($n \geq 2$) open-loop Nash strategies can not be discussed directly within the framework of usual variational principle. In this paper, by extending the original variables, it is shown that a maximizing problem under constraints in the strategies can be included in a problem (composite maximizing problem) in the usual variational principle. A model is given for the derivation of conservation laws, through which optimal paths are determined completely for finite horizon and then detailed for infinite horizon.

Introduction. Noether theorem (Noether [12]) concerning with symmetries of the action integral or its generalization (Bessel-Hagen [1]) with those up to divergence plays an effective role for discovering conservation laws from the Lagrangian or the Hamiltonian structures of considering problem. In contrast with Noether theorem, we built up a new operative procedure for the derivation of conservation laws (Mimura and Nôno [8]) and applied it to various economic growth models (Mimura and Nôno [9]; Mimura, Fujiwara and Nôno [10], [11]; Fujiwara, Mimura and Nôno [3]-[7]) to discover new economic conservation laws including non-Noether ones.

The application can be extended to the n-sector differential game in which the sectors (the players) are not able to make binding commitments in advance of play on the strategies they will employ. Such strategies are called open-loop Nash strategies. In the game, each sector has his own objective functional to maximize under a constraint leaving the strategies of other sectors out of account, i.e., regarding the variables with respect to other sectors as constants. This fact put difficulties for the application of, as well as Noether theorem, the new procedure in [8] to discover conservation laws in the open-loop Nash strategies.

Fershtman and Nitzan [2] introduced a model of the voluntary contributions to the provision of a collectively produced good in the dynamic framework of differential game. They compared the Parato-efficient and the level of the collective contributions in the corresponding steady states of the open-loop and the feedback Nash strategies. The model is interesting in the sense that the objective functional of each sector include both of the state variable (the stock of total contributions) and the control variable (the contribution rate of each sector). In this paper, the model is used with some generalization to formulate a way of discovering conservation laws in the n-sector open-loop Nash strategies (in which general derivation of conservation laws has been never discussed).

In section 1, we set objective functional of each sector i ($i = 1, \cdots, n$) whose maximizing problem will be discussed under a constraint. In the open-loop Nash strategies, we first show that the n functionals can be unified into a single objective functional. And then, introducing new variables, we give a composite maximizing problem (an extended maximizing problem in the usual variational principle), in which the optimal paths of the original variables are those in the maximizing problem of the single objective functional

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in the open-loop Nash strategies. In section 2, by appling the new procedure in [8] to the composite maximizing problem in the usual variational principle, we find three conserved quantities for a model in two-sector open-loop Nash strategies. Finally in section 3, through the obtained conserved quantities in section 2, optimal paths are determined completely for finite horizon and then detailed for infinite horizon. The determined optimal paths will illustrate the level of the collective contribution at the stationary open-loop Nash equibrium obtained in [2].

1. A composite maximizing problem. In the n-sector open-loop Nash strategies, we consider the following model which will be understood later as a generalization of the model of Fershtman and Nitzan [2]. Each sector i $(i = 1, \cdots, n)$ has a common state variable $x(t)$ and his own control variable $u^i(t)$. Leaving the behavior of $u^j(t)$ $(j \neq i)$ out of account, sector i seeks to maximize the integration over finite $(T < \infty)$ or infinite $(T = \infty)$ period of time:

$$\int_0^T e^{-\rho t}(\varphi(x) + \psi_i(u^1, \cdots, u^n))dt \tag{1-i}$$

under a constraint

$$\dot{x} = \alpha x + \sum_{k=1}^{n} \beta_k u^k \qquad (\alpha,\ \beta_k\text{: const.},\ k = 1, \cdots n;\ \alpha < 0), \tag{2}$$

where φ and ψ_i are assumed to be a monotonically increasing function and a concave function with respect to his control variable u^i, respectively, i.e.,

$$\varphi' > 0, \qquad \frac{\partial^2 \psi_i}{\partial u^i \partial u^i} < 0. \tag{3-i}$$

So that sector i has the following Lagrangian L_i with the multiplier π_i:

$$L_i = e^{-\rho t}(\varphi(x) + \psi_i(u^1, \cdots, u^n)) + \pi_i \left(\dot{x} - \alpha x - \sum_{k=1}^{n} \beta_k u^k \right).$$

Here keep in mind that sector i regards u^j and π_j $(j \neq i)$ as constants for the determination of his Euler-Lagrange equations which consist of (2) and

$$\dot{\pi}_i + \alpha \pi_i = e^{-\rho t} \varphi'(x), \tag{4-i}$$

$$\beta_i \pi_i = e^{-\rho t} \frac{\partial \psi_i(u^1, \cdots, u^n)}{\partial u^i}. \tag{5-i}$$

Then n systems of Euler-Lagrange equations (2), (4-i) and (5-i) of sector i $(i = 1, \cdots, n)$ are called together in the space of all variables x, $u = (u^1, \cdots, u^n)$ and $\pi = (\pi^1, \cdots, \pi^n)$ to determine the optimal paths.

Here assume that ψ_i in (1-i) (i=1,$\cdots$, n) satisfy

$$\frac{\partial^2 \psi_i}{\partial u^j \partial u^i} = \frac{\partial^2 \psi_j}{\partial u^i \partial u^j} \quad (i,\ j = 1, \cdots, n), \tag{6}$$

which guarantee an existence of function $\psi(u^1, \cdots, u^n)$ such that

$$\frac{\partial \psi}{\partial u^i} = \frac{\partial \psi_i}{\partial u^i} \quad (i = 1, \cdots, n). \tag{7}$$

Accordingly, in the open-loop Nash strategies, since the sector i regards u^j and π_j $(j \neq i)$ as constants, the relating Euler-Lagrange equations of sector i remains unchanged even if the utilities $\varphi + \psi_i$ are replaced with $\varphi + \psi$, which satisfies the similar condition of (3-i):

$$\varphi' > 0, \qquad \frac{\partial^2 \psi}{\partial u^i \partial u^i} < 0 \quad (i = 1, \cdots, n). \tag{8}$$

Consequently, it follows:

Theorem 1. *Let* $\varphi(x) + \psi_i(u^1, \cdots, u^n)$ *in* (1-i) (i= $1, \cdots, n$) *satisfy* (3-i) (i= $1, \cdots, n$) *and* (6). *Then, in the* n*-sector open-loop Nash strategies, the maximizing problem of* (1-i) (i= $1, \cdots, n$) *under the constraint* (2) *is equivalent to that of*

$$\int_0^T e^{-\rho t}(\varphi(x) + \psi(u^1, \cdots, u^n))dt \tag{9}$$

under the constraint (2), *where* $\psi(u^1, \cdots, u^n)$ *is a function satisfying* (7).

Remark 1. In the open-loop Nash strategies, since sector i regards u^j $(j \neq i)$ as constants for the determination of his Euler-Lagrange equations, the equations (4-i) and (5-i) are unchanged even if $\sum_{k=1}^n \beta_k u^k$ is replaced with $\beta_i u^i$ in the constraint (2) of the maximizing problem of sector i.

Remark 2. In the model of [2] in the open-loop Nash strategies, the utility V_i of sector i $(i = 1, \cdots, n)$ is of the form $V_i = \gamma f(x) - C(u^i)$ (γ: const.). By virtue of the theorem 1, V_i can be unified into $V = \gamma f(x) - \sum_{k=1}^n C(u^k)$, where $f(x)$ and $C(u^k)$ were specified respectively as $f(x) = ax - bx^2$ (a, b: const.) and $C(u^k) = \frac{1}{2}c(u^k)^2$ (c: const.) for the tractability.

Now introduce new variables $y^\sigma(t)$ $(\sigma = 1, \cdots, n-1)$ and put

$$z^1 = \frac{1}{n}\left(x + \sum_{\sigma=1}^{n-1} y^\sigma\right), \tag{10a}$$

$$z^{\sigma+1} = \frac{1}{n}(x - y^\sigma) \quad (\sigma = 1, \cdots, n-1). \tag{10b}$$

The equations (10a) and all of (10b) for $\sigma = 1, \cdots, n-1$ are added to see

$$\sum_{k=1}^n z^k = x, \tag{11}$$

so that the integration (9) is written as

$$\int_0^T e^{-\rho t}\left(\varphi(z^1 + \cdots + z^n) + \psi(u^1, \cdots, u^n)\right)dt. \tag{12}$$

So, in the usual variational principle, consider the maximizing problem of (12) under the reformed constraints

$$\dot{z}^i = \alpha z^i + \beta_i u^i \quad (i = 1, \cdots, n). \tag{13}$$

Here denote the principal k-th minor of Hessian matrix of $\psi(u^1, \cdots, u^n)$ by $D_k(\psi)$ $(k = 1, \cdots, n)$. Also, ψ in (12) is assumed to be a concave function, i.e., (see, e.g., [13])

$$(-1)^k D_k(\psi) > 0 \qquad (k = 1, \cdots, n). \tag{14}$$

Then the Lagrangian with the multipliers π_i $(i = 1, \cdots, n)$ is written as

$$L = e^{-\rho t}\left(\varphi(z^1 + \cdots + z^n) + \psi(u^1, \cdots, u^n)\right) + \sum_{k=1}^{n} \pi_k \left(\dot{z}^k - \alpha z^k - \beta_k u^k\right), \tag{15}$$

whose Euler-Lagrange equations consist of (13) and

$$\dot{\pi}_i + \alpha_i \pi_i = e^{-\rho t} \frac{\partial \varphi(z^1 + \cdots + z^n)}{\partial z^i} \qquad (i = 1, \cdots, n), \tag{16}$$

$$\beta_i \pi_i = e^{-\rho t} \frac{\partial \psi(u^1, \cdots, u^n)}{\partial u^i} \qquad (i = 1, \cdots, n). \tag{17}$$

All of (13) for $i = 1, \cdots, n$ are added and then (11) is used to derive (2). The equation (16) is equivalent to the collection of (4-i) (i = $1, \cdots, n$) by $\partial\varphi(z^1 + \cdots + z^n)/\partial z^i = \varphi'(x)$. The equation (17) is equivalent to the collection of (5-i) (i = $1, \cdots, n$) if (7) is satisfied. The equations (2), (16) and (17) are used to find optimal paths $x(t)$ and $u(t)$. The equation (10b) is substituted for (13) and then (2) is used to derive

$$\dot{y}^\sigma - \alpha y^\sigma = \sum_{k=1}^{n} \beta_k u^k - n\beta_{\sigma+1} u^{\sigma+1} \qquad (\sigma = 1, \cdots, n-1), \tag{18}$$

whose right hand side is written as $\varphi^\sigma(t)$ on the optimal path $u(t) = (u^1(t), \cdots, u^n(t))$. In the solution $y^\sigma = A^\sigma e^{\alpha t}$ (A^σ: const.) of the subsidiary equation $\dot{y}^\sigma - \alpha y^\sigma = 0$ of (18), the constant A^σ is replaced with an arbitrary function of t, and then determined as

$$A^\sigma(t) = \int \varphi^\sigma(t) e^{-\alpha t} dt.$$

Therefore, if the optimal paths $x(t)$ and $u(t)$ exist, the optimal path $y(t) = (y^1(t), \cdots, y^{n-1}(t))$ exists also, where $y^\sigma(t) = A^\sigma(t)e^{\alpha t}$ $(\sigma = 1, \cdots, n-1)$. Therefore, it is deduced:

Theorem 2. *Let $\varphi(x) + \psi(u^1, \cdots, u^n)$ in (9) satisfy (8) and $\varphi(z^1 + \cdots + z^n) + \psi(u^1, \cdots, u^n)$ in (12) satisfy $\varphi'(z^1 + \cdots + z^n) > 0$ and (14). Then the maximizing problem of (9) under the constraints (2) in the n-sector open-loop Nash strategies is included in that of (12) under the constraints (13) in the usual variational principle, i.e., the former and the latter expect an existence of the same optimal paths $x(t)$ and $u(t)$.*

2. Conservation laws for a model in the open-loop Nash strategies. Particularly in the two-sector open-loop Nash strategies, we consider the problem that each sector i $(i = 1, 2)$ seeks to maximize the integration

$$\int_0^T e^{-\rho t}(\varphi(x) + \psi_i(u^1, u^2))dt \tag{1-i)$'$}$$

under the constraints

$$\dot{x} = \alpha x + \beta_1 u^1 + \beta_2 u^2 \qquad (\alpha, \beta_1, \beta_2\text{: const.}, \ \alpha < 0); \tag{2)$'$}$$

where the utility $\varphi(x) + \psi_i(u^1, u^2)$ satisfying (3-i) and (6) is given by

(19a) $\varphi(x) = a_1 x + \frac{1}{2} a_2 x^2 \qquad (x < -a_1/a_2;\ a_1, a_2\text{: const.};\ a_1 > 0, a_2 < 0),$

(19b) $\psi_i(u^1, u^2) = b_i u^i + \frac{1}{2} b_{ii} (u^i)^2 + b_{12} u^1 u^2 + g_i(u^j) \qquad (b_i, b_{ij}\text{: const.},\ i, j = 1, 2;\ b_{ii} < 0),$

in which $g_i(u^j)$ is an arbitrary function of u^j $(j \neq i)$. By the theorem 1, the functions $\varphi + \psi_i$ $(i = 1, 2)$ are unified into $\varphi + \psi$ where

$$\psi(u^1, u^2) = b_1 u^1 + b_2 u^2 + \tfrac{1}{2} b_{11} (u^1)^2 + b_{12} u^1 u^2 + \tfrac{1}{2} b_{22} (u^2)^2, \tag{20}$$

which is also assumed to be provided with the concavity, i.e., $D_1(\psi) = b_{11} < 0$ and $D_2(\psi) = b_{11} b_{22} - b_{12}^2 > 0$. So it follows for $\varphi(x) + \psi(u^1, u^2)$ that

$$a_1 > 0,\ a_2 < 0,\quad b_{11} < 0,\quad b_{22} < 0,\quad b_{11} b_{22} - b_{12}^2 > 0. \tag{21}$$

Since $n = 2$, (10a) and (10b) are reduced respectively to

$$z^1 = \tfrac{1}{2}(x + y), \tag{10a$'$}$$
$$z^2 = \tfrac{1}{2}(x - y), \tag{10b$'$}$$

where $y \equiv y^1$ is the new variable. And the Lagrangian (15) is also to

$$L = e^{-\rho t} \left(\varphi(z^1 + z^2) + \psi(u^1, u^2) \right) + \pi_1 (\dot{z}^1 - \alpha z^1 - \beta_1 u^1) + \pi_2 (\dot{z}^2 - \alpha z^2 - \beta_2 u^2), \tag{15$'$}$$

where $\varphi(z^1 + z^2) + \psi(u^1, u^2)$ is given by (19a) and (20) with (21). The Euler-Lagrange equations for the Lagrangian (15)$'$ consist of (13) with $i = 1, 2$ and (see (16) and (17))

$$\dot{\pi}_i + \alpha \pi_i = e^{-\rho t} (a_1 + a_2 (z^1 + z^2)) \qquad (i = 1, 2), \tag{16$'$}$$
$$\beta_i \pi_i = e^{-\rho t} (b_i + b_{1i} u^1 + b_{i2} u^2) \qquad (i = 1, 2). \tag{17$'$}$$

In the situation, by the theorem 1 and the theorem 2, through the integration of $e^{-\rho t}(\varphi(x) + \psi(u^1, u^2))$, the maximizing problem of (1-i)$'$ under the constraints (2)$'$ in the two-sector open-loop Nash strategies is included in that of

$$\int_0^T e^{-\rho t} (\varphi(z^1 + z^2) + \psi(u^1, u^2)) dt \tag{12$'$}$$

under the constraints

$$\dot{z}^i = \alpha z^i + \beta_i u^i \quad (i = 1,\ 2), \tag{13$'$}$$

in the usual variational principle, where $\varphi(z^1 + z^2) + \psi(u^1, u^2)$ in (12)$'$ is given by (19a) and (20) with (21).

Now recall the theorem 2 in [4] for the derivation of conservation laws. By putting $q = (z^1, z^2, u^1, u^2)$ and $\lambda = (\pi_1, \pi_2)$ respectively, it shows here the following result:

On the optimal paths for the maxmizing problem of (12)$'$ under the constraints (13)$'$, let $\xi^i(\dot{q}, q, t)$ $(i = 1, 2, 3, 4)$ and $\eta_a(\dot{q}, q, t)$ $(a = 1,\ 2)$ satisfy the equations

$$\dot{\xi}^1 - \alpha \xi^1 = \beta_1 \xi^3, \tag{22a}$$
$$\dot{\xi}^2 - \alpha \xi^2 = \beta_2 \xi^4, \tag{22b}$$
$$\dot{\eta}_1 + \alpha \eta_1 = a_2 e^{-\rho t} (\xi^1 + \xi^2), \tag{22c}$$
$$\dot{\eta}_2 + \alpha \eta_2 = a_2 e^{-\rho t} (\xi^1 + \xi^2), \tag{22d}$$

$$\beta_1\eta_1 = e^{-\rho t}(b_{11}\xi^3 + b_{12}\xi^4), \tag{22e}$$

$$\beta_2\eta_2 = e^{-\rho t}(b_{12}\xi^3 + b_{22}\xi^4). \tag{22f}$$

Then the following conserved quantity Ω in the problem is constructed:

$$\Omega = \dot{z}^1\eta_1 + \dot{z}^2\eta_2 - (\dot{\pi}_1 + \rho\pi_1)\xi^1 - (\dot{\pi}_2 + \rho\pi_2)\xi^2. \tag{23}$$

The solutions ξ^i $(i = 1, 2, 3, 4)$ and η_a $(a = 1,\, 2)$ can be determined as follows. The difference $\dot{\eta}_2 - \dot{\eta}_1 = -\alpha(\eta_2 - \eta_1)$ of (22c) and (22d) is integrated as

$$\eta_2 = \eta_1 + C_1 e^{-\alpha t} \qquad (C_1\text{: const.}). \tag{24}$$

In view of $b_{11}b_{22} - b_{12}^2 \neq 0$ in (21), η_2 of (24) is substituted for the solutions ξ^3 and ξ^4 of (22e) and (22f) to see

$$\xi^3 = n_1 e^{\rho t}\eta_1 - \tfrac{b_{12}\beta_2}{B} C_1 e^{(\rho-\alpha)t}, \tag{25}$$

$$\xi^4 = n_2 e^{\rho t}\eta_1 + \tfrac{b_{11}\beta_2}{B} C_1 e^{(\rho-\alpha)t}, \tag{26}$$

where n_1 and n_2 are the constants:

$$n_1 = \tfrac{b_{22}\beta_1 - b_{12}\beta_2}{B}, \quad n_2 = \tfrac{b_{11}\beta_2 - b_{12}\beta_1}{B}, \qquad (B \equiv b_{11}b_{22} - b_{12}^2).$$

The above appearances of (25) and (26) are used in the addition of (22a) and (22b) to derive

$$(\dot{\xi}^1 + \dot{\xi}^2) - \alpha(\xi^1 + \xi^2) = (n_1\beta_1 + n_2\beta_2)e^{\rho t}\eta_1 + n_2\beta_2 C_1 e^{(\rho-\alpha)t}. \tag{27}$$

Moreover the identity $\xi^1 + \xi^2 = \frac{1}{a_2}e^{\rho t}(\dot{\eta}_1 + \alpha\eta_1)$ from (22c) is substituted for (27) to have

$$\ddot{\eta}_1 + \rho\dot{\eta}_1 + (\alpha(\rho - \alpha) - a_2(n_1\beta_1 + n_2\beta_2))\eta_1 = a_2 n_2\beta_2 C_1 e^{-\alpha t}. \tag{28}$$

By means of (21), which guarantees

$$N_\beta \equiv n_1\beta_1 + n_2\beta_2 = \tfrac{(b_{22}\beta_1 - b_{12}\beta_2)^2}{b_{22}(b_{11}b_{12} - b_{12}^2)} + \tfrac{\beta_2^2}{b_{22}} < 0,$$

the discriminant D of the subsidiary equation of (28) becomes positive:

$$D = (2\alpha - \rho)^2 + 4a_2 N_\beta > 0.$$

So, the solution η_1 of (28) can be determined as

$$\eta_1 = A_1 e^{(-\rho+\sqrt{D})t/2} + A_2 e^{(-\rho-\sqrt{D})t/2} - \tfrac{n_2\beta_2}{N_\beta} C_1 e^{-\alpha t} \quad (A_1,\, A_2,\, C_1\text{: const.}),$$

which is substituted for (24), (25) and (26) to obtain respectively

$$\eta_2 = A_1 e^{(-\rho+\sqrt{D})t/2} + A_2 e^{(-\rho-\sqrt{D})t/2} + \tfrac{n_1\beta_1}{N_\beta} C_1 e^{-\alpha t},$$

$$\xi^3 = n_1(A_1 e^{(\rho+\sqrt{D})t/2} + A_2 e^{(\rho-\sqrt{D})t/2}) - \tfrac{\beta_1\beta_2^2}{BN_\beta} C_1 e^{(\rho-\alpha)t},$$

$$\xi^4 = n_2(A_1 e^{(\rho+\sqrt{D})t/2} + A_2 e^{(\rho-\sqrt{D})t/2}) + \tfrac{\beta_1^2\beta_2}{BN_\beta} C_1 e^{(\rho-\alpha)t}.$$

Moreover, after substituting the above ξ^3 for (22a) and ξ^4 for (22b), ξ^1 and ξ^2 are determined respectively as

$$\xi^1 = \frac{2n_1\beta_1}{\rho-2\alpha+\sqrt{D}}A_1 e^{(\rho+\sqrt{D})t/2} + \frac{2n_1\beta_1}{\rho-2\alpha-\sqrt{D}}A_2 e^{(\rho-\sqrt{D})t/2} - \frac{\beta_1^2\beta_2^2}{BN_\beta(\rho-2\alpha)}C_1 e^{(\rho-\alpha)t} + C_2 e^{\alpha t},$$

$$\xi^2 = \frac{2n_2\beta_2}{\rho-2\alpha+\sqrt{D}}A_1 e^{(\rho+\sqrt{D})t/2} + \frac{2n_2\beta_2}{\rho-2\alpha-\sqrt{D}}A_2 e^{(\rho-\sqrt{D})t/2} + \frac{\beta_1^2\beta_2^2}{BN_\beta(\rho-2\alpha)}C_1 e^{(\rho-\alpha)t} - C_2 e^{\alpha t},$$

$$(A_1,\ A_2,\ C_1,\ C_2\text{: const.}).$$

Since A_1, A_2, C_1 and C_2 are arbitrary constants, the conserved quantity Ω of the form (23) yields the following four conserved quantities:

$$\Omega_1 = (\dot{z}^1 + \dot{z}^2)e^{(-\rho+\sqrt{D})t/2} - \frac{2}{\rho-2\alpha+\sqrt{D}}(n_1\beta_1(\dot{\pi}_1 + \rho\pi_1) + n_2\beta_2(\dot{\pi}_2 + \rho\pi_2))e^{(\rho+\sqrt{D})t/2},$$

$$\Omega_2 = (\dot{z}^1 + \dot{z}^2)e^{(-\rho-\sqrt{D})t/2} - \frac{2}{\rho-2\alpha-\sqrt{D}}(n_1\beta_1(\dot{\pi}_1 + \rho\pi_1) + n_2\beta_2(\dot{\pi}_2 + \rho\pi_2))e^{(\rho-\sqrt{D})t/2},$$

$$\Omega_3 = (-n_2\beta_2\dot{z}^1 + n_1\beta_1\dot{z}^2)e^{-\alpha t} + \frac{\beta_1^2\beta_2^2}{B(\rho-2\alpha)}((\dot{\pi}_1 + \rho\pi_1) - (\dot{\pi}_2 + \rho\pi_2))e^{(\rho-\alpha)t},$$

$$\Omega_4 = (\pi_1 - \pi_2)e^{\alpha t}.$$

Moreover, (10a)$'$, (10b)$'$ and(16)$'$ imply $\dot{\pi}_i = e^{-\rho t}(a_1 + a_2 x) - \alpha\pi_i$ $(i = 1, 2)$, which is substituted for $\dot{\pi}_i$ $(i = 1, 2)$ in Ω_j $(j = 1, 2, 3)$; $\dot{z}^1 + \dot{z}^2$ in Ω_j $(j = 1, 2)$ is written by (2)$'$, (10a)$'$ and (10b)$'$ as $\dot{z}^1 + \dot{z}^2 = \alpha x + \beta_1 u^1 + \beta_2 u^2$; (10a)$'$ and (10b)$'$ are substituted for $\dot{z}^i$ $(i = 1, 2)$ in Ω_3 and then, in the result, (2)$'$ and $\dot{y} = \alpha y + \beta_1 u^1 + \beta_2 u^2$ (see (18)) are substituted for $\dot{x}$ and $\dot{y}$; and finally (17)$'$ is used to eliminate u^i $(i = 1, 2)$. Consequently, by

$$N_b = b_1 n_1 + b_2 n_2, \qquad N_{\beta\pi} = n_1\beta_1\pi_1 + n_2\beta_2\pi_2,$$

above conserved quantities are written respectively as

$$\Omega_1 = \left(\frac{\rho-\sqrt{D}}{2}x + \frac{a_1(\rho-2\alpha-\sqrt{D})}{2a_2} - N_b\right)e^{(-\rho+\sqrt{D})t/2} - \frac{(\rho-\sqrt{D})N_{\beta\pi}}{\rho-2\alpha+\sqrt{D}}e^{(\rho+\sqrt{D})t/2}, \tag{29}$$

$$\Omega_2 = \left(\frac{\rho+\sqrt{D}}{2}x + \frac{a_1(\rho-2\alpha+\sqrt{D})}{2a_2} - N_b\right)e^{(-\rho-\sqrt{D})t/2} - \frac{(\rho+\sqrt{D})N_{\beta\pi}}{\rho-2\alpha-\sqrt{D}}e^{(\rho-\sqrt{D})t/2}, \tag{30}$$

$$\Omega_3 = \left(\frac{\alpha(-n_1\beta_1+n_2\beta_2)}{2}x + \frac{\alpha N_\beta}{2}y - \frac{\beta_1\beta_2(b_1\beta_2-b_2\beta_1)}{B}\right)e^{-\alpha t} - \frac{\alpha\beta_1^2\beta_2^2}{(\rho-2\alpha)B}(\pi_1 - \pi_2)e^{(\rho-\alpha)t}, \tag{31}$$

$$\Omega_4 = (\pi_1 - \pi_2)e^{\alpha t}. \tag{32}$$

Thus, by virtue of the theorem 2, it is concluded (Ω_3 is the conserved quantity with the new variable y):

Theorem 3. *In the two-sector open-loop Nash strategies, let each sector i $(i = 1, 2)$ seek to maximize:*

$$\int_0^T e^{-\rho t}(a_1 x + \tfrac{1}{2}a_2 x^2 + b_i u^i + \tfrac{1}{2}b_{ii}(u^i)^2 + b_{12}u^1u^2 + g_i(u^j))dt \qquad (j \neq i) \tag{1-i$''$}$$

$$(x < -a_1/a_2;\ a_i,\ b_i,\ b_{ij}\text{: const.},\ i, j = 1, 2;\ a_1 > 0,\ a_2 < 0,\ b_{ii} < 0)$$

under a constraint (2)$'$, *where $g_i(u^j)$ is an arbitrary function of u^j $(j \neq i)$. Then, there exist three conserved quantities* (29), (30) *and* (32).

3. Optimal paths for a model in the open-loop Nash strategies. In the conserved quantities (29) and (30), the term $N_{\beta\pi}$ can be eliminated to obtain the optimal path $x(t)$:

$$x(t) = \Xi_1 e^{(\rho-\sqrt{D})t/2} + \Xi_2 e^{(\rho+\sqrt{D})t/2} + \frac{a_1 N_\beta + (\rho-\alpha)N_b}{\alpha(\rho-\alpha) - a_2 N_\beta}, \tag{33}$$

where $\Xi_1 = \Omega_1/\sqrt{D}$ and $\Xi_2 = \Omega_2/\sqrt{D}$. Similarly, x in (29) and (30) is eliminated to have

$$N_{\beta\pi} = -2a_2 N_\beta \left(\frac{\Xi_1}{\rho-\sqrt{D}} e^{-(\rho+\sqrt{D})t/2} + \frac{\Xi_2}{\rho+\sqrt{D}} e^{(-\rho+\sqrt{D})t/2} \right) - \frac{N_\beta(a_1\alpha + a_2 N_b)}{\alpha(\rho-\alpha) - a_2 N_\beta} e^{-\rho t};$$

which and the conserved quantity (32) imply

$$\pi_1(t) = -\frac{2a_2\Xi_1}{\rho-\sqrt{D}} e^{-(\rho+\sqrt{D})t/2} + \frac{2a_2\Xi_2}{\rho+\sqrt{D}} e^{(-\rho+\sqrt{D})t/2} + n_2\beta_2\Xi_4 e^{-\alpha t} - \frac{a_1\alpha + a_2 N_b}{\alpha(\rho-\alpha) - a_2 N_\beta} e^{-\rho t}, \tag{34}$$

$$\pi_2(t) = -\frac{2a_2\Xi_1}{\rho-\sqrt{D}} e^{-(\rho+\sqrt{D})t/2} + \frac{2a_2\Xi_2}{\rho+\sqrt{D}} e^{(-\rho+\sqrt{D})t/2} - n_1\beta_1\Xi_4 e^{-\alpha t} - \frac{a_1\alpha + a_2 N_b}{\alpha(\rho-\alpha) - a_2 N_\beta} e^{-\rho t}, \tag{35}$$

where $\Xi_4 = \Omega_4/N_\beta$. The optimal paths (34) and (35) are substituted for $(17)'$ to obtain

$$u^1(t) = -\frac{2a_2 n_1 \Xi_1}{\rho-\sqrt{D}} e^{(\rho-\sqrt{D})t/2} + \frac{2a_2 n_1 \Xi_2}{\rho+\sqrt{D}} e^{(\rho+\sqrt{D})t/2} + \frac{\beta_1\beta_2^2\Xi_4}{B} e^{(\rho-\alpha)t} - \frac{n_1(a_1\alpha + a_2 N_b)}{\alpha(\rho-\alpha) - a_2 N_\beta} - \frac{b_{22}b_1 - b_{12}b_2}{B}, \tag{36}$$

$$u^2(t) = -\frac{2a_2 n_2 \Xi_1}{\rho-\sqrt{D}} e^{(\rho-\sqrt{D})t/2} + \frac{2a_2 n_2 \Xi_2}{\rho+\sqrt{D}} e^{(\rho+\sqrt{D})t/2} - \frac{\beta_1^2\beta_2\Xi_4}{B} e^{(\rho-\alpha)t} - \frac{n_2(a_1\alpha + a_2 N_b)}{\alpha(\rho-\alpha) - a_2 N_\beta} - \frac{b_{11}b_2 - b_{12}b_1}{B}. \tag{37}$$

Finally, the optimal paths (33), (34) and (35) are used in (31) to have

$$\begin{aligned} y(t) = {} & \frac{n_1\beta_1 - n_2\beta_2}{N_\beta}\left(\Xi_1 e^{(\rho-\sqrt{D})t/2} + \Xi_2 e^{(\rho+\sqrt{D})t/2}\right) \\ & + \Xi_3 e^{\alpha t} + \frac{2\beta_1^2\beta_2((\rho-\alpha)(b_{12}n_1 + b_{22}n_2) - (\rho-2\alpha)\beta_2)}{B\alpha(\rho-2\alpha)} \Xi_4 e^{(\rho-\alpha)t} \\ & + \frac{(n_1\beta_1 - n_2\beta_2)(a_1 N_\beta + (\rho-\alpha)N_b)}{N_\beta(\alpha(\rho-\alpha) - a_2 N_\beta)} + \frac{2\beta_1\beta_2(n_2(b_{22}b_1 - b_{12}b_2) - n_1(b_{11}b_2 - b_{12}b_1))}{B N_\beta \alpha}, \end{aligned} \tag{38}$$

where $\Xi_3 = 2\Omega_3/(N_\beta\alpha)$.

Theorem 4. *In the two-sector open-loop Nash strategies of finite horizon $T < \infty$, let each sector i $(i = 1, 2)$ seek to maximize* (1-i)$''$ *under the constraint* $(2)'$. *Then the optimal paths $x(t), u^1(t)$ and $u^2(t)$ are determined, completely as* (33), (36) *and* (37).

In the case of infinite horizon $T = \infty$, the optimal paths in the maximizing problem of $(12)'$ under the constraints $(13)'$ have to be feasible, i.e., they have to satisfy the transversality condition:

$$\lim_{t\to\infty} (\pi_1(t)z^1(t) + \pi_2(t)z^2(t)) = 0. \tag{39}$$

Such paths are called feasible optimal paths. By $(10\text{a})'$ and $(10\text{b})'$, the term $\pi_1(t)z^1(t) + \pi_2(t)z^2(t)$ is written as

$$\pi_1(t)z^1(t) + \pi_2(t)z^2(t) = \tfrac{1}{2}(\pi_1(t) + \pi_2(t))x(t) + \tfrac{1}{2}(\pi_1(t) - \pi_2(t))y(t),$$

for which the optimal paths (33), (34), (35) and (38) are substituted. Then, the result is a first order polynomial of $e^{(\rho-2\alpha-\sqrt{D})t/2}$, $e^{(\rho-2\alpha+\sqrt{D})t/2}$, $e^{(-\rho-\sqrt{D})t/2}$, $e^{(-\rho+\sqrt{D})t/2}$, $e^{\sqrt{D}t}$, $e^{-\sqrt{D}t}$, $e^{(\rho-2\alpha)t}$, $e^{-\alpha t}$ and $e^{-\rho t}$. Therefore, since $e^{(\rho-2\alpha-\sqrt{D})t/2}$, $e^{(-\rho-\sqrt{D})t/2}$, $e^{-\sqrt{D}t}$, $e^{-\alpha t}$ and $e^{-\rho t}$

go to zero as $t \to \infty$; the condition (39) requires that all of the coefficients of $e^{(\rho-2\alpha+\sqrt{D})t/2}$, $e^{(-\rho+\sqrt{D})t/2}$, $e^{\sqrt{D}t}$ and $e^{(\rho-2\alpha)t}$ (all of which go to ∞ as $t \to \infty$) and the constant term vanish:

$$(n_2\beta_2 - n_1\beta_1)\Xi_1\Xi_4 - \frac{n_2\beta_2 - n_1\beta_1}{N_\beta}\Xi_2 = 0,$$

$$\left(a_1\alpha + a_2N_b + \frac{2a_1a_2(N_\beta+(\rho-\alpha)N_b)}{\rho+\sqrt{D}}\right)\Xi_2 = 0,$$

$$\frac{4a_2}{\rho+\sqrt{D}}\Xi_2^2 = 0,$$

$$\frac{2\beta_1^2\beta_2((\rho-\alpha)(b_{12}n_1+b_{22}n_2)-(\rho-2\alpha)\beta_2)}{B\alpha(\rho-2\alpha)}\Xi_4 = 0,$$

$$\Xi_3 - \frac{2a_2\rho}{\alpha(\rho-\alpha)-a_2N_\beta}\Xi_1\Xi_2 = 0,$$

which imply that $\Xi_2 = \Xi_3 = \Xi_4 = 0$. Consequently, the optimal paths (33) (36) and (37) lead respectively to

$$(33)' \qquad x(t) = \Xi_1 e^{(\rho-\sqrt{D})t/2} + \frac{a_1N_\beta+(\rho-\alpha)N_b}{\alpha(\rho-\alpha)-a_2N_\beta},$$

$$(36)' \qquad u^1(t) = \frac{-2a_2n_1\Xi_1}{\rho-\sqrt{D}}e^{(\rho-\sqrt{D})t/2} - \frac{n_1(a_1\alpha+a_2N_b)}{\alpha(\rho-\alpha)-a_2N_\beta} - \frac{b_{22}b_1-b_{12}b_2}{B},$$

$$(37)' \qquad u^2(t) = -\frac{2a_2n_2\Xi_1}{\rho-\sqrt{D}}e^{(\rho-\sqrt{D})t/2} - \frac{n_2(a_1\alpha+a_2N_b)}{\alpha(\rho-\alpha)-a_2N_\beta} - \frac{b_{11}b_2-b_{12}b_1}{B}.$$

Theorem 5. *In the two-sector open-loop Nash strategies of infinite horizon $T = \infty$, let each sector i $(i = 1, 2)$ seek to maximize* (1-i)$''$ *under the constraints* (2)$'$. *Then there exist the feasible optimal paths of the form* (33)$'$, (36)$'$ *and* (37)$'$.

An Example. The results can be applied to the model of the voluntary contributions to the provision of a collectively produced good (Fershtman and Nitzan [2], in which the stock of total contributions K, the contribution rate of i-sector x_i, the discount rate r and the constant α are denoted here by x, u^i, ρ and γ respectively), by putting $a_1 = a\gamma$, $a_2 = -2b\gamma$, $b_1 = b_2 = 0$, $b_{11} = b_{22} = -c$, $b_{12} = 0$ and $g_1(u^2) = g_2(u^1) = 0$ in (1-i)$''$, i.e.,

$$(1\text{-i})''' \qquad \int_0^T \left(\gamma(ax - bx^2) - \tfrac{1}{2}c(u^i)^2\right) dt$$
$$(x < a/(2b),\ a,\ b,\ c,\ \gamma\text{: const.},\ a > 0,\ b > 0,\ c > 0,\ \gamma > 0),$$

and by putting $\alpha = -\delta$ and $\beta_1 = \beta_2 = 1$ in the constraint (2)$'$, i.e.,

$$(2)'' \qquad \dot{x} = -\delta x + u^1 + u^2 \qquad (\delta\text{: const.},\ \delta > 0).$$

Then, the conserved quantities (29)-(32) are reduced respectively to

$$\Omega_1 = \left(\frac{\rho-\sqrt{D}}{2}x - \frac{a\gamma(\rho+2\delta-\sqrt{D})}{4b\gamma}\right)e^{(-\rho+\sqrt{D})t/2} + \frac{\rho-\sqrt{D}}{c(\rho+2\delta+\sqrt{D})}(\pi_1+\pi_2)e^{(\rho+\sqrt{D})t/2},$$

$$\Omega_2 = \left(\frac{\rho+\sqrt{D}}{2}x - \frac{a\gamma(\rho+2\delta+\sqrt{D})}{4b\gamma}\right)e^{-(\rho+\sqrt{D})t/2} + \frac{\rho+\sqrt{D}}{c(\rho+2\delta-\sqrt{D})}(\pi_1+\pi_2)e^{(\rho-\sqrt{D})t/2},$$

$$\Omega_3 = \frac{1}{c}(\delta x - \delta y - u^1 - u^2)e^{\delta t} - \frac{\rho+\delta}{(\rho+2\delta)c^2}(\pi_1-\pi_2)e^{(\rho+\delta)t},$$

$$\Omega_4 = (\pi_1 - \pi_2)e^{-\delta t},$$

in which Ω_1, Ω_2 and Ω_4 are the conserved quantities in the maximizing problem of (1-i)$'''$ (i$= 1, 2$) under the constraint $(2)''$ in the two-sector open-loop Nash strategies. Moreover, the optimal path $(33)'$ in the infinite horizon $T = \infty$ is also to

$$x(t) = \Xi_1 e^{(\rho-\sqrt{D})t/2} + \tfrac{2a\gamma}{\delta(\rho+\delta)c+4b\gamma}.$$

So, it follows that

$$\lim_{t\to\infty} x(t) = \tfrac{2a\gamma}{\delta(\rho+\delta)c+4b\gamma},$$

which is the level of the collective contribution at the stationary open-loop Nash equilibrium K^* appeared in ([2], Theorem 2, in which n is given here as $n = 2$).

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