COMMON FIXED POINTS THEOREM USING MINIMAL COMMUTATIVITY AND RECIPROCAL CONTINUITY CONDITIONS IN METRIC SPACE

Sanjay Kumar and Renu Chugh

Received November 12, 1999

Abstract. The purpose of this paper is to prove a common fixed point theorem, from the class of compatible continuous mappings to a larger class of mappings having noncompatible and discontinuous mappings which generalizes the result of G. Jungck, B. Fisher, S.M. Kang and Y.P. Kim, Jachymski, Rhoades and Pant.

1 Introduction In 1976, G. Jungck [6] proved a common fixed point theorem for commuting maps generalizing the Banach's fixed point theorem, which states that, "let (X, d) be a complete metric space. If T satisfies $d(Tx, Ty) \leq kd(x, y)$ for each $x, y \in X$ where $0 \leq k < 1$, then T has a unique fixed point in X". This theorem has many applications, but suffers from one drawback-the definition requires that T be continuous throughout X. There then follows a flood of papers involving contractive definition that do not require the continuity of T. This result was further generalized and extended in various ways by many authors. On the other hand S. Sessa [16] defined weak commutativity and proved common fixed point theorem for weakly commuting maps. Further G. Jungck [7] introduced more generalized commutativity, so called compatibility, which is more general than that of weak commutativity. Since then various fixed point theorems, for compatible mappings satisfying contractive type conditions and assuming continuity of at least one of mappings, have been obtained by many authors.

It has been known from the paper of Kannan [9] that there exists maps that have a discontinuity in the domain but which have fixed points, moreover, the maps involved in every case were continuous at the fixed point. In 1994, R. Pant [12] introduced the notion of pointwise R-weak commutativity and show that compatible maps are pointwise R-weak commutativity but converse need not be true. In this paper, we present a common fixed point theorem, in which the fixed point may be point may point of discontinuity, by using a minimal commutativity condition and reciprocal continuity which generalizes the result of Jungck *et al.* [8], Fisher [3], Kang and Kim [10], Jachymski [4], Rhoades *et al.*[14] and Pant [11, 13].

2. Preliminaries

Definition 2.1. Let A and S be mappings from a metric space (X, d) into itself. Then A and S are said to be compatible mappings on X if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$, when $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$. If A and S are compatible mappings on X, then $d(ASt, SAt) = 0$, when $d(At, St) = 0$ for some t in X.

Definition 2.2. Two self maps S and T of a metric space (X, d) are called pointwise R-weakly commuting on X if given x in X there exists $R > 0$ such that $d(ASx, SAx) \leq Rd(Ax, Sx)$.

2000 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. R-weak commutativity, reciprocal continuity, compatible mappings.

It is obvious that pointwise R-weakly commuting maps commute at their coincidence points but maps A and S can fail to be pointwise R-weakly commuting only if there exists some x in X such that $Ax = Sx$ but $ASx \neq SAx$. Therefore, the notion of pointwise R-weak commutativity is equivalent to commutativity at coincidence points. Moreover, since contractive conditions exclude the possibilities of the existence of a common fixed point together with existence of a coincidence point at which the mappings do not commute, poinwise R-weak commutativity is a necessary condition for the existence of common fixed points of contractive type mapping pairs.

It is to be noted that compatible maps are necessarily pointwise R-weakly commuting since compatible maps commute at their coincident points but converse may not be true.

Example 2.1. Let $X = [2, 20]$ and d be usual metric on X.

Define $A, S : X \to X$ by $Ax = 2$ if $x = 2$ or $x > 5$, $Sx = 2$ if $x = 2$

$Ax = 6$ if $2 < x \leq 5$, $Sx = x - 3$ if $x > 5$, $Sx = 12$ if $2 < x \leq 5$.

The mappings A and S are non-compatible but they are pointwise R-weakly commuting. A and S are pointwise R-weakly commuting since they commute at their coincidence points. Let us consider the sequence $\{x_n\}$ defined by $x_n = 5 + 1/n,\ n \geq 1$.
Then $Sx_n \to 2,\ Ax_n = 2,\ SAx_n = 2$ and $ASx_n = 6$. Hence A and S are non-compatible. Moreover, mappings A and S are discontinuous at $x = 2$.

Example 2.2. Let $X = [4, 30]$ and d be the usual metric defined on X. i.e. $d(x, y) = |x - y|$ for all $x, y \in X$.

Define $Ax = x$ if $x = 4$ or $x > 5$, $Ax = 10$ if $4 < x \leq 5$.

$Sx = x$ if $x = 4$, $Sx = 20$ if $4 < x \leq 5$, $Sx = x - 1$ if $x > 5$.

Here A and S pointwise R-weakly commuting, since they commute at their coincidence points. Let us consider the sequence $\{x_n\}$ defined by $5 + 1/n,\ n \geq 1$. Then $Sx_n \to 4,\ Ax_n = 4,\ SAx_n = 4$ and $ASx_n = 10$. Hence mappings A and S are non-compatible. The mappings involved in this example are discontinuous at $x = 4$.

Definition 2.3. Let A and S be mappings from a metric space (X, d) into itself. Then A and S are said to be reciprocally continuous if $\lim_{n\to\infty} ASx_n = At$ and $\lim_{n\to\infty} SAx_n = St$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some t in X.

Continuous mappings are reciprocally continuous on (X, d) but converse may not be true.

Example 2.3 Let $X = [2, 20]$ and d be usual metric on X.

Define mappings $A, S : X \to X$ by $Ax = 2$ if $x = 2$, $Sx = 2$ if $x = 2$
$Ax = 3$ if $x > 2$ $Sx = 6$ if $x > 2$.
It is noted that A and S are reciprocally continuous mapings but they are not continuous.

Example 2.4 Let $X = [4, 30]$ and d be usual metric on X.

Define mappings $A, S : X \to X$ by $Ax = x$ if $x = 4$ $Sx = x$ if $x = 4$
$Ax = 5$ if $x > 4$ $Sx = 10$ if $x > 4$.

Here A and S are reciprocally continuous mappings but A and S are not continuous. D. Delbosco [2] considered the set S of all real continuous functions $g : [0, \infty)^3 \to [0, \infty)$ satisfying the following properties :

(i) $g(1, 1, 1) = h < 1$,

(ii) If $u, v \geq 0$ are such that $u \leq g(u,v,v)$ or $u \leq g(v,u,v)$ or $u \leq g(v,v,u)$, then $u \leq hv$.

But later on Constantin [1] considered the family G of all continuous functions g, where $g : [0,\infty)^5 \to [0,\infty)$ satisfies the following properties :

(g_1) g is non-decreasing in the 4th and 5th variable,

(g_2) If $u, v \in [0,\infty)$ are such that $u \leq g(v,v,u,u+v,0)$ or $u \leq g(v,u,v,u+v,0)$ or $u \leq g(v,u,v,u+v,0)$ or $u \leq g(v,u,v,0,u+v)$, then $u \leq hv$ where $0 < h < 1$ is a given constant,

(g_3) If $u \in [0,\infty)$ is such that $u \leq g(u,0,0,u,u)$ or $u \leq g(0,u,0,u,u)$ or $u \leq g(0,0,u,u,u)$, then $u = 0$.

3. Fixed Point Theorem

Let A, B, S and T be mappings from a metric space (X,d) into itself satisfying the following conditions :
Let (A,S) and (B,T) be pointwise R-weakly commuting pairs of self mappings of a complete metric space (X,d) such that

$$A(X) \subset T(X), \ B(X) \subset S(X) \quad \text{and} \tag{3.1}$$

$$d(Ax,By) \leq g(d(Sx,Ty), d(Ax,Sx), d(By,Ty), d(Ax,Ty), d(By,Sx)) \tag{3.2}$$

for all $x, y \in X$, where $g \in G$.
Then for an arbitrary point x_0 in X, by (3.1), we choose a point x_1 such that $Tx_1 = Ax_0$ and for this point x_1, there exists a point x_2 in X such that $Sx_2 = Bx_1$ and so on. Continuing in this manner, we can define a sequence $\{y_n\}$ in X such that

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \ \text{ and } \ y_{2n+1} = Bx_{2n+1} = Sx_{2n+1}, \quad n = 1,2,3,\ldots \tag{3.3}$$

Lemma 3.1. *Let A, B, S and T be mappings from a complete metric space (X,d) into itself satisfying the conditions* (3.1) *and* (3.2). *Then the sequence $\{y_n\}$ defined by* (3.3) *is a Cauchy sequence in X.*

Proof. From (3.2) we have

$$\begin{aligned} d(Ax_{2n}, Bx_{2n+1}) \ &\leq \ g(d(Sx_{2n}, Tx_{2n+1}), d(Ax_{2n}, Sx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), \\ &\qquad d(Ax_{2n}, Tx_{2n+1}), d(Bx_{2n+1}, Sx_{2n})) \end{aligned}$$

$$\begin{aligned} d(y_{2n}, y_{2n+1}) \ &\leq \ g(d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), d(y_{2n}, y_{2n}), d(y_{2n+1}, y_{2n-1})) \\ &\leq \ g(d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), 0, [d(y_{2n+1}, y_{2n}) + d(y_{2n}, y_{2n-1})]) \end{aligned}$$

By (g_2), we obtain, $d(y_{2n}, y_{2n+1}) \leq hd(y_{2n-1}, y_{2n})$.
But $d(y_n, y_{n+1}) \leq hd(y_{n-1}, y_n) \leq \ldots \leq h^n d(y_0, y_1)$.
Moreover, for every integer $m > 0$, we get

$$\begin{aligned} d(y_n, y_{n+m}) \ &\leq \ d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2}) + \ldots d(y_{n+m-1}, y_{n+m}) \\ &\leq \ h^n d(y_0, y_1) + h^{n+1} d(y_0, y_1) + \ldots h^{n+m-1} d(y_0, y_1) \\ &= \ h^n d(y_0, y_1)(1 + h + h^2 + \ldots h^{m-1}) \\ d(y_n, y_{n+m}) \ &\leq \ h^n/(1-h) d(y_0, y_1) \end{aligned}$$

On proceeding limit as $n \to \infty$, we have $d(y_n, y_{n+m}) \to 0$. Therefore, $\{y_n\}$ is a Cauchy sequence in X.

Theorem 3.1. *Let (A, S) and (B, T) be pointwise R-weakly commuting pairs of self mappings of a complete metric space (X, d) satisfying (3.1) and (3.2).*
Suppose that (A, S) or (B, T) is a compatible pair of reciprocally continuous mappings. Then A, B, S and T have a unique common fixed point in X.

Proof. By lemma 3.1, $\{y_n\}$ is a Cauchy sequence in X. Since X is complete. So there exists a point z in X such that $\lim_{n\to\infty} y_n = z$. $\mathrm{Lim}_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z$ and $\lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z$. Suppose A and S are compatible and reciprocally continuous. Then by reciprocally continuous of A and S, we have $\lim_{n\to\infty} ASx_{2n} = Az$ and $\lim_{n\to\infty} SAx_{2n} = Sz$. Also, by compatibility of A and $S, Az = Sz$. Since $AX \subset TX$, so there exists a point v in X such that $Az = Tv$.

$$\begin{aligned} d(Az, Bv) &\leq g(d(Sz, Tv), d(Az, Sz), d(Bv, Tv), d(Az, Tv), d(Bv, Sz)) \\ &= g(d(Tv, Tv), d(Sz, Sz), d(Az, Bv), d(Az, Az), d(Az, Bv)) \\ &= g(0, 0, d(Az, Bv), 0, d(Az, Bv)) \end{aligned}$$

Then by (g_3) $Az = Bv$.
Thus $Az = Sz = Tv = Bv$.
Since A and S are pointwise R-weak commutativity, there exist $R > 0$ such that $d(ASz, SAz) \leq Rd(Az, Sz) = 0$, this implies that $ASz = SAz$ and $AAz = ASz = SAz = SSz$. Also, B and T are pointwise R-weak commutative, so we have $BBv = BTv = TBv = TTv$. From (3.2), we get

$$\begin{aligned} d(Az, AAz) = d(AAz, Bv) &\leq g(d(SAz, Tv), d(AAz, SAz), d(Bv, Tv), \\ &\quad\ d(AAz, Tv), d(Bv, SAz)) \\ &= g(d(AAz, Bv), 0, 0, d(AAz, Bv), d(AAz, Bv)) \end{aligned}$$

By (g_3) $AAz = Az$, so $Az = AAz = SAz$.
Thus Az is a common fixed point of A and S. Similarly, we can prove that $Bv(= Az)$ is a common fixed point of B and T.
Finally, in order to prove uniqueness of Az, suppose that Az and Aw, $Az \neq Aw$, are common fixed points of A, B, S and T. Then by (3.2), we obtain

$$\begin{aligned} d(Az, Aw) &= d(AAz, BAw) \\ &\leq g(d(SAz, TAw), d(AAz, SAz), d(BAw, TAw), \\ &\quad\ d(AAz, TAw), d(BAw, SAz)) \\ &= g(d(Az, Aw), d(Az, Az), d(Aw, Aw), d(Az, Aw), d(Aw, Az)) \\ &= g(d(Az, Aw), 0, 0, d(Az, Aw), d(Az, Aw)) \end{aligned}$$

By (g_3), we have $Az = Aw$. This completes the proof.

The following corollaries follow immediately from Theorem 3.2.

Corollary 3.1 [12]. *Let (A, S) and (B, T) be pointwise R-weakly commuting pairs of self mapping of a complete metric space (X, d) satisfying (3.1), (3.3) and (3.4)*

$$d(Ax, By) \leq hM(x, y), \ 0 \leq h < 1, \ x, y \in X, \quad \text{where} \tag{3.4}$$

$M(x,y) = \max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), [d(Ax,Ty) + d(By,Sx)]/2\}$

Suppose that (A,S) or (B,T) is a compatible pair of reciprocally continuous mappings. Then A, B, S and T have a unique common fixed point in X.

Proof. We consider the function $g : [0,\infty)^5 \to [0,\infty)$ defined by
$g(x_1,x_2,x_3,x_4,x_5) = h \ \max\{x_1,x_2,x_3,1/2(x_4+x_5)\}$.
Since $g \in G$. We can apply Theorem 3.1 and deduce the Corollary.

Example 3.1. Let $X = [2,20]$ and d be the usual metric on X.

Define mappings $A, B, S, T : X \to X$ by

$$\begin{array}{lll} Ax = x \text{ if } x = 2, & Sx = x \text{ if } x = 2, & Sx = 6 \text{ if } x > 2 \\ Ax = 3 \text{ if } x > 2, & Bx = x \text{ if } x = 2 \text{ or } x > 5, & Bx = 6 \text{ if } 2 < x \le 5 \\ Tx = x \text{ if } x = 2, & Tx = 12 \text{ if } 2 < x \le 5, & Tx = x - 3 \text{ if } x > 5 \end{array}$$

It may be noted that A and S are reciprocally continuous mappings. But neither A nor S is continuous not even at the common fixed point $x = 2$. The mappings B and T are non-compatible but pointwise R-weakly commuting. B and T are pointwise R- weakly commuting, since they commute at their coincidence points. Let us consider the sequence $\{x_n\}$ defined by $x_n = 5+(1/n),\ n \ge 1$. Then $Tx_n \to 2,\ Bx_n = 2,\ TBx_n = 2$ and $BTx_n = 6$. Hence B and T are non-compatible. Thus A, B, S and T satisfy all the conditions of the Corollary 3.1 with $h = 2/3$ and have a unique common fixed point $x = 2$. Moreover, all the mappings involved in this example are discontinuous at $x = 2$.

Remark 3.1. Kang, Kim [10] stated that theorem 3.1 is no longer true if we do not assume that any of the mappings to be continuous (see page 1037). Now we have shown in example 3.1 that continuity of any one of the maps is not necessary for the existence of common fixed points if we assume poinwise R-weak commutativity, reciprocal continuity and any one of the pairs to be compatible.

Corollary 3.2. *Let (A,S) and (B,T) be pointwise R-weakly commuting pairs of self mapping of a complete metric (X,d) satisfying* (3.1), (3.3) *and* (3.5). *Suppose that (A,S) or (B,T) is a compatible pair of reciprocally continuous mappings.*
Then A, B, S and T have a unique common fixed point in X.

(3.5) $\quad d(Ax,By) \le h \ \max\{d(Ax,Sx), d(By,Ty), 1/2d(Ax,Ty), 1/2d(By,Sx), d(Sx,Ty)\}$

for all x, y in X, where $0 \le h < 1$.

Then A, B, S and T have a unique common fixed point in X.

Proof. We consider the function $g[0,\infty)^5 \to [0,\infty)$ defined by
$g(x_1,x_2,x_3,x_4,x_5) = h \ \max\{x_1,x_2,x_3,1/2x_4,1/2x_5\}$. Since $g \in G$, we can apply Theorem 3.1 to obtain this Corollary.

Remark 3.2. Theorem 3.1, generalizes the result of Jungck [5] by using a minimal commutativity condition and a new type of continuity condition as opposed to the continuity of both S and T. Theorem 3.1 and Corollary 3.2 also generalize the result of Fisher [3] by employing fewer compatibility and new type of continuity condition instead of commutativity of maps. Further, the condition (3.2) is more general that the condition of Fisher [3]. Moreover, the results of Jachymski [4], Jungck *et al.* [8], Pant [11, 13], Rhoades *et al.* [14], Kang & Kim [10] are also generalized in two ways. Firstly, by using both compatibility and pointwise R-weak commutativity. Secondly, our theorem does not force the maps to be continuous at the fixed point (see Example 3.1).

Remark 3.3 We shall show that under contractive condition (3.2) continuity of one of the mappings in the compatible pair implies their reciprocally continuity. Let us assume that A and S are compatible and S is continuous. Let $\{x_n\}$ be sequence such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t \text{ for some } t \text{ in } X.$$

Now we shall prove that $\lim_{n\to\infty} ASx_n = At$ and $\lim_{n\to\infty} SAx = St$. Since $AX \subset TX$, so for each n, there exist y_n in X such that $ASx_n = Ty_n$.
Also $\lim_{n\to\infty} SSx_n = \lim_{n\to\infty} SAx_n = \lim_{n\to\infty} ASx_n = \lim_{n\to\infty} Ty_n = St$.
We claim that $\lim_{n\to\infty} By_n = St$. If not possible, then there exist a subsequence $\{By_m\}$ of $\{By_n\}$, a number $r > 0$ and a positive integer N such that for each $m \geq N$, we have $\lim_{m\to\infty} d(ASx_m, By_m) \geq r$, i.e., $\lim_{m\to\infty} d(By_m, St) \geq r$.
From (3.2), we have

$$\begin{aligned} d(ASx_m, By_m) \quad \leq \quad & g(d(SSx_m, Ty_m), d(ASx_m, SSx_m), d(By_m, Ty_m), \\ & d(ASx_m, Ty_m), d(By_m, SSx_m)) \end{aligned}$$

Letting limit as $m \to \infty$, we have

$$\lim_{m\to\infty} d(St, By_m) \leq \lim_{m\to\infty} g(0, 0, d(By_m, St), 0, d(By_m, St))$$

By (g_3), we have $\lim_{m\to\infty} d(St, By_m) = 0$, i.e., $\lim_{n\to\infty} By_n = St$.

Again from (3.2), we get

$$d(At, By_n) \leq g(d(St, Ty_n), d(At, St), d(By_n, Ty_n), d(At, Ty_n), d(By_n, St)$$

Taking limit as $n \to \infty$, we get $At = St$.
Thus $\lim_{n\to\infty} SAx_n = St$ and $\lim_{n\to\infty} ASx_n = St = At$. Hence the assertion. A similar argument applies when A and S are compatible and A is assumed to be continuous.

References

[1] A. Constantin, Common fixed points of weakly commuting mappings in 2-metric spaces, **Math. Japon., 38(3)** (1991), 507–514.

[2] D. Delbosco, Un'estesione di un teorema sul punto fisso di **S. Reich Rend. Semin. Math. Univers. Pol. Torino., 35**(1976–1977), 233-238.

[3] Fisher, B., Common fixed points of four mappings, **Bull. Inst. Math. Acad. Scinica, 11 (1983), 103–113.**

[4] Jachymski, J., Common fixed point theorems for some families of maps, **Indian J. Pure Appl. Math., 25** (1994), 925-934.

[5] Jungck, G., Compatible mappings and common fixed points (2), **Internat. J. Math. and Math. Sci., 11** (1988), 285–288.

[6] Jungck, G., Commuting maps and fixed points, **Amer. Math. Monthly 83** (1976), 261-263.

[7] Jungck, G., Compatible mappings and common fixed points, **Internat. J. Math. & Math. Sci., 9** (1986), 771–779.

[8] Jungck, G., Moon, K.B., Park, S. and Rhoades, B.E., On generalizations of the Meir-Keeler tpe contraction maps : Correction, **J. Math. Anal. Appl. 180** (1993), 221.

[9] Kannan, R., Some results on fixed points, **Bull. Cal. Math. Soc., 60** (1968), 71-76.

[10] Kang, S.M and Kim, Y.P., Common fixed points theorems, **Math. Janonica, 37,** No. 6(1992), 1031–1039.

[11] Pant, R.P., Common fixed points of weakly commuting mappings, **Math. Student 62** (1993), 97-102.

[12] Pant, R.P., Common fixed points of non-commuting mappings, **J. Math. Anal. Appl. 188** (1994), 436-440.

[13] Pant, R.P., Common fixed points of four mappings, **Bull. Cal. Math. Soc. 90** (1998), 281–286.

[14] Rhoades, B.E., Park, S. and Moon, K. B., On generalizations of the Meir-Keeler type contraction maps, **J. Math. Anal. Appl. 146** (1990), 482-487.

[15] Rhoades, B.E., Contractive definitions and continuity, **Contemporary Mathematics, Vol. 72** (1988), 233–245.

[16] Sessa, S., On a weak commutativity condition of mappings in fixed point considerations, **Pub. Inst. Math., 32(46),** (1982), 149–153.

Department of Mathematics, M.D.Univeersity, Rohtak - 124001, India