ON L- AND D-SPACES

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Abstract. We show that L-spaces in the sense of Nagami are hereditary, answering his question positively. As the application, we give the metrizability of a space in terms of its hyperspaces. We also show that both L-spaces and D-spaces coincide with each other under the condition of their large inductive dimension 0.

1. Introduction.

All spaces are assumed to be regular T_1. The letter $\mathbb{N}$ denotes all positive numbers. For a space X, $\tau(X)$ denotes the topology of X. For brevity, let the letters CP and IP stand for the terms "closure-preserving" and "interior-preserving", respectively.

In 1980, Nagami defined two classes of L-spaces and D-spaces between Lašnev spaces and M_1-spaces from the point of view of dimension theory. L-spaces are preserved by closed mappings, but not closed under finite products. As for the subspaces, he asked whther L-spaces are hereditary, [4, p.241], while the open and closed subspaces are L-spaces, [4, Theorem 1.8].

In this paper, first we show that L-spaces are hereditary, answering his question positively. Also, we apply the result to the metrizability of a space in terms of its hyperspaces. Second, we show that if a space X is an L-space with $\mathrm{Ind}X=0$, then X is a D-space. This is a partial answer to his question whether L-spaces are D-spaces, [4, Problem 4.8]. This has been already answered negatively by Okuyama [5], where he actually constructed an L-space, but not a D-space X with $\mathrm{Ind}X=1$.

2. L-spaces.

A space X is said to be an *L-space* if X is a paracomapct σ-space such that each closed subset F of X has a CP closed neighborhood base and at the same time has an IP (in $X \setminus F$) open neighborhood base in X, [4, Definition 1.2]. Equivalently, a space X is an L-space if and only if X is a paracompact σ-space such that each closed subset has an approaching anti-cover, [4, Theorem 1.3]. Any L-space is an M_1-space, [4, Theorem 1.7], and hence stratifiable in the sense of Borges [1]. So, any L-space X has the stratification S :{closed subsets of X} $\times \mathbb{N} \longrightarrow \tau(X)$ satisfying the following:

(i) For each closed set F of X,

$$\bigcap_n S(F,n) = \bigcap_n \overline{S(F,n)} = F;$$

(ii) if F and G are closed sets of X such that $F \subset G$, then $S(F,n) \subset S(G,n)$ for each n.

Theorem 2.1. *L-spaces are hereditary.*

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Proof. Since L-spaces are hereditary with respect to closed subspaces, [4, Theorem 1.8], it suffices to show that any dense subspace X of an L-space Y is an L-space. Let M be a closed subset of X. Since $\overline{M}$ is a σ-space, there exists a σ-discrete closed network $\mathcal{F}$ for $\overline{M}$. Let $\mathcal{F} = \bigcup\{\mathcal{F}_n \mid n \in \mathbb{N}\}$, where each $\mathcal{F}_n$ is discrete in Y. Let $\{O(F) \mid F \in \mathcal{F}_n\}$ be a discrete open expansion of $\mathcal{F}_n$ in Y such that

$$F \subset O(F) \subset S(F, n). \tag{1}$$

Since Y is an L-space, F has an IP (in $Y \setminus F$) base $\mathcal{U}(F)$ such that $\bigcup \mathcal{U}(F) \subset O(F)$. Let τ_0 be the totality of open subsets of Y containing M. For each $O \in \tau_0$, let

$$\mathcal{F}(O) = \{F \in \mathcal{F} \mid F \subset O\}.$$

For each $F \in \mathcal{F}(O)$, take $U_F \in \mathcal{U}(F)$ such that $U_F \subset O$. Then it is easy to see that

$$W(O) = \bigcup\{U_F \cap X \mid F \in \mathcal{F}(O)\}$$

is an open neighborhood of M in X. By (1) it is easily checked that

$$\mathcal{W} = \{W(O) \mid O \in \tau_0\}$$

is an open neighborhood base of M in X which is IP in $X \setminus M$. Since L-spaces are stratifiable, M has a CP closed neighborhood base in X. This proves that X is an L-space. $\square$

In order to give an application of the result above, we recall the definition of finite topology to the hyperspace of a space X. Let $\mathcal{K}(X)$ and $\mathcal{F}(X)$ be the set of all non-empty compact subsets and non-empty finite subsets of X, respectively. The hyperspace $\mathcal{K}(X)$ is topologized by a base consisting of subsets of the form:

$$\langle U_1, \cdots, U_k \rangle = \{K \in \mathcal{K}(X) \mid K \subset \bigcup_{i=1}^{k} U_i,\ K \cap U_i \neq \emptyset,\ i = 1, \cdots, k\},$$

where $\{U_1, \cdots, U_k\}$ is a finite family of open subsets of X. Note that the notation $\langle \cdots \rangle$ is used later in the same sense as above for any finite family of subsets (not necessarily open) of X. $\mathcal{F}(X)$ is considered the subspace of $\mathcal{K}(X)$. In $\mathcal{F}(X)$, we write again $\langle U_1, \cdots, U_k \rangle$ in place of $\langle U_1, \cdots, U_k \rangle \cap \mathcal{F}(X)$.

Theorem 2.2. *For a countable space X, the following are equivalent:*

(i) *$\mathcal{K}(X)$ is an L-space.*
(ii) *$\mathcal{F}(X)$ is an L-space.*
(iii) *X is metrizable.*

Proof. (i) $\to$ (ii) is obvious from the theorem above. (iii) $\to$ (i) is obvious, because the metrizability of X implies the metrizability of $\mathcal{K}(X)$, [2, Theorem 4.9.13]. We show (ii) $\to$ (iii): Without loss of generality we can assume that X is not discrete. Then there exists a sequence (p_n) of points which has an accumulation point p, where $p \neq p_n \neq p_m$ for each pair (n, m) with $n \neq m$. Since $\mathcal{F}(X)$ is an L-space, $\hat{A}(x) = \big\{\{p, x\}\big\}$ has an approaching anti-cover $\hat{\mathcal{U}}(x)$ in $\mathcal{F}(X)$. For each pair (n, m) with $n \neq m$ and each point $y \in X$, there exists $\hat{U}(n, m, y) \in \hat{\mathcal{U}}(x)$ with $\{p_n, p_m, y\} \in \hat{U}(n, m, y)$, because $\{p_n, p_m, y\} \notin \hat{A}(x)$. This implies that

$$\langle \{p_n\}, \{p_m\}, U(n, m, y) \rangle \subset \hat{U}(n, m, y)$$

for some open neighborhood $U(n, m, y)$ of y in X. If we let

$$\mathcal{U}(n, m, x) = \{U(n, m, y) \mid y \in X\},$$

then

$$\mathcal{U} = \{\mathcal{U}(n, m, x) \mid n, m \in \mathbb{N},\ x \in X\}$$

is a sequence of open covers of X. To show the metrizability of X, it suffices to show that $\mathcal{U}$ satisfies the criteria of the Moore metrization theorem. Suppose that O is an open neighborhood of x in X. First, we consider the case $p \neq x$. Take disjoint open neighborhoods $O_1,\ O_2$ of $p,\ x$ in X, respectively, in X such that $O_2 \subset O$. Since $\hat{\mathcal{U}}(x)$ is approaching to $\hat{A}(x)$ in $\mathcal{F}(X)$, there exists an open neighborhood $\hat{U} = \langle U_1, U_2 \rangle$ of $\{p, x\}$ in $\mathcal{F}(X)$ such that

$$St\big(\mathcal{F}(X) \setminus \langle O_1, O_2 \rangle, \hat{\mathcal{U}}(x)\big) \cap \hat{U} = \emptyset, \qquad U_1 \subset O_1,\ U_2 \subset O_2. \tag{2}$$

There exist $p_n,\ p_m \in U_1$ with $n \neq m$. For these n and m, we show that $St(U_2, \mathcal{U}(n, m, x)) \subset O$. Suppose

$$U(n, m, z) \cap U_2 \neq \emptyset,\ U(n, m, z) \cap (X \setminus O) \neq \emptyset$$

for some member $U(n, m, z) \in \mathcal{U}(n, m, x)$. But this is a contradiction to (2). Next, we consider the case $x = p$. Since $\hat{\mathcal{U}}(p)$ is approaching to $\{p\}$ in $\mathcal{F}(X)$, there exists an open neighborhood U of p in X such that

$$St\big(\mathcal{F}(X) \setminus \langle O \rangle, \hat{\mathcal{U}}(p)\big) \cap \langle U \rangle = \emptyset. \tag{3}$$

Take $p_n,\ p_m \in U$ with $n \neq m$. By (3), we easily have $St(U, \mathcal{U}(n, m, p)) \subset O$. Hence by the criteria above, X is metrizable. □

A space X has a *countable tightness*, denoted by $t(X) = \aleph_0$, if $p \in \overline{A} \setminus A,\ A \subset X$ implies that there exists a countable subset $A_0 \subset A$ such that $p \in \overline{A_0}$. According to his Definition, [4, Definition 4.4], an anti-cover $\mathcal{U}$ of a closed subset F of a space X is said to be *uniformly approaching* (to F) in X if for each open set V of X

$$\overline{St(X \setminus V, \mathcal{U})} \cap V \cap F = \emptyset.$$

A space X is said to be a *D-space* if X is a paracompact σ-space such that each closed set of X has a uniformly approaching anti-cover. D-spaces are between Lašnev spaces and L-spaces.

Theorem 2.3. *Let X be a space with $t(X) = \aleph_0$. Then the following are equivalent:*

(i) *$\mathcal{K}(X)$ is a D-space.*
(ii) *$\mathcal{F}(X)$ is a D-space.*
(iii) *X is metrizable.*

Proof. By the discussion above, it suffices to show that if X is not discrete and $\mathcal{F}(X)$ is a D-space, then X has a sequence of open covers satisfying the criteria above. Since X has a countable tightness, there exists a sequence $\{p_n \,|\, n \in \mathbb{N}\}$ of points of X with its accumulation point p such that $p \neq p_n \neq p_m$ if $n \neq m$. Since

$$\hat{X} = \big\{\{p, x\} \,|\, x \in X\big\}$$

is a closed set of a D-space $\mathcal{F}(X)$, there exists a uniformly approaching anti-cover $\hat{\mathcal{U}}$ of $\hat{X}$ in $\mathcal{F}(X)$. Let $z \in X$ be arbitrary, and let $n \neq m$. Take $\hat{U}(n, m, z) \in \hat{\mathcal{U}}$ such that $\{p_n, p_m, z\} \in \hat{U}(n, m, z)$. Since $\hat{U}(n, m, z)$ is open in $\mathcal{F}(X)$, there exists an open neighborhood $U(n, m, z)$ of z in X such that

$$\langle \{p_n\}, \{p_m\}, U(n, m, z) \rangle \subset \hat{U}(n, m, z).$$

Set

$$\mathcal{U}(n, m) = \{U(n, m, z) \,|\, z \in X\}.$$

Then $\mathcal{U}(n, m)$ is an open cover of X. By the same way as above proof, we can show that the sequence $\{\mathcal{U}(n, m) \,|\, n, m \in \mathbb{N}, n \neq m\}$ satisfies the same criteria. This completes the proof. □

3. The coincidence of L-spaces and D-spaces.

To show the coincidence theorem under the additional condition, we recall the following fact:

Fact 3.1. ([3, Fact 4]) Let $\mathcal{B}$ be a CP family of closed subsets of a stratifiable space X. Then there exists a pair $\langle \mathcal{F}, \mathcal{V} \rangle$ of a σ-discrete closed cover $\mathcal{F}$ and a point-finite, σ-discrete open cover $\mathcal{V} = \{V(F) \,|\, F \in \mathcal{F}\}$ of X satisfying the following:

(i) If $F \in \mathcal{F}$ and $B \in \mathcal{B}$, then $F \cap B \neq \emptyset$ if and only if $F \subset B$;
(ii) for each $F \in \mathcal{F}$, $F \subset V(F)$ and if $F \cap B = \emptyset$, then $V(F) \cap B = \emptyset$.

(We call $\mathcal{F}$ the *mosaic* on $\mathcal{B}$ and $\mathcal{V}$ the *frill* of $\mathcal{F}$ in X.)

Fact 3.2. ([4, Theorem 1.3]) Let F be a closed subset of an L-space X with Ind $X = 0$. Then F has a neighborhood base $\mathcal{U}$ in X, consisting of clopen subsets of X such that $\mathcal{U}$ is CP in X and at the same time IP in $X \setminus F$.

Theorem 3.3. *If X is an L-space and Ind$X = 0$, then X is a D-space.*

Proof. It suffices to show that each closed subset M of X has a uniformly approaching anti-cover. Since M is a σ-space as the subspace, there exists a closed network $\mathcal{F} = \bigcup\{\mathcal{F}_n \,|\, n \in \mathbb{N}\}$, where each $\mathcal{F}_n$ is discrete in M. For each n, let $\mathcal{U}_n = \{U(F) \,|\, F \in \mathcal{F}_n\}$ be a discrete clopen expansion of $\mathcal{F}_n$ in X such that $U(F) \subset S(F, n)$ for each $F \in \mathcal{F}_n$, where $S : \{\text{closed subsets of } X\} \times \mathbb{N} \longrightarrow \tau(X)$ denotes a stratification of X. By Fact 3.2, for each $F \in \mathcal{F}$ there exists a neighborhood base $\mathcal{V}(F)$ of F in X consisting of clopen subsets of X which is both CP in X and IP in $X \setminus F$. Without loss of generality, we can assume $\bigcup \mathcal{V}(F) \subset U(F)$. For each $O \in \tau(X)$, let

$$\mathcal{F}(O) = \{F \in \mathcal{F} \,|\, F \subset O\}.$$

For each $F \in \mathcal{F}(O)$, take $V_F \in \mathcal{V}(F)$ such that $V_F \subset O$. Then

$$W(O) = \bigcup\{V_F \,|\, F \in \mathcal{F}(O)\}$$

is an open neighborhood of $O \cap M$ in X. Since $\{U(F) \,|\, F \in \mathcal{F}_n\}$, $n \in \mathbb{N}$, are stratified with n, it is easily checked that

$$\mathcal{W} = \{W(O) \,|\, O \in \tau(X)\}$$

is a family of open subsets of X which is both CP and IP in $X \setminus M$. Note that for each $W(O) \in \mathcal{W}$, $W(O) \cup M$ is closed in X. By Fact 3.1, there exists a pair $\langle \mathcal{H}, \mathcal{G} \rangle$ of the mosaic $\mathcal{H}$ and the frill $\mathcal{G} = \{G(H) \,|\, H \in \mathcal{H}\}$ on the CP family

$$\{W \cup M \,|\, W \in \mathcal{W}\} \cup \{(X \setminus W) \cup M \,|\, W \in \mathcal{W}\} \cup \{M\}$$

of closed subsets of X. Then

$$\mathcal{G}^* = \{G \in \mathcal{G} \,|\, G \cap M = \emptyset\}$$

is an anti-cover of M in X. Since $\mathcal{G}$ is the frill of $\mathcal{H}$, it is easy to see that $\mathcal{G}^*$ is a uniformly approaching anti-cover of M in X. This completes the proof. □

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