

WEAK AND STRONG CONVERGENCE THEOREMS FOR NONEXPANSIVE MAPPINGS

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ABSTRACT. In this paper, we shall first show that the iteration $\{x_n\}$ defined by (1.2) below converges weakly to a fixed point of T when E is a uniformly convex Banach space with Opial's condition, which generalizes the recent theorem due to Takahashi and Kim [11]. Next, we show that the weak limit points of subsequences of the iteration $\{x_n\}$ defined by (1.5) are fixed points of T (or S) when E is a uniformly convex Banach space, which generalizes the recent theorem due to Takahashi and Tamura [12].

1. INTRODUCTION

Let E be a real Banach space and let C be a closed convex subset of E . Then a mapping T of C into itself is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Throughout this paper, we denote by \mathbf{N} and \mathbf{R} the set of positive integers and the set of real numbers respectively. For a mapping T of C into itself, we consider the following iteration process:

$$(1.1) \quad \begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n T[\beta_n T x_n + (1 - \beta_n)x_n] + (1 - \alpha_n)x_n \end{cases}$$

for all $n \in \mathbf{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$. Such an iteration process was introduced by Ishikawa [4]; see also Mann [6]. We consider a more general iterative process of the type (cf., Xu [14]) emphasizing the randomness of errors as follows:

$$(1.2) \quad \begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n x_n + \beta_n T y_n + \gamma_n u_n, \\ y_n = \alpha'_n x_n + \beta'_n T x_n + \gamma'_n v_n, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}$ are real sequences in $[0, 1]$ satisfying

$$(1.3) \quad \alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1 \text{ for all } n \in \mathbf{N},$$

$$(1.4) \quad \sum_{n=1}^{\infty} \gamma_n < \infty \text{ and } \sum_{n=1}^{\infty} \gamma'_n < \infty,$$

and $\{u_n\}, \{v_n\}$ are two bounded sequences in C . If $\gamma_n = \gamma'_n = 0$ for all $n \in \mathbf{N}$, then the iteration process (1.2) reduces to the Ishikawa iteration process, while setting $\beta'_n = 0$ and $\gamma'_n = 0$ for all $n \in \mathbf{N}$ reduces to the Mann iteration process with errors, which is a generalized case of the Mann iteration process [6]. For two mappings S, T of C into itself, we also consider a more general iterative process of the type (cf. Das and Debata [2] and

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Xu [14]) emphasizing the randomness of error as follows:

$$(1.5) \quad \begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n x_n + \beta_n S y_n + \gamma_n u_n, \\ y_n = \alpha'_n x_n + \beta'_n T x_n + \gamma'_n v_n, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}$ are real sequences in $[0, 1]$ satisfying (1.3) and (1.4) and $\{u_n\}, \{v_n\}$ are bounded sequences in C . If $S = T$, then the iterations (1.5) are reduced to (1.2).

Recently Takahashi and Kim [11] proved the following result: Let C be a closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition and let T be a nonexpansive mapping of C into itself with a fixed point. Then for any initial data $x_1 \in C$, the iteration $\{x_n\}$ defined by (1.1) converges weakly to a fixed point of T under the assumption that $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen satisfying that either $\alpha_n \in [a, b]$ and $\beta_n \in [0, b]$ or $\alpha_n \in [a, 1]$ and $\beta_n \in [a, b]$ for some $a, b \in \mathbf{R}$ with $0 < a \leq b < 1$. For other related results, see Reich [8] and Tan and Xu [13]. On the other hand, Takahashi and Tamura [12] proved the following result: Let E be a uniformly convex Banach space. Let C be a closed convex subset of E and let S, T be nonexpansive mappings of C into itself with a common fixed point. Suppose that $\{x_n\}$ is given by $x_1 \in C$ and

$$x_{n+1} = \alpha_n S[\beta_n T x_n + (1 - \beta_n)x_n] + (1 - \alpha_n)x_n$$

for all $n \in \mathbf{N}$, where $\alpha_n, \beta_n \in [0, 1]$. Then the following hold:

1. If $\alpha_n \in [a, b]$ and $\beta_n \in [0, b]$ for some $a, b \in \mathbf{R}$ with $0 < a \leq b < 1$, $x_{n_i} \rightharpoonup y$ implies $y \in F(S)$;
2. If $\alpha_n \in [a, 1]$ and $\beta_n \in [a, b]$ for some $a, b \in \mathbf{R}$ with $0 < a \leq b < 1$, $x_{n_i} \rightharpoonup y$ implies $y \in F(T)$;
3. If $\alpha_n, \beta_n \in [a, b]$ for some $a, b \in \mathbf{R}$ with $0 < a \leq b < 1$, $x_{n_i} \rightharpoonup y$ implies $y \in F(S) \cap F(T)$.

In this paper, we shall first show that the iteration $\{x_n\}$ defined by (1.2) converges weakly to a fixed point of T when E is a uniformly convex Banach space with Opial's condition, which generalizes the recent theorem due to Takahashi and Kim [11]. Next, we show that the weak limit points of subsequences of the iteration $\{x_n\}$ defined by (1.5) are fixed points of T (or S) when E is a uniformly convex Banach space, which generalizes the recent theorem due to Takahashi and Tamura [12]. Finally, we shall show that if E is uniformly convex and the union of the range of T and $\{u_n\}$ is contained in a compact subset of C , the iteration $\{x_n\}$ defined by (1.2) converges strongly to a fixed point of T .

2. PRELIMINARIES

Throughout this paper we denote by E a real Banach space. Let C be a closed convex subset of E and let T be a mapping of C into itself. Then we denote by $F(T)$ the set of all fixed points of T , i.e., $F(T) = \{x \in C \mid Tx = x\}$. A Banach space E is called *uniformly convex* if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for $x, y \in E$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$, it holds that $\|x + y\| \leq 2(1 - \delta)$. When $\{x_n\}$ is a sequence in E , then $x_n \rightarrow x$ ($x_n \rightharpoonup x$) will denote strong (weak) convergence of the sequence $\{x_n\}$ to x . A Banach space E is said to satisfy *Opial's condition* [7] if for any sequence $\{x_n\}$ in E , $x_n \rightharpoonup x$ implies that

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$. All Hilbert spaces and l^p ($1 < p < \infty$) satisfy Opial's condition, while L^p with $1 < p \neq 2 < \infty$ do not.

Let C be a subset of a Banach space E . A mapping T of C into E is said to be *demiclosed* if $x_n \rightharpoonup x$ in C and $Tx_n \rightarrow y$ imply $Tx = y$.

Theorem 2.1 ([1]). *Let C be a bounded closed convex subset of a uniformly convex Banach space E and let T be a nonexpansive mapping of C into E . Then $I - T$ is demiclosed.*

We immediately get the following:

Proposition 2.2. *Let C be a closed convex subset of a uniformly convex Banach space E and let T be a nonexpansive mapping of C into E . If $\{x_n\}$ is a bounded sequence in C and $\{x_n - Tx_n\}$ converges strongly to 0 as $n \rightarrow \infty$, then $F(T)$ is nonempty.*

Proof. Take $C_0 = \overline{\text{co}}\{x_n\}$, where $\overline{\text{co}}A$ means the closed convex hull of a subset A of E . Then C_0 is bounded closed convex and $T|_{C_0}$ is a nonexpansive mapping of C_0 into C . Let $x_{n_i} \rightharpoonup z$. Then we obtain $z \in F(T)$ by Theorem 2.1. \square

Lemma 2.3 ([13]). *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} b_n < \infty$ and*

$$a_{n+1} \leq a_n + b_n$$

for all $n \in \mathbf{N}$. Then $\lim_{n \rightarrow \infty} a_n$ exists.

Lemma 2.4 ([9]). *Let E be a uniformly convex Banach space, let $0 < b \leq t_n \leq c < 1$ for all $n \in \mathbf{N}$, and let $\{x_n\}$ and $\{y_n\}$ be sequences of E such that $\overline{\lim}_{n \rightarrow \infty} \|x_n\| \leq a$, $\overline{\lim}_{n \rightarrow \infty} \|y_n\| \leq a$ and $\overline{\lim}_{n \rightarrow \infty} \|t_n x_n + (1 - t_n)y_n\| = a$ for some $a \geq 0$. Then, it holds that $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.*

Now, we'll prepare to discuss the convergences for the iterations defined by (1.2) and (1.5). In this paper, the iterations defined by (1.2) and (1.5) are always assumed that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}$ are real sequences in $[0, 1]$ satisfying (1.3) and (1.4) and $\{u_n\}, \{v_n\}$ are bounded sequences in C .

Lemma 2.5. *Let C be a closed convex subset of a Banach space E and let S, T be nonexpansive mappings of C into itself with $F(S) \cap F(T) \neq \emptyset$. Suppose a sequence $\{x_n\}$ is defined by (1.5), then $\lim_{n \rightarrow \infty} \|x_n - z\|$ exists for any $z \in F(S) \cap F(T)$.*

Proof. Since $\{u_n\}$ and $\{v_n\}$ are bounded in C , for a fixed $z \in F(S) \cap F(T)$, let

$$M = \sup_{n \in \mathbf{N}} \|u_n - z\| \vee \sup_{n \in \mathbf{N}} \|v_n - z\| (< \infty).$$

Since

$$\begin{aligned} (2.1) \quad \|Sy_n - z\| &\leq \|y_n - z\| \\ &= \|\alpha'_n x_n + \beta'_n T x_n + \gamma'_n v_n - z\| \\ &\leq \alpha'_n \|x_n - z\| + \beta'_n \|T x_n - z\| + \gamma'_n \|v_n - z\| \\ &\leq \alpha'_n \|x_n - z\| + \beta'_n \|x_n - z\| + \gamma'_n \|v_n - z\| \\ &\leq (1 - \gamma'_n) \|x_n - z\| + \gamma'_n \|v_n - z\|, \end{aligned}$$

we have

$$\begin{aligned} \|x_{n+1} - z\| &\leq \|\alpha_n x_n + \beta_n S y_n + \gamma_n u_n - z\| \\ &\leq \alpha_n \|x_n - z\| + \beta_n \|S y_n - z\| + \gamma_n \|v_n - z\| \\ &\leq \alpha_n \|x_n - z\| + \beta_n \{(1 - \gamma'_n) \|x_n - z\| + \gamma'_n \|v_n - z\|\} + \gamma_n \|v_n - z\| \\ &\leq (1 - (\gamma_n + \beta_n \gamma'_n)) \|x_n - z\| + \gamma'_n M + \gamma_n M \end{aligned}$$

$$\leq \|x_n - z\| + (\gamma'_n + \gamma_n)M.$$

By Lemma 2.3, we readily see that $\lim_{n \rightarrow \infty} \|x_n - z\|$ exists. \square

Using Lemma 2.4, we have the following:

Lemma 2.6. *Let C be a closed convex subset of a uniformly convex Banach space E and let T be nonexpansive mapping of C into itself with a fixed point. Suppose the sequence $\{x_n\}$ defined by (1.2) satisfies that either*

1. $\alpha_n \in [a, 1]$, $\beta_n \in [a, b]$, $\beta'_n \in [0, b]$ for some $a, b \in \mathbf{R}$ with $0 < a \leq b < 1$, or
2. $\alpha'_n, \beta_n \in [a, 1]$, $\beta'_n \in [a, b]$ for some $a, b \in \mathbf{R}$ with $0 < a \leq b < 1$.

Then $\{x_n - Tx_n\}$ converges strongly to 0 as $n \rightarrow \infty$.

Proof. Let $r = \lim_{n \rightarrow \infty} \|x_n - z\|$ which exists for a fixed $z \in F(T)$ by Lemma 2.5. If $r = 0$, it is clear that $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ and so we assume $r > 0$. Note that $d_n \equiv \max\{\gamma'_n, \gamma_n/a\} \rightarrow 0$ as $n \rightarrow \infty$. Since $\{u_n\}$ and $\{v_n\}$ are bounded in C , let

$$M = \sup_{n \in \mathbf{N}} \|u_n - z\| \vee \sup_{n \in \mathbf{N}} \|v_n - z\| \quad (< \infty).$$

Now, we assume (1). Since $\|Ty_n - z\| \leq \|x_n - z\| + d_n M$ by the same calculus as (2.1) and $\left\| \frac{\alpha_n x_n}{\alpha_n + \gamma_n} + \frac{\gamma_n u_n}{\alpha_n + \gamma_n} - z \right\| \leq \|x_n - z\| + d_n M$, we have $\overline{\lim}_{n \rightarrow \infty} \|Ty_n - z\| \leq r$ and $\overline{\lim}_{n \rightarrow \infty} \left\| \frac{\alpha_n x_n}{\alpha_n + \gamma_n} + \frac{\gamma_n u_n}{\alpha_n + \gamma_n} - z \right\| \leq r$. On the other hand, it holds that

$$\begin{aligned} r &= \lim_{n \rightarrow \infty} \|x_{n+1} - z\| \\ &= \lim_{n \rightarrow \infty} \|\alpha_n x_n + \beta_n Ty_n + \gamma_n u_n - z\| \\ &= \lim_{n \rightarrow \infty} \left\| \beta_n (Ty_n - z) + (1 - \beta_n) \left(\frac{\alpha_n x_n}{\alpha_n + \gamma_n} + \frac{\gamma_n u_n}{\alpha_n + \gamma_n} - z \right) \right\|. \end{aligned}$$

Using Lemma 2.4, it holds that $\lim_{n \rightarrow \infty} \left\| Ty_n - \frac{\alpha_n x_n}{\alpha_n + \gamma_n} - \frac{\gamma_n u_n}{\alpha_n + \gamma_n} \right\| = 0$, and so we obtain $\lim_{n \rightarrow \infty} \|Ty_n - x_n\| = 0$ by virtue of $\sup_{n \in \mathbf{N}} \|x_n - u_n\| < \infty$. Since

$$\begin{aligned} \|Tx_n - x_n\| &\leq \|Tx_n - Ty_n\| + \|Ty_n - x_n\| \\ &\leq \|x_n - y_n\| + \|Ty_n - x_n\| \\ &= \|x_n - \alpha'_n x_n - \beta'_n Tx_n - \gamma'_n v_n\| + \|Ty_n - x_n\| \\ &\leq \beta'_n \|Tx_n - x_n\| + \gamma'_n \|x_n - v_n\| + \|Ty_n - x_n\|, \end{aligned}$$

we have

$$(2.2) \quad \begin{aligned} (1 - b) \|Tx_n - x_n\| &\leq (1 - \beta'_n) \|Tx_n - x_n\| \\ &\leq \gamma'_n \|x_n - v_n\| + \|Ty_n - x_n\| \\ &\leq \gamma'_n M' + \|Ty_n - x_n\|, \end{aligned}$$

where $M' = \sup_{n \in \mathbf{N}} \|x_n - v_n\| (< \infty)$. It easily follows from (2.2) that

$$(2.3) \quad \lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0.$$

Next, assuming (2), we have

$$\begin{aligned} \|x_{n+1} - z\| &= \|\alpha_n x_n + \beta_n Ty_n + \gamma_n u_n - z\| \\ &\leq \alpha_n \|x_n - z\| + \beta_n \|Ty_n - z\| + \gamma_n \|u_n - z\| \end{aligned}$$

$$\begin{aligned} &\leq \alpha_n \|x_n - z\| + \beta_n \|y_n - z\| + \gamma_n M \\ &\leq (1 - \beta_n) \|x_n - z\| + \beta_n \|y_n - z\| + \gamma_n M \end{aligned}$$

and hence

$$\frac{\|x_{n+1} - z\| - \|x_n - z\|}{\beta_n} + \|x_n - z\| \leq \|y_n - z\| + \frac{\gamma_n}{a} M.$$

So, using $\|y_n - z\| \leq \|x_n - z\| + d_n M$ obtained by (2.1), we have

$$r \leq \underline{\lim}_{n \rightarrow \infty} \|y_n - z\| \leq \overline{\lim}_{n \rightarrow \infty} \|y_n - z\| \leq \overline{\lim}_{n \rightarrow \infty} \{\|x_n - z\| + d_n M\} = r.$$

Hence

$$\begin{aligned} r &= \lim_{n \rightarrow \infty} \|y_n - z\| \\ &= \lim_{n \rightarrow \infty} \|\alpha'_n x_n + \beta'_n T x_n + \gamma'_n v_n - z\| \\ &= \lim_{n \rightarrow \infty} \left\| \beta'_n (T x_n - z) + (1 - \beta'_n) \left(\frac{\alpha'_n x_n}{\alpha'_n + \gamma'_n} + \frac{\gamma'_n v_n}{\alpha'_n + \gamma'_n} - z \right) \right\|. \end{aligned}$$

By using Lemma 2.4 and $\sup_{n \in \mathbf{N}} \|x_n - v_n\| < \infty$, we have (2.3). \square

The following is useful for weak and strong convergence theorems for the iteration defined by (1.5).

Theorem 2.7. *Let C be a closed convex subset of a Banach space E . Let S, T be non-expansive mappings of C into itself such that $F(S) \cap F(T)$ is nonempty. Suppose that $\{x_n\}$ is defined by (1.5) and for every $n \in \mathbf{N}$ a mapping T_n of C into itself is defined by $T_n x = \alpha_n x + \beta_n S[\alpha'_n x + \beta'_n T x + \gamma'_n x] + \gamma_n x$ for all $x \in C$. If there are $\beta_n, \beta'_n \in [a, b]$ for some $a, b \in \mathbf{R}$ with $0 < a \leq b < 1$, then $\{T_n T_{n-1} \cdots T_1 x_1 - x_{n+1}\}$ converges strongly to 0 as $n \rightarrow \infty$.*

Proof. Since

$$\begin{aligned} &\|T_n T_{n-1} \cdots T_1 x_1 - x_{n+1}\| \\ &\leq \alpha_n \|T_{n-1} \cdots T_1 x_1 - x_n\| \\ &\quad + \beta_n \|S[\alpha'_n T_{n-1} \cdots T_1 x_1 + \beta'_n T T_{n-1} \cdots T_1 x_1 + \gamma'_n T_{n-1} \cdots T_1 x_1] \\ &\quad - S[\alpha'_n x_n + \beta'_n T x_n + \gamma'_n v_n]\| + \gamma_n \|T_{n-1} \cdots T_1 x_1 - u_n\| \\ &\leq \alpha_n \|T_{n-1} \cdots T_1 x_1 - x_n\| \\ &\quad + \beta_n \|\alpha'_n T_{n-1} \cdots T_1 x_1 + \beta'_n T T_{n-1} \cdots T_1 x_1 + \gamma'_n T_{n-1} \cdots T_1 x_1 \\ &\quad - [\alpha'_n x_n + \beta'_n T x_n + \gamma'_n v_n]\| + \gamma_n \|T_{n-1} \cdots T_1 x_1 - u_n\| \\ &\leq \alpha_n \|T_{n-1} \cdots T_1 x_1 - x_n\| \\ &\quad + \beta_n \{(\alpha'_n + \beta'_n) \|T_{n-1} \cdots T_1 x_1 - x_n\| + \gamma'_n \|T_{n-1} \cdots T_1 x_1 - v_n\|\} \\ &\quad + \gamma_n \|T_{n-1} \cdots T_1 x_1 - u_n\| \\ &\leq \alpha_n \|T_{n-1} \cdots T_1 x_1 - x_n\| \\ &\quad + \beta_n \{\|T_{n-1} \cdots T_1 x_1 - x_n\| + \gamma'_n \|x_n - v_n\|\} \\ &\quad + \gamma_n \{\|T_{n-1} \cdots T_1 x_1 - x_n\| + \|x_n - u_n\|\} \\ &\leq \|T_{n-1} \cdots T_1 x_1 - x_n\| + (\gamma_n + \gamma'_n) M, \end{aligned}$$

where $M = \sup_{n \in \mathbf{N}} \|x_n - u_n\| \vee \sup_{n \in \mathbf{N}} \|x_n - v_n\|$ which is finite by Lemma 2.5, we have the desired result by Lemma 2.3. \square

3. WEAK CONVERGENCE THEOREMS

In this section, we treat the weak convergences of the iterations defined by (1.2) and (1.5). Our Theorem 3.2 carries over Theorem 1 of Takahashi and Kim [11] to a more general Ishikawa type iteration.

Lemma 3.1. *Let C be a closed convex subset of a uniformly convex Banach space E satisfying Opial's condition and let T be a nonexpansive mapping of C into itself. If the sequence $\{x_n\}$ defined by (1.2) satisfies that $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$, then $\{x_n\}$ converges weakly to some fixed point of T .*

Proof. Let z_1, z_2 be weak subsequential limits of the sequence $\{x_n\}$. We claim that the conditions $x_{n_i} \rightharpoonup z_1$ and $x_{n_j} \rightharpoonup z_2$ imply $z_1 = z_2 \in F(T)$. We first show that $z_1, z_2 \in F(T)$. If $Tz_1 \neq z_1$, then, by Opial's condition, we have

$$\begin{aligned} \underline{\lim}_{i \rightarrow \infty} \|x_{n_i} - z_1\| &< \underline{\lim}_{i \rightarrow \infty} \|x_{n_i} - Tz_1\| \\ &\leq \underline{\lim}_{i \rightarrow \infty} \{\|x_{n_i} - Tx_{n_i}\| + \|Tx_{n_i} - Tz_1\|\} \\ &\leq \underline{\lim}_{i \rightarrow \infty} \{\|x_{n_i} - Tx_{n_i}\| + \|x_{n_i} - z_1\|\} \\ &= \underline{\lim}_{i \rightarrow \infty} \|x_{n_i} - z_1\|. \end{aligned}$$

This is a contradiction. Hence we have $Tz_1 = z_1$ and so $z_1 \in F(T)$. Similarly, we have $z_2 \in F(T)$. Next, we show $z_1 = z_2$. If not, by Lemma 2.5 and Opial's condition, we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - z_1\| &= \lim_{i \rightarrow \infty} \|x_{n_i} - z_1\| < \lim_{i \rightarrow \infty} \|x_{n_i} - z_2\| \\ &= \lim_{j \rightarrow \infty} \|x_{n_j} - z_2\| < \lim_{j \rightarrow \infty} \|x_{n_j} - z_1\| \\ &= \lim_{n \rightarrow \infty} \|x_n - z_1\|. \end{aligned}$$

This is a contradiction. Hence we have $z_1 = z_2$, which implies that $\{x_n\}$ converges weakly to a fixed point of T . \square

Theorem 3.2. *Let C be a closed convex subset of a uniformly convex Banach space E satisfying Opial's condition and let T be a nonexpansive mapping of C into itself with a fixed point. Assume that the sequence $\{x_n\}$ defined by (1.2) satisfies that either*

1. $\alpha_n \in [a, 1]$, $\beta_n \in [a, b]$, $\beta'_n \in [0, b]$ for some $a, b \in \mathbf{R}$ with $0 < a \leq b < 1$, or
2. $\alpha'_n, \beta_n \in [a, 1]$, $\beta'_n \in [a, b]$ for some $a, b \in \mathbf{R}$ with $0 < a \leq b < 1$.

Then $\{x_n\}$ converges weakly to some fixed point of T .

Proof. Since $\lim_{n \rightarrow \infty} \|x_n - z\|$ exists for any $z \in F(T)$ by Lemma 2.5 and $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ by Lemma 2.6, the conclusion follows from Lemma 3.1. \square

Our Theorem 3.3 carries over Theorem 1 of Takahashi and Tamura [12] to a more general Ishikawa type iteration.

Theorem 3.3. *Let E be a uniformly convex Banach space. Let C be a closed convex subset of E and let S, T be nonexpansive mappings of C into itself with a common fixed point. Suppose that a sequence $\{x_n\}$ is defined by (1.5). Then the following hold:*

1. *If $\alpha_n, \alpha'_n \in [a, 1]$, $\beta_n \in [a, 1]$, $\beta'_n \in [0, b]$ for some $a, b \in \mathbf{R}$ with $0 < a \leq b < 1$, $x_{n_i} \rightharpoonup y$ implies $y \in F(S)$;*
2. *If $\alpha'_n, \beta_n \in [a, 1]$ and $\beta'_n \in [a, b]$ for some $a, b \in \mathbf{R}$ with $0 < a \leq b < 1$, $x_{n_i} \rightharpoonup y$ implies $y \in F(T)$;*
3. *If $\alpha_n, \alpha'_n \in [a, 1]$ and $\beta_n, \beta'_n \in [a, b]$ for some $a, b \in \mathbf{R}$ with $0 < a \leq b < 1$, $x_{n_i} \rightharpoonup y$ implies $y \in F(S) \cap F(T)$.*

Proof. Let $x \in C$ and set $U = F(S) \cap F(T)$. Then for $w \in U$ and $r = \|x - w\|$, $D = C \cap B_r[w]$ is a bounded closed convex subset of C which is invariant under S and T , where $B_r[w]$ denotes the closed ball centered at w with radius r . So, we may assume that C is bounded. For a fixed $z \in F(S) \cap F(T)$, let $r = \lim_{n \rightarrow \infty} \|x_n - z\|$ which exists by Lemma 2.5. If $r = 0$, the conclusions are clear. So, we assume $r > 0$. We obtain

$$(3.1) \quad \begin{aligned} \|Sx_n - x_n\| &\leq \|Sx_n - Sy_n\| + \|Sy_n - x_n\| \\ &\leq \|x_n - y_n\| + \|Sy_n - x_n\| \\ &= \|x_n - \alpha'_n x_n - \beta'_n T x_n - \gamma'_n v_n\| + \|Sy_n - x_n\| \\ &\leq \beta'_n \|T x_n - x_n\| + \gamma'_n \|x_n - v_n\| + \|Sy_n - x_n\| \\ &\leq \beta'_n \|T x_n - x_n\| + \gamma'_n M' + \|Sy_n - x_n\|, \end{aligned}$$

where $M' = \sup_{n \in \mathbf{N}} \|x_n - v_n\| (< \infty)$. Since $\{u_n\}$ and $\{v_n\}$ are bounded in C , let

$$M = \sup_{n \in \mathbf{N}} \|u_n - z\| \vee \sup_{n \in \mathbf{N}} \|v_n - z\| (< \infty).$$

First, we assume that $0 < a \leq \alpha_n \leq 1$. Note that $d_n \equiv \max\{\gamma'_n, \gamma_n/a\} \rightarrow 0$ as $n \rightarrow \infty$. Since $\|Sy_n - z\| \leq \|x_n - z\| + d_n M$ by (2.1) and $\left\| \frac{\alpha_n x_n}{\alpha_n + \gamma_n} + \frac{\gamma_n u_n}{\alpha_n + \gamma_n} - z \right\| \leq \|x_n - z\| + d_n M$, we have $\overline{\lim}_{n \rightarrow \infty} \|Sy_n - z\| \leq r$ and $\overline{\lim}_{n \rightarrow \infty} \left\| \frac{\alpha_n x_n}{\alpha_n + \gamma_n} + \frac{\gamma_n u_n}{\alpha_n + \gamma_n} - z \right\| \leq r$. Hence

$$\begin{aligned} r &= \lim_{n \rightarrow \infty} \|x_{n+1} - z\| \\ &= \lim_{n \rightarrow \infty} \|\alpha_n x_n + \beta_n S y_n + \gamma_n u_n - z\| \\ &= \lim_{n \rightarrow \infty} \left\| \beta_n (S y_n - z) + (1 - \beta_n) \left(\frac{\alpha_n x_n}{\alpha_n + \gamma_n} + \frac{\gamma_n u_n}{\alpha_n + \gamma_n} - z \right) \right\|. \end{aligned}$$

Using Lemma 2.4, it holds that $\lim_{n \rightarrow \infty} \left\| S y_n - \frac{\alpha_n x_n}{\alpha_n + \gamma_n} - \frac{\gamma_n u_n}{\alpha_n + \gamma_n} \right\| = 0$, and so we obtain $\lim_{n \rightarrow \infty} \|S y_n - x_n\| = 0$ by virtue of $\sup_{n \in \mathbf{N}} \|x_n - u_n\| < \infty$. Next, if $0 < a \leq \beta_n \leq 1$, we have

$$\begin{aligned} \|x_{n+1} - z\| &= \|\alpha_n x_n + \beta_n S y_n + \gamma_n u_n - z\| \\ &\leq \alpha_n \|x_n - z\| + \beta_n \|S y_n - z\| + \gamma_n \|u_n - z\| \\ &\leq \alpha_n \|x_n - z\| + \beta_n \|y_n - z\| + \gamma_n M \\ &\leq (1 - \beta_n) \|x_n - z\| + \beta_n \|y_n - z\| + \gamma_n M \end{aligned}$$

and hence

$$\frac{\|x_{n+1} - z\| - \|x_n - z\|}{\beta_n} + \|x_n - z\| \leq \|y_n - z\| + \frac{\gamma_n}{a} M.$$

So, using $\|y_n - z\| \leq \|x_n - z\| + d_n M$ obtained by (2.1), we have

$$r \leq \underline{\lim}_{n \rightarrow \infty} \|y_n - z\| \leq \overline{\lim}_{n \rightarrow \infty} \|y_n - z\| \leq \overline{\lim}_{n \rightarrow \infty} \{\|x_n - z\| + d_n M\} = r.$$

Hence

$$(3.2) \quad \begin{aligned} r &= \lim_{n \rightarrow \infty} \|y_n - z\| \\ &= \lim_{n \rightarrow \infty} \|\alpha'_n x_n + \beta'_n T x_n + \gamma'_n v_n - z\| \\ &= \lim_{n \rightarrow \infty} \left\| \beta'_n (T x_n - z) + (1 - \beta'_n) \left(\frac{\alpha'_n x_n}{\alpha'_n + \gamma'_n} + \frac{\gamma'_n v_n}{\alpha'_n + \gamma'_n} - z \right) \right\|. \end{aligned}$$

Further, if $0 < a \leq \alpha'_n \leq 1$, we get

$$(3.3) \quad \overline{\lim}_{n \rightarrow \infty} \|Tx_n - z\| \leq r \text{ and } \overline{\lim}_{n \rightarrow \infty} \left\| \frac{\alpha'_n x_n}{\alpha'_n + \gamma'_n} + \frac{\gamma'_n v_n}{\alpha'_n + \gamma'_n} - z \right\| \leq r$$

as above. Now we prove (1). Assume $x_{n_i} \rightharpoonup y$. Then, since $0 \leq \beta'_n \leq b < 1$ we have $\underline{\lim}_{i \rightarrow \infty} \beta'_{n_i} = 0$ or $\underline{\lim}_{i \rightarrow \infty} \beta'_{n_i} > 0$. If $\underline{\lim}_{i \rightarrow \infty} \beta'_{n_i} > 0$, by using (3.2), (3.3) and Lemma 2.4, it follows that

$$\lim_{i \rightarrow \infty} \left\| Tx_{n_i} - \frac{\alpha'_{n_i} x_{n_i}}{\alpha'_{n_i} + \gamma'_{n_i}} - \frac{\gamma'_{n_i} u_{n_i}}{\alpha'_{n_i} + \gamma'_{n_i}} \right\| = 0,$$

and so we obtain $\lim_{i \rightarrow \infty} \|Tx_{n_i} - x_{n_i}\| = 0$ by virtue of $\sup_{n \in \mathbf{N}} \|x_n - u_n\| < \infty$. Further, from

(3.1), we have $\lim_{i \rightarrow \infty} \|Sx_{n_i} - x_{n_i}\| = 0$, which implies $y \in F(S)$ by the demiclosedness of

$I - S$. If $\underline{\lim}_{i \rightarrow \infty} \beta'_{n_i} = 0$, then by (3.1) we have a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that

$\lim_{j \rightarrow \infty} \|Sx_{n_{i_j}} - x_{n_{i_j}}\| = 0$. Since $I - S$ is demiclosed, we have $y \in F(S)$. Next, we prove (2).

By using Lemma 2.4 with (3.2) and (3.3), we have $\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$ similarly to the argument above. Since $x_{n_i} \rightharpoonup y$ and $I - T$ is demiclosed, we have $y \in F(T)$. (3) is obvious from (1) and (2). □

Theorem 3.4 ([12]). *Let C be a closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition or whose norm is Fréchet differentiable. Let S, T be nonexpansive mappings of C into itself such that $F(S) \cap F(T)$ is nonempty. Suppose that $\{x_n\}$ is given by $x_1 \in C$ and $x_{n+1} = \alpha_n S[\beta_n T x_n + (1 - \beta_n)x_n] + (1 - \alpha_n)x_n$ for all $n \in \mathbf{N}$, where $\alpha_n, \beta_n \in [a, b]$ for some $a, b \in \mathbf{R}$ with $0 < a \leq b < 1$. Then $\{x_n\}$ converges weakly to a common fixed point of S and T .*

Combining Theorem 3.4 and Theorem 2.7, we immediately get the following.

Theorem 3.5. *Let C be a closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition or whose norm is Fréchet differentiable. Let S, T be nonexpansive mappings of C into itself such that $F(S) \cap F(T)$ is nonempty. Suppose that $\{x_n\}$ is defined by (1.5). If $\beta_n, \beta'_n \in [a, b]$ for some $a, b \in \mathbf{R}$ with $0 < a \leq b < 1$, then $\{x_n\}$ converges weakly to a common fixed point of S and T .*

4. STRONG CONVERGENCE THEOREMS

In this section, we consider strong convergences of the iterations defined by (1.2) and (1.5) in a Banach space.

Theorem 4.1. *Let C be a closed convex subset of a uniformly convex Banach space E and let T be a nonexpansive mapping of C into itself with a fixed point. Suppose that the sequence $\{x_n\}$ defined by (1.2) satisfies that either*

1. $\alpha_n \in [a, 1]$, $\beta_n \in [a, b]$, $\beta'_n \in [0, b]$ for some $a, b \in \mathbf{R}$ with $0 < a \leq b < 1$, or
2. $\alpha'_n, \beta_n \in [a, 1]$, $\beta'_n \in [a, b]$ for some $a, b \in \mathbf{R}$ with $0 < a \leq b < 1$.

If $T(C) \cup \{u_n\}$ is contained in a compact subset of C , then $\{x_n\}$ converges strongly to some fixed point of T as $n \rightarrow \infty$.

Proof. By Mazur's theorem [3], $\overline{\text{co}}(\{x_1\} \cup T(C) \cup \{u_n\})$ is a compact subset of C containing $\{x_n\}$. Then, there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and a point $z \in C$ such that $x_{n_i} \rightarrow z$. Since

$$\begin{aligned} \|z - Tz\| &\leq \|z - x_{n_i}\| + \|x_{n_i} - Tx_{n_i}\| + \|Tx_{n_i} - Tz\| \\ &\leq 2\|z - x_{n_i}\| + \|x_{n_i} - Tx_{n_i}\| \end{aligned}$$

$$\rightarrow 0 \quad (i \rightarrow \infty)$$

by using Lemma 2.6, we have $z = Tz$. By Lemma 2.5 $\lim_{n \rightarrow \infty} \|x_n - z\|$ exists, and so we have $\lim_{n \rightarrow \infty} \|x_n - z\| = 0$. \square

Theorem 4.2 ([12]). *Let C be a closed convex subset of a strictly convex Banach space E . Let S, T be nonexpansive mappings of C into itself such that $S(C) \cup T(C)$ is contained in a compact subset of C and $F(S) \cap F(T)$ is nonempty. Suppose that $\{x_n\}$ is given by $x_1 \in C$ and $x_{n+1} = \alpha_n S[\beta_n T x_n + (1 - \beta_n)x_n] + (1 - \alpha_n)x_n$ for all $n \in \mathbf{N}$, where $\alpha_n, \beta_n \in [a, b]$ for some $a, b \in \mathbf{R}$ with $0 < a \leq b < 1$. Then $\{x_n\}$ converges strongly to a common fixed point of S and T .*

Combining Theorem 4.2 and Theorem 2.7, we immediately get the following.

Theorem 4.3. *Let C be a closed convex subset of a strictly convex Banach space E . Let S, T be nonexpansive mappings of C into itself such that $S(C) \cup T(C)$ is contained in a compact subset of C and $F(S) \cap F(T)$ is nonempty. Suppose that $\{x_n\}$ is defined by (1.5). If $\beta_n, \beta'_n \in [a, b]$ for some $a, b \in \mathbf{R}$ with $0 < a \leq b < 1$, then $\{x_n\}$ converges strongly to a common fixed point of S and T .*

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