GRAPH COLORING COMPACTNESS THEOREMS EQUIVALENT TO BPI

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Abstract. We introduce compactness theorems for generalized colorings and derive several particular compactness theorems from them. It is proved that the theorems and many of their consequences are equivalent in ZF set theory to BPI, the Prime Ideal Theorem for Boolean algebras.
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1 Introduction In 1951, de Bruijn and Erdös [3], proved that a graph is k-colorable if every finite subgraph is k-colorable. J. Mycielski seems to have been the first to raise the question as to the strength of the de Bruijn and Erdös result as a set theory axiom in a lecture delivered in 1959, pointing out, in particular, that it follows from the Prime Ideal Theorem for Boolean algebras (BPI) (see [25],[27]), and this was further explored by A. Levy [23]. BPI is weaker than the Axiom of Choice (AC), a result proved by Halpern [18][19], but has many useful equivalent forms which can often be substituted for AC in mathematical proofs (see [26], [20]). Twenty years after the appearance of the de Bruijn and Erdös paper, Läuchli [22] showed that their theorem is in fact one of the equivalent forms of BPI, even for fixed k, $k \geq 3$. A simplified proof of Läuchli's result, due to Mycielski, can be found in [14].

Notions of generalized colorings have been introduced by different authors (see, e.g., the Surveys [1, 2] and [24]) and it is natural to ask if a compactness theorem can be formulated for these colorings. We show that this is indeed the case for both generalized vertex colorings and generalized edge colorings. In addition we show that these general compactness theorems, along with several of their consequences are equivalent to BPI.

2 Generalized Colorings Let $\mathcal{I}^*$ be the class of all simple graphs (finite or infinite). A property $\mathcal{P}$ of simple graphs is any isomorphism-closed nonempty subclass of $\mathcal{I}^*$. Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be properties of simple graphs; then a vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-coloring of a graph $G = < V, E >$ is a mapping $f : V \to \{1, 2, \ldots, n\}$ of the vertex set V of G into the set of colors $\{1, 2, \ldots, n\}$ such that the induced subgraph, $G[f^{-1}(i)]$, has property $\mathcal{P}_i$ for all $i \in Ran(f)$, the range of f. Thus a vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-coloring of G is a partition $(V_1, ..., V_n)$ of V such that each partition class, V_i, induces a subgraph $G[V_i]$ having property $\mathcal{P}_i$. We explictly allow empty partition classes in the partition sequence and these induce the null graph $K_0 = < \emptyset, \emptyset >$. If each of the $\mathcal{P}_i$, $i = 1, 2, \ldots, n$, is the property of being edgeless, the ordinary regular n-coloring is obtained; while if the $\mathcal{P}_i$ is the property of having maximum degree at most k, the defective (n, k)-colorings of [9] are defined. Many other examples, references and results on generalized colorings of finite graphs may be found e.g. in [1, 2], [24] and in the book [21] by Jensen and Toft.

Generalized edge colorings can be defined analagously. An edge $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-coloring of a graph $G = < V, E >$ is a mapping $f : E \to \{1, 2, \ldots, n\}$ of the edge set E of G into the set of colors $\{1, 2, \ldots, n\}$ such that the subgraph induced by the edges $f^{-1}(i)$ has property $\mathcal{P}_i$ for all $i \in Ran(f)$. Thus an edge $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-coloring of G is a partition $(E_1, ..., E_n)$ of E such that each partition class, E_i, induces a subgraph having property $\mathcal{P}_i$.

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3 Compactness Theorems for Generalized Colorings

Our intention is to formulate a compactness theorem for both vertex and edge $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colorings. The key concept is that of a property being of *finite character*. (Connections between compactness theorems, the prime ideal theorem, and properties of finite character have also been made in [12],[13], [16] and [29].) We will first prove an abstract compactness result regarding set partitions and properties of finite character. Let S be a set and P a property of subsets of S. Then P is a property of S of *finite set character* or, simply, of *finite character* if a set belongs to P if and only if every finite subset of the set belongs to P. Surely, if P is of finite character, any subset of a set having property P, also has property P. A $(P_1, ..., P_n)$-partition of S is a partition of S as $(S_1, \ldots, S_n)$, where S_i belongs to P_i, $i = 1, \ldots, n$. The following theorem generalizes a result from [12].

Theorem 3.1 (Set Partition Theorem) *Let S be a set and suppose $P_1, ...P_n$ are properties of S of finite character. Then S has a $(P_1, ..., P_n)$-partition if every finite subset of S has a $(P_1, ..., P_n)$-partition.*

We will give a proof of the Set Partition Theorem from the following lemma which is known as Rado's Selection Lemma [28](see also [11],[29]).

Lemma 3.2 (Selection Lemma) *Let $\{A_v | v \in I\}$ be a set of finite sets. Suppose for every finite $W \subset I$ there exists a function f_W, with domain W, such that $f_W(v) \in A_v, v \in W$. Then there exists a function f, with domain I, such that for every finite $W \subset I$, there exists a finite $W', W \subset W' \subset I$ with $f(v) = f_{W'}(v), v \in W$.*

Proof of the Set Partition Theorem. Assume that every finite subset of S has a $(P_1, \ldots, P_n)$ partition. Then for each finite $W \subset S$ there exists a $(P_1, \ldots, P_n)$ partition of W–that is, there exists a partition function, $f_W : W \to \{1, \ldots, n\}$, such that $f_W^{-1}(i)$ has property $P_i, i = 1, \ldots, n$.

Rado's lemma, now gives a function $f : S \to \{1, \ldots, n\}$ such that for any finite $W \subset S$, there exists finite W' with $W \subset W'$ and $f(s) = f_{W'}(s), s \in W$. We claim that f is a $(P_1, \ldots, P_n)$ partition of S. It must be shown that $f^{-1}(i)$ has property P_i, $i = 1, \ldots, n$. Since P_i is of finite character it suffices to show that every finite subset of $f^{-1}(i)$ has property P_i.

Let W be a finite subset of $f^{-1}(i)$; then there exists W' with $W \subset W'$ and $f(s) = f_{W'}(s), s \in W$. But $f(s) = i$, for $s \in W \subset f^{-1}(i)$; hence $f_{W'}(s) = i, s \in W$, as well. Thus $W \subset f_{W'}^{-1}(i)$. However W' is $(P_1, \ldots, P_n)$ partitioned by $f_{W'}$ Thus, $f_{W'}^{-1}(i)$ has property P_i; but W is a subset of $f_{W'}^{-1}(i)$ and hence has property P_i, as well.

Since every finite subset of $f^{-1}(i)$ has property P_i and P_i is of finite character, $f^{-1}(i)$ has property P_i as desired.

Let $\mathcal{P}$ be a property of graphs; $\mathcal{P}$ is of *finite character with respect to vertices* or, simply, *finite vertex character* if a graph in $\mathcal{I}^*$ has property $\mathcal{P}$ if and only if every finite vertex induced subgraph has property $\mathcal{P}$. It is easy to see that if $\mathcal{P}$ is of finite vertex character and a graph has $\mathcal{P}$ then so does every induced subgraph. A property $\mathcal{P}$ is said to be *induced-hereditary* if $G \in \mathcal{P}$ and $H < G$ implies $H \in \mathcal{P}$, that is $\mathcal{P}$ is closed under taking induced subgraphs. Thus properties of finite vertex character are induced hereditary. However not all induced hereditary properties are of finite vertex character; for example the graph property P of not containing a vertex of infinite degree is induced-hereditary but not of finite vertex character. Let us also remark that every property which is hereditary with respect to every subgraph is induced-hereditary as well; however the converse is false–for example, the property of being a complete subgraph or clique is induced-hereditary but not hereditary.

The properties of being edgeless, of maximum degree at most k, K_n-free, acyclic, perfect, claw-free, etc. are properties of finite vertex character. We next state and prove a compactness theorem for $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colorings, where the $\mathcal{P}_i$ are of finite vertex character.

Theorem 3.3 (Vertex Coloring Compactness Theorem) *Let G be a graph in $\mathcal{I}^*$ and suppose $\mathcal{P}_1, ...\mathcal{P}_n$ are properties of graphs of finite vertex character. Then G is $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$-colorable if every finite induced subgraph of G is $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$-colorable.*

Proof. Let S be the set of vertices of G and let P_i hold for a subset of S if and only if the graph induced by the subset has property $\mathcal{P}_i$, $1 \leq i \leq n$. Then the P_i are set properties of finite character. Since every finite induced subgraph of G is $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$-colorable, every finite subset of S has a $(P_1, ..., P_n)$-partition. Thus, by the Set Partition Theorem, S has a $(P_1, ..., P_n)$-partition, that is, G is $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$-colorable.

If G is a graph and $\mathcal{S}$ is a set of graphs, then G will be said to be *$\mathcal{S}$-free* if no induced-subgraph of G is (isomorphic to) a member of $\mathcal{S}$. If the graphs in $\mathcal{S}$ are all finite graphs, then being $\mathcal{S}$-free is a property of finite vertex character. If, on the other hand, P is a property of finite vertex character and $\mathcal{S}$ is the set of all finite graphs which do not have property P, then a graph will have property P if and only if it is $\mathcal{S}$- free. This characterization leads directly to an equivalent formulation of the preceding theorem in terms of $\mathcal{S}$-free properties. Let $G =< V, E >$ be a simple graph and let $\Sigma = \{\mathcal{S}_i : i = 1, ..., n\}$, where $\mathcal{S}_i$ is a set of graphs, $i = 1, ..., n$. Then a Σ-free partition of G is a partition $(V_1, ..., V_n)$ of the vertex set $V(G)$, where $G[V_i]$ is $\mathcal{S}_i$-free, $i = 1, ..., n$.

Theorem 3.4 (Vertex Coloring Compactness Theorem) *Let G be a simple graph and let $\Sigma = \{\mathcal{S}_i : i = 1, ..., n\}$, where $\mathcal{S}_i$ is a set of finite graphs, $i = 1, ..., n$. Then if every finite induced subgraph of G has a Σ-free partition, then G has a Σ-free partition.*

It is natural to replace 'finite induced subgraph' by 'finite subgraph' in the above theorems, if the properties are hereditary as well as induced hereditary. Thus the theorem of de Bruijn and Erdös [3] as stated requires all finite subgraphs to be k-colorable rather than all finite induced subgraphs, since being 'edgeless' is a hereditary property. We shall, for the most part, adhere to this convention.

We next present a few corollaries as examples of applications of our result. As already mentioned, the property of having maximum degree at most k is of finite vertex character. A graph $G =< V, E >$ will be called $(n; k_1, ..., k_n)$-colorable if the vertex set V can be partitioned as $(V_1, ..., V_n)$ and the induced graphs, $G[V_i]$ have maximum degree at most k_i, $i = 1, ..., n$. Thus we have the following corollary, which, if all $k_i = 0$, is the theorem of deBruijn and Erdös [3].

Corollary 3.5 *Let G be a simple graph and suppose that every finite subgraph of G is $(n; k_1, ..., k_n)$-colorable. Then G is $(n; k_1, ..., k_n)$-colorable.*

If all of the k_i are equal to k, the graph is said to be (n, k)-colorable [9]. It is a theorem of [9] that any finite simple planar graph is $(3, 2)$- colorable; the corollary implies that this is true for infinite planar graphs as well.

The property of being acyclic is of finite vertex character. If the vertex set of a graph can be partitioned into at most k sets, each of which induces an acyclic graph, we say the graph is *partitionable into k forests* or, simply, *k-forestable*. Thus we have the following corollary.

Corollary 3.6 (Forest Compactness Theorem - FCT) *Let G be a simple graph and suppose that every finite subgraph of G is k-forestable. Then G is k-forestable.*

The *point-aboricity,* $\varrho(G)$, of a graph G is the minimum number of sets the vertices can be partitioned into such that each set of the partition induces an acyclic graph. Chartrand and Kronk [8] proved that if G is a finite simple planar graph $\varrho(G) \leq 3$; the corollary implies that the same is true for infinite planar graphs.

A graph will be said to have a *k-clique coloring* if its vertex set can be partitioned into k sets, each of which induces a clique. Being a clique is a property of finite vertex character and hence, induced hereditary, although it is not hereditary. Thus we have the following corollary.

Corollary 3.7 *Let G be a simple graph and suppose that every finite induced subgraph of G has a k-clique coloring. Then G has a k-clique coloring.*

We next turn to edge colorings. Let $\mathcal{P}$ be a property of graphs; $\mathcal{P}$ is of *finite character with respect to edges* or *finite edge character* if a graph in $\mathcal{I}^*$ has property $\mathcal{P}$ if and only if every finite edge induced subgraph has property $\mathcal{P}$. It is easy to see that if $\mathcal{P}$ is of finite edge character and a graph has $\mathcal{P}$ then so does every induced subgraph. Also, the property of an edge set inducing a subgraph which has a property which is of finite edge character is a property of finite character. Hence the following theorem is an immediate consequence of the Set Partition Theorem.

Theorem 3.8 (Edge Coloring Compactness Theorem) *Let G be a graph in $\mathcal{I}^*$ and suppose $\mathcal{P}_1, ...\mathcal{P}_n$ are properties of graphs of finite edge character. Then G is $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colorable if every finite edge-induced subgraph of G is $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colorable.*

We can, as in the case of vertex colorings, state the preceding theorem in an equivalent form where now $\mathcal{S}$*-free* means no *edge* induced subgraph of G is (isomorphic to) a member of $\mathcal{S}$ and a Σ-free edge partition entails a partition of the *edges*.

Theorem 3.9 (Edge Coloring Compactness Theorem) *Let G be a simple graph and let $\Sigma = \{\mathcal{S}_i : i = 1, ..., n\}$, where $\mathcal{S}_i$ is a set of finite graphs, $i = 1, ..., n$. If every finite edge-induced subgraph of G has a Σ-free edge partition, then G has a Σ-free edge partition.*

Let F, G, H be graphs. An edge coloring of F with colors red and blue is a (G, H)*-good coloring* if F contains neither a red copy of G nor a blue copy of H. If G and H are finite graphs then (G, H)-goodness is a property of finite edge character. Thus we have the following corollary.

Corollary 3.10 (Ramsey Compactness Theorem - RCT) *Let G, H be finite graphs. Then a graph F has a (G, H)-good coloring if every finite subgraph of F has a (G, H)-good coloring.*

In order to prove most of the above theorems in Zermelo-Fraenkel set theory (ZF) some form of AC, the Axiom of Choice, is required; however the full strength of AC is not needed; BPI suffices. This follows since the Set Partition Theorem was proved from Rado's lemma along with ACF, the axiom of choice for families of finite sets (which was used to select one coloring, f_W from the finite set of colorings $f : W \to \{1, \ldots, n\}$, with $f^{-1}(i) \in \mathcal{P}_i$, $i \in Ran(f)$), and Rado's Lemma + ACF is known to be equivalent to BPI. (See the excellent book of P. Howard and J. Rubin [20] for current information on AC, BPI and related principles.) In fact all of the Colloraries proved above are equivalent to BPI; this follows since all, except Corollary 3.6 (FCT) and Corollary 3.10 (RCT), easily imply the original theorem of deBruijn and Erdös (Corollary 3.7 is the deBruijn- Erdös theorem for the complementary graph), which, in turn, implies BPI (see [20]). The proof that FCT implies BPI is given in section 5. In the next section, we prove that RCT implies BPI, for each pair of graphs, G, H– if they satisfy certain conditions.

4 Ramsey compactness and BPI

Let F, G, H be graphs. Recall that an edge coloring of F with colors red and blue is (G, H)-good if F contains neither a red copy of G nor a blue copy of H. Let Γ_3 be the class of finite graphs that are 3-connected or are triangles. Burr [6] proved that (G, H)-goodness is NP-Complete if G, H belong to Γ_3. We will prove that the corresponding compactness theorem is equivalent to BPI. Since, as mentioned above, RCT has been proved from theorems which follow from BPI, it suffices to prove BPI from RCT when $G, H \in \Gamma_3$. We shall need some machinery from [7]. A (G, H, e)*-determiner* with *determined* edge e is a finite graph F which has (G, H)-good colorings, but in any (G, H)-good coloring, e is red. Note that if $G = H$, determiners do not exist. Also note that if F is an (H, G) determiner with determined edge e, then in any (G, H)-good coloring of F, e is blue. A *positive* (G, H) *sender* with *signal edges* e, f is an F which has (G, H)-good colorings, but in any (G, H)-good coloring e and f receive the same color; also F is not a determiner for the signal edges. A *negative* (G, H) is defined similarly, but with "opposite colors" replacing "the same color." The following results are proved in [7].

Lemma 4.1 *If $G, H \in \Gamma_3$, then there exits both positive and negative (G, H) senders whose signal edges are arbitrarily distant from each other.*

Instead of proving BPI directly we prove an equivalent form concerning the compactness for 2-colorings of a hypergraph. A *hypergraph*, $\mathcal{H} =< V, E >$, is an ordered pair consisting of a set V, called the *vertices* and a set E of finite subsets of V, called the *edges.* A *2-coloring* of a hypergraph is an assignment of one of the colors, {red, blue}, to each element in V, such that no edge is monochromatic. Then the following theorem was proved equivalent to BPI in [14].

Theorem 4.2 *Let $\{E_i\}_{i \in I}$, be the edges of a hypergraph, H where each E_i has at most three elements. If for every finite $I_0 \subset I$, $\{E_i\}_{i \in I_0}$ has a hypergraph 2-coloring, then H has a hypergraph 2-coloring.*

Theorem 4.2 remains equivalent to BPI even if each E_i is an ordered set having exactly three elements; this follows from noticing that Theorem 4.2 is used in [14] to prove another equivalent form of BPI, the compactness of propositional logic when each propositional formula is a disjunction of 3 literals, which entails an ordered triple. For each pair of finite graphs, G,H, let RCT[G,H] stand for the statement of the Ramsey Compactness Theorem with these particular graphs.

Theorem 4.3 *For each $G, H \in \Gamma_3$, RCT[G,H] $\Leftrightarrow$ BPI, in ZF.*

Proof. We prove Theorem 4.2 is implied by RCT[G,H]. Let H be the given hypergraph. As mentioned above we can assume that each edge is an ordered triple, $E_i =< x_{i1}, x_{i2}, x_{i3} >$. Suppose that in H, for every finite $I_0 \subset I$, $\{E_i\}_{i \in I_0}$ has a hypergraph 2-coloring.

We wish to show that the entire hypergraph H has a hypergraph 2-coloring. Suppose then that G, H and all copies of them that we shall construct have three of their edges labeled $< g_1, g_2, g_3 >, < h_1, h_2, h_3 >$, respectively. We shall construct a new graph which corresponds to, but is separate from, the given hypergraph H. First, for each edge, $E_i =< x_{i1}, x_{i2}, x_{i3} >$, of the hypergraph, associate copies, G_i of G with the labeled edges denoted by $< g_{i1}, g_{i2}, g_{i3} >$, and H_i of H with the labeled edges denoted by $< h_{i1}, h_{i2}, h_{i3} >$.

Next, join g_{ik} to h_{ik}, $k = 1, 2, 3$, by a positive sender with these two edges as signal edges. Join the other edges of G_i, if any, to g_{i1} by positive (G, H) senders with g_{i1} and the other edge as signal edges. Join the other edges of H_i, if any, to h_{i1} by positive (G, H) senders with h_{i1} and the other edge as signal edges. Finally, if $x_{ik} = x_{jl}$, join edge g_{ik} to edge g_{jl} by a positive sender, with these two as signal edges. We can assume that the length of all these senders is greater than the maximum of length of G, H. Call the resulting graph K. It should be noted that no additional copies of G or H are formed in this construction, which are not entirely contained in a sender. This is clearly the case for triangles; if, however, the new copy of G or H were 3-connected but not entirely contained in a sender, the removal of the signal edge it must contain would disconnect it!

Every finite subgraph K' of K has a (G, H)-good coloring, since if E' is the finite set of edges of the hypergraph used in the construction of K', E' is hypergraph 2-colorable and we can color each labeled g_{ik}, h_{ik} in K' the same color as x_{ik}, $x_{ik} \in E_i \in E'$. Color the other edges of G_i and the other edges of H_i, the same as g_{i1}. Then, G_i, H_i have both red and blue edges and the edges of the senders can be colored appropriately to yield a (G, H)-good coloring of K'.

By Ramsey Compactness, K has a (G, H)-good coloring. This induces a 2-coloring of the hypergraph: color $x_{ik} \in E_i$, the color received by g_{ik}. Since the edges of G_i can't all be red,

$< g_{i1}, g_{i2}, g_{i3} >$ can't all be red and since the edges of H_i can't all be blue, $< h_{i1}, h_{i2}, h_{i3} >$ can't all be blue. Also if $< g_{i1}, g_{i2}, g_{i3} >$ were all blue, $< h_{i1}, h_{i2}, h_{i3} >$ would be all blue as well, since g_{ik} and h_{ik} are connected by a positive sender. Thus $< g_{i1}, g_{i2}, g_{i3} >$ and hence $< x_{i1}, x_{i2}, x_{i3} >$ have red and blue elements. Also, if $x_{ik} = x_{jl}$, we joined edge g_{ik} to edge g_{jl} by a positive sender and this insures that the same element receives the same color in each hyperedge in which it occurs. Thus the hypergraph is 2-colored.

If G and H are stars (graphs of the form $K_{1,n}$), Burr [6] shows that a determination whether a graph F is (G, H) good can be made in polynomial time; we conjecture that the Ramsey Compactness Theorem, restricted to stars, is weaker than BPI.

5 Partition into Forests and BPI The next problem to be considered is partitioning a graph into forests. We prove that $BPI \Leftrightarrow FCT(n)$, for each $n \geq 2$, where $FCT(n)$ is the Forest Compactness Theorem in the case $k = n$. We accomplish this by first proving $FCT(n+1) \Rightarrow FCT(n)$, for all $n > 1$ and then we prove $FCT(2) \Rightarrow BPI$. This suffices since we already know that FCT follows from BPI.

The proof that $FCT(n+1) \Rightarrow FCT(n)$ will be broken up into a few pieces. For each $n > 2$, we now consider a graph which we will call H_n. To construct H_n, we begin with n disjoint paths, each of length 4, which we call $P_1, \dots, P_n$. We complete H_n by adding edges between every pair of vertices not in the same P_n.

Lemma 5.1 *The only decomposition of H_n into n or fewer subforests is the decomposition of H_n into the n paths from which it is constructed.*

Proof. Note that if P is any path of length 4, and we delete one vertex and its associated edges, there will be at least one edge remaining.

Next, if u,v and w are vertices in H_n, and each is in a different P_k, these vertices induce a triangle. Also, if u, v, w and x are vertices of H_n, and u, v belong to one P_k while w, x belong to another, then u, w, v, x, u is a 4-cycle. Thus subgraphs of H_n which are forests contain vertices from at most two paths, and contain more than one vertex from at most one path. In particular, if such a subforest contains at least three vertices, then all but one, at most, will come from the same path, say P_k.

It now follows that if a subforest of H_n has 4 vertices, then the subforest must be one of the P_k, because, by the above, it must have at least three of the vertices in P_k. But if the fourth vertex were not in this P_k, it plus the two connected vertices would form a triangle. In particular, this implies that no subforest of H_n can have more than four vertices.

Finally, suppose we decompose H_n into n or fewer subforests. Each has at most four vertices. Therefore each must have exactly four vertices and, by the above, each must be one of the P_k.

Theorem 5.2 *For all $n > 1$, $FCT(n+1) \Rightarrow FCT(n)$, in ZF.*

Proof. Asume $FCT(n+1)$, and let G be a graph with the property that each of its finite subgraphs can be decomposed into at most n subforests. We must show that G can be decomposed into at most n subforests.

Let T be the graph consisting of two vertices, u, v and an edge connecting these vertices. We construct a graph G^* as follows. The set of vertices of G^* is the union of the sets of vertices of G, H_{n+1}, and T. The edges of G^* are all the edges of the following three types: 1) the edges in G, H_{n+1}, and T, 2) edges connecting each vertex of T to each vertex of G, and 3) edges connecting each vertex of T to each vertex in $H_{n+1} - P_{n+1}$.

We first show that every finite subgraph of G^* can be decomposed into at most $n+1$ subforests. Let K be any finite subgraph of G^*. By hypothesis $K \cap G$ can be decomposed into at most n subforests: $K_1^G, \dots, K_n^G$. Now decompose K, itself, into the following $n+1$ subforests: $K_1^G \cup (P_1 \cap K), \dots, K_n^G \cup (P_n \cap K), (P_{n+1} \cap K) \cup (T \cap K)$. Therefore, by $FCT(n+1)$, G^* can be decomposed into $n+1$ subforests: $G_1^*, \dots, G_{n+1}^*$. This, of course, induces a similar decomposition of H_{n+1} into $n+1$ forests, and, as we have shown above, these must be the paths, P_i. Thus, by renumbering, we may assume that for each k, $P_k \subset G_k^*$.

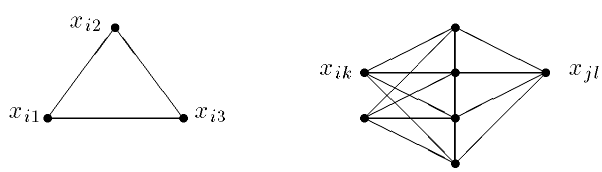
Since both vertices of T are connected to all vertices of P_k, except P_{n+1}, we must have $T \subset G_{n+1}^*$. But, since both vertices of T are also connected to all vertices of G, we must have $G \cap G_{n+1}^* = \emptyset$. However, this implies that $G \cap G_1^*, \ldots, G \cap G_n^*$ is a decomposition of G into n forests.

Theorem 5.3 *In ZF,* $FCT(2) \Leftrightarrow BPI$.

Proof. It suffices to prove Theorem 4.2 from $FCT(2)$. Again, as mentioned above, we can assume that each edge of hypergraph $\mathcal{H}$, with edges $\{E_i\}_{i \in I}$, is an ordered triple, $E_i = < x_{i1}, x_{i2}, x_{i3} >$. Suppose that for every finite $I_0 \subset I$, $\{E_i\}_{i \in I_0}$ has a hypergraph 2-coloring and suppose the colors are red and blue.

We wish to show that the entire hypergraph has a red-blue coloring.

We associate a graph G with the hypergraph as follows. For each edge $< x_{i1}, x_{i2}, x_{i3} >$, take a triangle with vertices labeled, x_{i1}, x_{i2}, x_{i3} as shown on the left in the figure below. If $x_{ik} = x_{jl}$ connect the two vertices, labeled, x_{ik}, x_{jl} as shown in the figure, on the right. Note that this "connector" has the property that if the labeled vertices are colored the same, the red-blue coloring can be completed with no monochrome cycles and no monocrome path between the labeled vertices. However, if the labeled vertices are oppositely colored, any 2-coloring must contain a monochrome cycle.



We claim next that any finite subgraph of G is 2-forestable. A finite subgraph contains triangles built from a finite number of edges of the hypergraph; these finite edges can be properly 2-colored. Transfer the colors to the labeled vertices of the finite subgraph of G. Since no hyperedge is monochromatic, the labeled triangles have both red and blue vertices. Vertices that come from the same element in different edges of the hypergraph will naturally receive the same color and this allows us to properly color the connectors as mentioned above. Note also that no "accidental" monochromatic cycles are formed, since there can be no monochrome path connecting the labeled vertices under this coloring. Hence every finite subgraph of G is 2-forestable.

The forest compactness theorem now yields a red-blue coloring of the vertices of G which has no red or blue cycles. Color the hypergraph edges using the coloring of the associated labeled triangles, that is, each vertex, x_{ij}, of the hyperedge receives the color assigned to the vertex labeled x_{ij}. Since no triangle can be monochromatic, each edge gets both colors; also, the connectors insure that the same elements in different edges receive the same color. Thus the hypergraph has been properly 2-colored.

Finally, putting together the results above gives the following theorem.

Theorem 5.4 *For each* $n > 1$, $FCT(n) \Leftrightarrow BPI$, *in ZF.*

We note that the same constructions can be utilized to establish that partition into k-forests (GT 14 of [17]) is an NPC property, even for fixed $k = 2, 3, \ldots$.

6 Defective Colorings and BPI

We return next to the (n,k)-colorings introduced in [9]. These are also called "defective colorings" and the number of neighboring vertices receiving the same color as a vertex in a coloring of the graph is referred to as the *defect* of that vertex in that coloring. It is an easy consequence of our results in section 3 that the following theorem is equivalent to BPI.

Theorem 6.1 (Defective Coloring Compactness Theorem-DCCT) *Let G be a simple graph and suppose that every finite subgraph of G is (n,k)-colorable. Then G is (n,k)-colorable.*

Let $DC(n,k)$ be the statement of the DCCT for fixed n,k and we ask the question: for which n,k is $DC(n,k)$ equivalent to BPI? It has been proved in [15] that $DC(2,1)$ is equivalent to BPI and Läuchli [22] has shown that $DC(n,0)$ is equivalent to BPI, for $n \geq 3$. It is also well known that $DC(2,0)$ is much weaker than BPI in ZF, being equivalent to the Axiom of Choice for unordered pairs (see [25],[27]). We will show that this is an exception, in that $DC(n,k)$ is equivalent to BPI if $n=2$ and $k \geq 1$ or $n=3$ and $k \geq 0$. The idea for our proof comes from the proof of NP-Completeness of $(3,1)$-colorings, Theorem 4.2b, of [10]. We use their reduction method.

Theorem 6.2 $DC(n,k+1) \Rightarrow DC(n,k)$*, for* $n \geq 2$*,* $k \geq 0$*.*

Proof. Assume $DC(n,k+1)$. Let G be a simple graph such that every finite subgraph is (n,k)-colorable. We wish to show G is (n,k)-colorable. For each vertex v of G associate a new graph H_v which is isomorphic to $K_{(n-1)(k+2)+1}$, the complete graph on $(n-1)(k+2)+1$ vertices and designate one of its vertices, $v*$. Then the H_v have the following two properties: 1)H_v is not $(n-1,k+1)$-colorable, 2) H_v is $(n,k+1)$ colorable, and, for any color c, there is a $(n,k+1)$-coloring of H_v with $v*$ being the unique vertex receiving the color c.

Let G' be the graph obtained from G by taking the union of G, the H_v and edges to connect the vertices of H_v to v, for all v in G. Then we claim that every finite subgraph H of G' is $(n,k+1)$-colorable. (We can, without loss of generality, assume H contains all of H_v if it contains a vertex v from G and if it contains any part of some H_v, it contains v as well.) Then the claim follows, since the part of H that comes from G is (n,k)-colorable and then, by 2), any H_v from H can be $(n,k+1)$-colored with $v*$ being the only vertex to receive the color of v. The defect in the vertices from G is increased by 1 in this coloring and $v*$ is the only vertex in H_v whose defect is increased–from 0 to 1.

It follows, by $DC(n,k+1)$, that G' is $(n,k+1)$-colorable. Since H_v is not $(n-1,k+1)$ colorable, all n colors must occur in each H_v and then whatever color is assigned to v its defect in G is at most $(k+1)-1=k$. Thus G is (n,k)-colored.

Corollary 6.3 $DC(n,k) \Rightarrow$ *BPI, if* $n \geq 2$ *and* $n+k \geq 3$*.*

Proof. Follows from the theorem and the aforementioned results of Cowen [15] and Läuchli [22].

This is an example of a case when a proof of NPCompleteness easily suggests a proof that the corresponding compactness theorem is equivalent to BPI. For more on the connections between BPI and NPC see [5],[14], [15].

7 Hypergraph Compactness

The notion of generalized coloring has recently been extended in [4] to hypergraphs. We wish to point out here that a compactness theorem is available in this context as well. Let $\mathcal{P}_1,...,\mathcal{P}_n$ be properties of hypergraphs. A hypergraph H is $(\mathcal{P}_1,...,\mathcal{P}_n)$-*colorable* if the vertex set X of H can be partitioned into sets $X_1,\dots,X_n$, such that the induced subhypergraphs $< X_i, E(X_i) >$ of H, where $E(X_i)$ consists of all hyperedges of H all of whose vertices belong to X_i, has property $\mathcal{P}_i$; $i=1,2,\dots,n$. We shall denote by $H[X_i]$ the induced subhypergraph $< X_i, E(X_i) >$. A property $\mathcal{P}$ of hypergraphs is of *finite vertex character* if a hypergraph has property $\mathcal{P}$ if and only if every finite induced subhypergraph has $\mathcal{P}$. Then the following compactness theorem also follows immediately from the Set Partition Theorem.

Theorem 7.1 *Let H be a simple hypergraph and suppose $\mathcal{P}_1,...\mathcal{P}_n$ are properties of hypergraphs of finite vertex character. Then H is $(\mathcal{P}_1,\mathcal{P}_2,\dots,\mathcal{P}_n)$-colorable if every finite induced subhypergraph of H is $(\mathcal{P}_1,\mathcal{P}_2,\dots,\mathcal{P}_n)$-colorable.*

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