MISSPECIFIED INTERPOLATION FOR TIME SERIES AND ITS NUMERICAL STUDIES

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ABSTRACT. Let $\{X_t\}$ be a stationary process with zero mean and true spectral density $g(\lambda)$. We assume that all the values $\{X_t\}$ are known, except for the value X_0. It is important to interpolate missing value X_0 by linear combination of $\{X_t : t \neq 0\}$. In this paper, we consider the misspecified interpolation problems for $\{X_t\}$ under the condition that the true structure $g(\lambda)$ is not completely specified. Next we shall discuss the interpolation problems for square-transformed process, $Y_t = X_t^2$ and Hermite-transformed process, $Y_t = H_q(X_t)$. Moreover, we compare the mean square interpolation error for the respective results. Also, we numerically plot interpolators for missing values, and illuminiate unexpected effects from the misspecification of spectra. Finally, as an example, we conclude the paper with applications to real data.

1 Introduction. An important problem of stationary processes is that of interpolating a missing value of the process, which for some reason cannot be observed completely, in terms of observed values of the same process. It should be noted that statistical analysis of the data may require deletion of some of the observed values, for example, in outlier detection, bootstrapping, and cross-validation etc. Suppose that $\{X_t\}$ is a stationary process with zero mean and spectral density $g(\lambda)$. We shall interpolate the unknown value X_0 by linear combination of $\{X_t,\, t \neq 0\}$ where all values X_t are known, except for the value X_0. However, it is often that the true structure $g(\lambda)$ is not completely specified. This leads us to a misspecified interpolation problem when a conjectured structure $f(\lambda)$ is fitted to $g(\lambda)$. It is Grenander and Rosenblatt (1957) that first discussed the misspecified prediction problem which is evaluated on the basis of the conjectured spectral density $f(\lambda)$ although the true one is $g(\lambda)$ (also see Choi and Taniguchi (2001a, 2001b)).

Regarding general polynomial transformations Hannan (1970) derived the autocovariance function for Hermite polynomials of a Gaussian process. Granger and Newbold (1976) discussed the problems of prediction for a class of nonlinear transformation $T = T(\cdot)$ of a Gaussian process. Here T is assumed to be approximated by Hermite polynomials. Hannan (1970) mentioned the usual interpolation problem for a general vector linear process. Taniguchi (1980) evaluated the interpolation error for stationary process with multiple missing time points when the interpolators are constructed by the true spectral density and a conjectured spectral density, respectively. Pourahmadi (1989) gave an interpolator and its error for a class of nondeterministic stationary processes with arbitrary number of missing values.

The primary purpose of this paper is to evaluate the mean square error of misspecified interpolation for square-transformed process and Hermite-transformed process and to examine the behavior of MSE and interpolator numerically. This paper is organized as follows. In the next section we give the asymptotic mean square error of interpolation for linear

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process with a single or multiple time points when an interpolator is constructed by a conjectured parametric spectral density. Here the parameter is estimated by the quasi-MLE. In Section 3, under the assumption that $\{X_t\}$ is a Gaussian stationary process we evaluate the mean square interpolation error of the squared process X_t^2 by a naive interpolator, i.e., (the best linear interpolator of $X_t)^2$ and the bias adjusted mean square interpolation error for square-transformed process when the best linear interpolators are constructed by the true spectral density and a conjectured spectral density, respectively. Futhermore, this result is extended to Hermite-transformed process. Section 4 illuminates some unexpected aspects of the interpolation problems numerically. We actually interpolate missing data for a process when the interpolators are constructed by the true spectral density and a conjectured spectral density, respectively. Finally, we present results of analysis of a monthly accidental death data in the United States.

2 Misspecified Interpolation for Stationary Processes.

We consider a misspecified interpolation of a missing value in time series. Let $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ be a nondeterministic stationary process with mean zero and spectral density

$$g(\lambda) = \frac{1}{2\pi}|c(e^{-i\lambda})|^2, \quad |c(0)|^2 = \sigma^2, \tag{1}$$

which belongs to the class $D = \{g : g(\lambda)$ is continuous and piecewise smooth on $[\pi, \pi]$, $g(\lambda) = g(-\lambda), g(\lambda) > 0$ for all $\lambda \in [-\pi, \pi]\}$. Here $c(z)$ is a polynomial of z, i.e., $c(z) = \sum_{j=0}^{\infty} c_j z^j$. Suppose that all values X_t are known, except for the value X_0. We shall interpolate the unknown value X_0 by linear combination of $X_t, t \neq 0$. We write the spectral representation of X_t as

$$X_t = \int_{-\pi}^{\pi} e^{it\lambda} dz(\lambda). \tag{2}$$

Let $\mathcal{M}_0\{\dots\}$ denote the closed linear manifold generated by elements in the braces with respect to the mean-square norm. Then it is well known that the response function $h(\lambda) \in \mathcal{M}_0\{e^{-ij\lambda} : j \neq 0\}$ of the best linear interpolator for X_0 is given by

$$h(\lambda) = 1 - g(\lambda)^{-1}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\lambda)^{-1} d\lambda\right\}^{-1}$$

(e.g. Hannan (1970, p.163)). We can write the best linear interpolator of X_0 as

$$\hat{X}_0 = \int_{-\pi}^{\pi} h(\lambda) dz(\lambda). \tag{3}$$

Then the interpolation error is

$$\begin{aligned} E|X_0 - \hat{X}_0|^2 &= E\left|\int_{-\pi}^{\pi} dz(\lambda) - \int_{-\pi}^{\pi} h(\lambda) dz(\lambda)\right|^2 \\ &= \int_{-\pi}^{\pi} |1 - h(\lambda)|^2 g(\lambda) d\lambda \\ &= 2\pi\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\lambda)^{-1} d\lambda\right]^{-1}. \end{aligned} \tag{4}$$

We examine a simple numerical evaluation for the above result. Let $g(\lambda) = \frac{1}{2\pi}|1 - \theta e^{-i\lambda}|^{-2}$, $|\theta| < 1$, i.e., $X_t = \theta X_{t-1} + \epsilon_t$, where ϵ_t's are i.i.d. $(0, 1)$ random variables. In

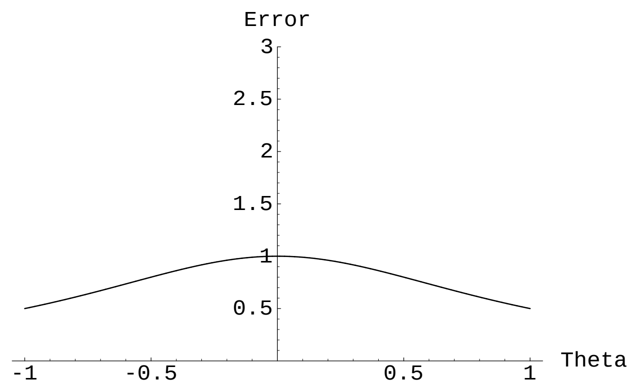

Figure 1. The interpolation error (4) for $g(\lambda) = \frac{1}{2\pi}|1-\theta e^{-i\lambda}|^{-2}$.

Figure 1 we plotted the interpolation error (4) for the model. Figure 1 shows that the interpolation errors in this case are relatively small and stable with respect to θ.

In most natural phenomena, it is often that the true spectral density $g(\lambda)$ is not completely specified. Thus it is of considerable interest to see what happens when an interpolator is computed on the basis of a conjectured spectral density $f_\theta(\lambda)$ although the true one is $g(\lambda)$. Here θ is a parameter vector satisfying $\theta \in \Theta \subset \boldsymbol{R}^p$. The response function $h_\theta(\lambda)$ of the interpolator based on $f_\theta(\lambda)$ is given by

$$h_\theta(\lambda) = 1 - f_\theta(\lambda)^{-1}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} f_\theta(\lambda)^{-1}d\lambda\right\}^{-1}.$$

Hence, the best linear interpolator $\hat{X}_0$ based on $f_\theta(\lambda)$ is

$$\hat{X}_0 = \int_{-\pi}^{\pi} h_\theta(\lambda)dz(\lambda). \tag{5}$$

Then the interpolation error is

$$\begin{aligned} E|X_0 - \hat{X}_0|^2 &= E\left|\int_{-\pi}^{\pi} dz(\lambda) - \int_{-\pi}^{\pi} h_\theta(\lambda)dz(\lambda)\right|^2 \\ &= \int_{-\pi}^{\pi} |1-h_\theta(\lambda)|^2 g(\lambda)d\lambda \\ &= \left\{\int_{-\pi}^{\pi}\frac{g(\lambda)}{f_\theta(\lambda)^2}d\lambda\right\}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} f_\theta(\lambda)^{-1}d\lambda\right]^{-2} \equiv M(\theta). \end{aligned} \tag{6}$$

To see effect of misspecification of spectra for the interpolation, we also examine the following example.

Example 1. Let the true spectral density and conjectured spectral density be $g(\lambda) = \frac{1-\theta^2}{2\pi}|1-\theta e^{-i\lambda}|^{-2}$ and $f(\lambda) = \frac{1-\theta^2}{2\pi}|1-\theta e^{-i\lambda}|^{2}$, $|\theta| < 1$, respectively. Then the interpolation error (6) is $\frac{2(1+4\theta^2+\theta^4)}{(1-\theta^2)^3}$. In Figure 2 we plotted $M(\theta)$. From Figure 2, we see that if $|\theta| \nearrow 1$, interpolation errors become very large. This implies that we have to be careful about the misspecification of spectra for interpolation. Therefore investigation of the misspecified interpolation problem seems important.

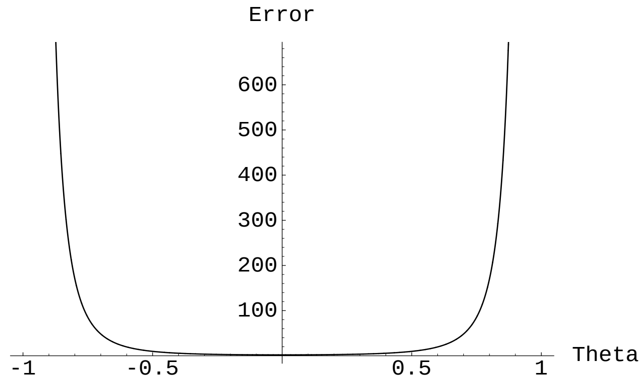

Figure 2. The interpolation error (6) for Example 1.

Actually the parameter θ in $f_\theta(\lambda)$ is unknown. Suppose that we have an observed stretch $X'_1, \ldots, X'_T$ which has the same probability structure as $\{X_t\}$ and is independent of $\{X_t\}$. The unknown parameter θ is estimated by a quasi-MLE $\hat{\theta}_Q = (\hat{\theta}_{n,1}, \ldots, \hat{\theta}_{n,p})'$ which minimizes

$$\int_{-\pi}^{\pi} \{\log f_\theta(\lambda) + I_T(\lambda) f_\theta(\lambda)^{-1}\} d\lambda,$$

with respect to $\theta \in \Theta$, where

$$I_T(\lambda) = \frac{1}{2\pi T} |\sum_{t=1}^{T} X'_t e^{it\lambda}|^2.$$

Then the estimated interpolator is given by

$$\hat{\hat{X}}_0 = \int_{-\pi}^{\pi} h_{\hat{\theta}_Q}(\lambda) dz(\lambda), \tag{7}$$

and the interpolation error of $\hat{\hat{X}}_0$ is

$$E_X[\{X_0 - \hat{\hat{X}}_0\}^2] = M(\hat{\theta}_Q),$$

where E_X is the expectation with respect to $\{X_t\}$. Here we set down some assumption on f_θ.

Assumption 1. (i) *The parameter θ is innovation free.*
(ii) *f_θ is continuously three times differentiable with respect to $\theta \in \Theta$.*

Expanding $M(\hat{\theta}_Q)$ at $\theta = \underline{\theta}$ in a Taylor series we obtain

$$\begin{aligned} M(\hat{\theta}_Q) &= M(\underline{\theta}) + \frac{\partial}{\partial \theta'} M(\underline{\theta})(\hat{\theta}_Q - \underline{\theta}) \\ &\quad + \frac{1}{2} \mathrm{tr} \left\{ \frac{\partial^2}{\partial \theta \partial \theta'} M(\underline{\theta})(\hat{\theta}_Q - \underline{\theta})(\hat{\theta}_Q - \underline{\theta})' \right\} + \text{lower order terms}, \end{aligned} \tag{8}$$

where $\underline{\theta} = (\underline{\theta}_1, \ldots, \underline{\theta}_p)' = \arg\min_{\theta \in \Theta} \int_{-\pi}^{\pi} [\log f_\theta(\lambda) + \{g(\lambda) f_\theta(\lambda)^{-1}\}] d\lambda$. To evaluate the second term of the right hand side (RHS) of (8), we make the following assumption.

Assumption 2. *For $\alpha = 1, \dots, p$*

$$E[\sqrt{T}(\hat{\theta}_{n,\alpha} - \underline{\theta}_\alpha)] = T^{-1/2}\mu^\alpha + o(T^{-1}), \tag{9}$$

where μ^α's are constants.

This assumption is natural because most of regular estimators satisfy it. For example, Taniguchi and Watanabe (1994) evaluated $E[\sqrt{T}(\hat{\theta}_{n,\alpha} - \theta_\alpha)]$ in the form of (9) for generalized curved probability densities. To describe the asymptotics of $\hat{\theta}_Q$ we need the following regularity conditions.

Assumption 3. (A.1) *$\{X_t\}$ is a linear process generated by*

$$X_t = \sum_{j=0}^{\infty} a_j e_{t-j}, \quad \sum_{j=0}^{\infty} a_j^2 < \infty \text{ and } a_0 = 1,$$

where $\{e_t\}$ is a sequence of i.i.d. random variables with $Ee_t = 0$, $Ee_t^2 = \sigma^2$ and finite fourth moment $Ee_t^4 < \infty$;
(A.2) *The spectral density $g(\lambda)$ of $\{X_t\}$ is square-integrable;*
(A.3) $g(\lambda) \in \mathrm{Lip}(\alpha)$, $\alpha > 1/2$;
(A.4) *$\underline{\theta}$ exists uniquely and $\underline{\theta} \in Int\Theta$;*
(A.5) *The matrix*

$$M_f = \int_{-\pi}^{\pi} \left[\frac{\partial^2}{\partial\theta\partial\theta'} f_\theta(\lambda)^{-1} g(\lambda) + \frac{\partial^2}{\partial\theta\partial\theta'} \log f_\theta(\lambda) \right]_{\underline{\theta}} d\lambda$$

is nonsingular.

By means of (8) we evaluate $\overline{E}\{M(\hat{\theta}_Q)\}$, where $\overline{E}\{\cdot\}$ is the expectation with respect to the asymptotic distribution of $\sqrt{T}(\hat{\theta}_Q - \underline{\theta})$. We have placed the proofs of the propositions in Appendix.

Proposition 1. *Under the Assumptions 1, 2 and 3,*

$$\begin{aligned} \overline{E}\{M(\hat{\theta}_Q)\} &= M(\underline{\theta}) + \frac{1}{T}\sum_{\alpha=1}^{q} \mu^\alpha \left\{ \frac{\partial}{\partial\theta_\alpha} M(\underline{\theta}) \right\} \\ &\quad + \frac{1}{2T} \mathrm{tr} \left\{ \frac{\partial^2}{\partial\theta\partial\theta'} M(\underline{\theta}) M_f^{-1} \tilde{V} M_f^{-1} \right\} + o(T^{-1}), \end{aligned} \tag{10}$$

where

$$M_f = \int_{-\pi}^{\pi} \left[\frac{\partial^2}{\partial\theta\partial\theta'} f_\theta(\lambda)^{-1} g(\lambda) + \frac{\partial^2}{\partial\theta\partial\theta'} \log f_\theta(\lambda) \right]_{\underline{\theta}} d\lambda,$$

$$\begin{aligned} \tilde{V} &= 4\pi \int_{-\pi}^{\pi} \left[g(\lambda) \frac{\partial}{\partial\theta} \{f_\theta(\lambda)\}^{-1} g(\lambda) \frac{\partial}{\partial\theta} \{f_\theta(\lambda)\}^{-1} \right]_{\underline{\theta}} d\lambda \\ &\quad + 2\pi \int\int_{-\pi}^{\pi} \left\{ \frac{\partial}{\partial\theta} f_\theta^{-1}(\lambda_1) \frac{\partial}{\partial\theta} f_\theta^{-1}(\lambda_2) \right\}_{\underline{\theta}} \\ &\quad \times \tilde{Q}^X(-\lambda_1, \lambda_2, -\lambda_2) d\lambda_1 d\lambda_2, \end{aligned}$$

and $\tilde{Q}^X(-\lambda_1, \lambda_2, -\lambda_2)$ *is the fourth-order cumulant spectral density of the process* $\{X_t\}$.

Next, we consider a misspecified interpolation problem with multiple missing time points. We denote by Z the set of all integers, and put $A_p = \{1, \dots, p\}$. Let $\{X_t : t \in Z\}$ be a non-deterministic stationary process with true spectral density $g(\lambda)$ and mean zero. Denote a conjectured spectral density by

$$f_\theta(\lambda) = \frac{1}{2\pi}|c_\theta(e^{-i\lambda})|^2. \tag{11}$$

We suppose that all values X_t are known, except for the values $X_t, t \in A_p$. We shall interpolate the unknown value $\boldsymbol{X}_{A_p} \equiv (X_1, \dots, X_p)'$. That is, we seek a vector whose elements are linear combination of $X_t, t \in \boldsymbol{Z} - A_p$, and which minimize the trace of the mean square interpolation error.

We can write the spectral representation of $\boldsymbol{X}_{A_p}$ as

$$\boldsymbol{X}_{A_p} = \int_{-\pi}^{\pi} \boldsymbol{e}(\lambda) dz(\lambda), \tag{12}$$

where $\boldsymbol{e}(\lambda) = (e^{-i\lambda}, \dots, e^{-ip\lambda})'$. It is easy to see that the response function $\boldsymbol{h}_\theta(\lambda)$ of the interpolator based on $f_\theta(\lambda)$ is given by

$$\boldsymbol{h}_\theta(\lambda) = \left\{ \boldsymbol{I}_p - f_\theta(\lambda)^{-1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f_\theta(\lambda)^{-1} \boldsymbol{F}(\lambda) d\lambda \right)^{-1} \right\} \boldsymbol{e}(\lambda), \tag{13}$$

where $\boldsymbol{F}(\lambda) = \boldsymbol{e}(\lambda)\boldsymbol{e}(\lambda)^*$, and each component of $\boldsymbol{h}_\theta(\lambda)$ belongs to $\mathcal{M}_0\{e^{-ij\lambda} : j \in \boldsymbol{Z} - A_p\}$.

We can construct the best interpolator of $\boldsymbol{X}_{A_p}$ by

$$\hat{\boldsymbol{X}}_t = \int_{-\pi}^{\pi} \boldsymbol{h}_\theta(\lambda) dz(\lambda). \tag{14}$$

Then the interpolation error is

$$\begin{aligned}
&\mathrm{tr} E[\{\boldsymbol{X}_{A_p} - \hat{\boldsymbol{X}}_t\}\{\boldsymbol{X}_{A_p} - \hat{\boldsymbol{X}}_t\}'] \\
&\quad = \mathrm{tr} E\left[\left\{ \int_{-\pi}^{\pi} \boldsymbol{e}(\lambda) dz(\lambda) - \int_{-\pi}^{\pi} \boldsymbol{h}_\theta(\lambda) dz(\lambda) \right\} \left\{ \int_{-\pi}^{\pi} \boldsymbol{e}(\lambda) dz(\lambda) - \int_{-\pi}^{\pi} \boldsymbol{h}_\theta(\lambda) dz(\lambda) \right\}^* \right] \\
&\quad = \mathrm{tr} \int_{-\pi}^{\pi} [\boldsymbol{e}(\lambda) - \boldsymbol{h}_\theta(\lambda)] g(\lambda) [\boldsymbol{e}(\lambda) - \boldsymbol{h}_\theta(\lambda)]^* d\lambda \\
&\quad = \mathrm{tr} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} f_\theta(\lambda)^{-1} \boldsymbol{F}(\lambda) d\lambda \right\}^{-1} \left\{ \int_{-\pi}^{\pi} \frac{g(\lambda)}{f_\theta(\lambda)^2} \boldsymbol{F}(\lambda) d\lambda \right\} \\
&\qquad \times \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} f_\theta(\lambda)^{-1} \boldsymbol{F}(\lambda) d\lambda \right\}^{-1} \\
&\quad \equiv \boldsymbol{M}(\theta), \ (\text{say}), \qquad (15)
\end{aligned}$$

(see Taniguchi (1980)).

Repeating the same arguments as in Proposition 1, we have the following proposition.

Proposition 2. *Under the Assumptions 1, 2 and 3,*

$$\overline{E}\{\boldsymbol{M}(\hat{\theta}_Q)\} = \boldsymbol{M}(\underline{\theta}) + \frac{1}{T}\sum_{\alpha=1}^{q}\mu^{\alpha}\left\{\frac{\partial}{\partial\theta_{\alpha}}\boldsymbol{M}(\underline{\theta})\right\} + \frac{1}{2T}\mathrm{tr}\left\{\frac{\partial^2}{\partial\theta\partial\theta'}\boldsymbol{M}(\underline{\theta})M_f^{-1}\tilde{V}M_f^{-1}\right\} + o(T^{-1}). \tag{16}$$

Pourahmadi (1989) discussed how to interpolate missing observations and obtain the mean square error of the interpolator. Let $\{X_t\}$ be a nondeterministic stationary process with AR parameter $\{a_k\}$, and let $\hat{X}'_{0,r}$ be the best linear interpolator of X_0 based on $\{X_t; t \leq r, t \neq 0\}$, where r is a positive integer. Then he showed

$$\hat{X}'_{0,r} = \hat{X}_0 + \sum_{k=1}^{r} c_{k,r}(X_k - \hat{X}_k), \tag{17}$$

where $c_{k,r}$ is the regression coefficient of X_0 on $X_k - \hat{X}_k$ given by

$$c_{k,r} = \left(1 + \sum_{i=1}^{r} a_i^2\right)^{-1}\left(a_k - \sum_{i=1}^{r-k} a_i a_{i+k}\right), \quad (k = 1, 2, \ldots, r),$$

and $\hat{X}_k$ $(k \geq 0)$ is the best linear predictor of X_k based on the infinite past $X_{-1}, X_{-2}, \ldots$.

Using our approach, we can grasp the results of Pourahmadi's interpolator and interpolation error. Let X_k be a stationary process with zero mean and spectral density $g(\lambda)$. For $g(\lambda)$, we fit an AR-type spectral density

$$f_k(\lambda) = \frac{1}{2\pi}|1 - \theta_1 e^{i(k+1)\omega} - \cdots - \theta_q e^{i(k+q+1)\omega} - \cdots|^{-2} \tag{18}$$

as the conjectured spectral density $f_k(\lambda)$. Here, $\{\theta_q : q = 1, 2, \ldots\}$ are defined by $(\theta_1, \ldots, \theta_q, \ldots) \equiv \mathrm{argmin}_{\theta} \int_{-\pi}^{\pi} g(\lambda)/f_k(\lambda)d\lambda$. Writing

$$f_k(\lambda) = \frac{1}{2\pi}|c_{f_k}(e^{-i\lambda})|^2,$$

we can construct the best linear predictor of X_k based on $f_k(\lambda)$ as

$$\hat{X}_k = \int_{-\pi}^{\pi} e^{ik\lambda}\frac{c_{f_k}(e^{-i\lambda}) - c_{f_k}(0)}{c_{f_k}(e^{-i\lambda})}dz(\lambda), \quad c_{f_k}(0) = 1, \quad k = 1, 2, \ldots. \tag{19}$$

Also, for $k = 0$,

$$\hat{X}_0 = \int_{-\pi}^{\pi} \frac{c_{f_0}(e^{-i\lambda}) - c_{f_0}(0)}{c_{f_0}(e^{-i\lambda})}dz(\lambda), \quad c_{f_0}(0) = 1. \tag{20}$$

Next we seek the coefficients $c_{k,r}$ in our expression. Let $\boldsymbol{a}' = [E\{X_0(X_1 - \hat{X}_1)\}, \ldots, E\{X_0(X_r - \hat{X}_r)\}]'$. Then the k-th element of $\boldsymbol{a}$ is given by

$$E\{X_0(X_k - \hat{X}_k)\} = E\left[\overline{\int_{-\pi}^{\pi} dz(\lambda)}\int_{-\pi}^{\pi} e^{ik\lambda}\frac{1}{c_{f_k}(e^{-i\lambda})}dz(\lambda)\right] = \int_{-\pi}^{\pi} e^{ik\lambda}\frac{g(\lambda)}{c_{f_k}(e^{-i\lambda})}d\lambda. \tag{21}$$

Let A be the $r \times r$ matrix whose (l,k)th element is

$$\begin{aligned} E\{(X_l - \hat{X}_l)(X_k - \hat{X}_k)\} &= E\left[\int_{-\pi}^{\pi} e^{il\lambda}\frac{1}{c_{f_l}(e^{-i\lambda})}dz(\lambda)\overline{\int_{-\pi}^{\pi} e^{ik\lambda}\frac{1}{c_{f_k}(e^{-i\lambda})}dz(\lambda)}\right] \\ &= \int_{-\pi}^{\pi} e^{i(l-k)\lambda}\frac{g(\lambda)}{c_{f_l}(e^{-i\lambda})\overline{c_{f_k}(e^{-i\lambda})}}d\lambda. \qquad (22) \end{aligned}$$

Thus the coefficients $c_{k,r}$'s are given by the equation

$$\begin{pmatrix} c_{1,r} \\ \vdots \\ c_{r,r} \end{pmatrix} = A^{-1}\boldsymbol{a}. \qquad (23)$$

Hence the best linear interpolator of Pourhmadi (1981) is expressed as

$$\hat{X}'_{0,r} = \int_{-\pi}^{\pi} \frac{c_{f_0}(e^{-i\lambda}) - c_{f_0}(0)}{c_{f_0}(e^{-i\lambda})}dz(\lambda) + \sum_{k=1}^{r} c_{k,r}\int_{-\pi}^{\pi} e^{ik\lambda}\frac{1}{c_{f_k}(e^{-i\lambda})}dz(\lambda), \qquad (24)$$

where $c_{k,r}$'s are defined by (23).

3 Interpolation Problems for Transformed Processes. In this section we consider the problem of misspecified interpolation for transformed process. First, we discuss the case of square-transformation $Y_t = X_t^2$. Let $\{X_t\}$ be a Gaussian stationary process with zero mean, and spectral density

$$g(\lambda) = \frac{1}{2\pi}|c(e^{-i\lambda})|^2 \qquad |c(0)|^2 = \sigma^2.$$

For simplicity we assume $E(X_t^2) = 1$. We write the spectral representation of $\{X_t\}$ as

$$X_t = \int_{-\pi}^{\pi} e^{it\lambda}dz(\lambda). \qquad (25)$$

We suppose that all the values X_t are known, except for the value X_0. As we saw in Section 2 the response function $h(\lambda)$ of the best linear interpolator for X_0 is given by

$$h_g(\lambda) = 1 - g(\lambda)^{-1}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\lambda)^{-1}d\lambda\right\}^{-1}.$$

To interpolate X_0^2 we use a naive interpolator $\hat{X}_0^2$ where

$$\hat{X}_0 = \int_{-\pi}^{\pi} h_g(\lambda)dz(\lambda). \qquad (26)$$

Then the interpolation error of $\hat{X}_0^2$ for X_0^2 is obtained

Proposition 3.

$$\begin{aligned} &E[\{X_0^2 - \hat{X}_0^2\}^2] \\ &\quad = 8\pi\left\{\int_{-\pi}^{\pi} g(\lambda)d\lambda\right\}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\lambda)^{-1}d\lambda\right\}^{-1} - 4\pi^2\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\lambda)^{-1}d\lambda\right\}^{-2} \\ &\quad \equiv IE_1, \text{(say)}. \qquad (27) \end{aligned}$$

Next we consider the bias adjusted interpolation. Solving $E[\hat{X}_0^2] - b = 0$ with respect to b, we have

$$\begin{aligned} b &= E\left[\int_{-\pi}^{\pi} h_g(\lambda)dz(\lambda)\overline{\int_{-\pi}^{\pi} h_g(\mu)dz(\mu)}\right] = \int_{-\pi}^{\pi} h_g(\lambda)^2 g(\lambda)d\lambda \\ &= \int_{-\pi}^{\pi} g(\lambda)d\lambda - 2\pi\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\lambda)^{-1}d\lambda\right\}^{-1}. \end{aligned} \tag{28}$$

Then the interpolation error is given by

Proposition 4.

$$E[\{X_0^2 - 1 - \hat{X}_0^2 + b\}^2] = IE_1 - (1-b)^2 \equiv IE_2, (\text{say}). \tag{29}$$

From the above result we observe that the bias adjusted interpolator is better than the naive interpolator.

In most of natural phenomena, the true spectral density $g(\lambda)$ is not known a priori. If an interpolator is computed on the basis of a conjectured spectral density $f(\lambda)$ instead of the true one, then the response function of the interpolator based on $f(\lambda)$ is given by

$$h_f(\lambda) = 1 - f(\lambda)^{-1}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\lambda)^{-1}d\lambda\right\}^{-1}. \tag{30}$$

Letting

$$\hat{X}_0 = \int_{-\pi}^{\pi} h_f(\lambda)dz(\lambda), \tag{31}$$

we can evaluate the interpolation error of $\hat{X}_0^2$. As in (27), it is seen that the interpolation error of $\hat{X}_0^2$ for X_0^2 is evaluated as follows.

Proposition 5.

$$\begin{aligned} E[\{X_0^2 - \hat{X}_0^2\}^2] &= \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\lambda)^{-1}d\lambda\right\}^{-2}\left[8\left\{\int_{-\pi}^{\pi}\frac{g(\lambda)}{f(\lambda)}d\lambda\right\}^2\right. \\ &\quad - 12\left\{\int_{-\pi}^{\pi}\frac{g(\lambda)}{f(\lambda)}d\lambda\right\}\left\{\int_{-\pi}^{\pi}\frac{g(\lambda)}{f(\lambda)^2}d\lambda\right\}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\lambda)^{-1}d\lambda\right\}^{-1} \\ &\quad + 3\left\{\int_{-\pi}^{\pi}\frac{g(\lambda)}{f(\lambda)^2}d\lambda\right\}^2\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\lambda)^{-1}d\lambda\right\}^{-2} \\ &\quad \left. +4\left\{\int_{-\pi}^{\pi} g(\lambda)d\lambda\right\}\left\{\int_{-\pi}^{\pi}\frac{g(\lambda)}{f(\lambda)^2}d\lambda\right\}\right] \equiv MIE_3, (\text{say}). \end{aligned} \tag{32}$$

Similarly we can discuss the bias adjusted misspecified interpolation problem. Evaluating $b = E[\hat{X}_0^2]$ we obtain

$$\begin{aligned} b &= \int_{-\pi}^{\pi} g(\lambda)d\lambda - 2\left\{\int_{-\pi}^{\pi}\frac{g(\lambda)}{f(\lambda)}d\lambda\right\}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\lambda)d\lambda\right\}^{-1} \\ &\quad + \left\{\int_{-\pi}^{\pi}\frac{g(\lambda)}{f(\lambda)^2}d\lambda\right\}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\lambda)d\lambda\right\}^{-2}. \end{aligned} \tag{33}$$

Then we have

Proposition 6.

$$E[\{X_0^2 - 1 - \hat{X}_0^2 + b\}^2] = MIE_3 - (1-b)^2 \equiv MIE_4, (\text{say}). \tag{34}$$

That is, the bias adjusted interpolator is better than the naive interpolator.

Let $\{X_t\}$ be a stationary Gaussian process with $E(X_t) = 0$ and $EX_t^2 = 1$. We now turn our attention to an interpolation problem for Hermite-transformed process, $Y_t = H_q(X_t)$. We write the spectral representation of Y_t as

$$Y_t = \int_{-\pi}^{\pi} e^{it\lambda} d\omega(\lambda).$$

It is known that the spectral density of $\{Y_t\}$ is given by $p(\lambda) = q!g^{*q}(\lambda)$, where g^{*q} is the q-fold convolution of $g(\lambda)$ (see Hannan (1970, p.83)). We suppose that all values of Y_t are known, except for the value Y_0. Then the response function $h_g(\lambda)$ of the best linear interpolator based on $Y_t, t \neq 0$ for Y_0 is given by

$$h_g(\lambda) = 1 - p(\lambda)^{-1} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} p(\lambda)^{-1} d\lambda \right\}^{-1}. \tag{35}$$

We can write the best linear interpolator of Y_0 as

$$\hat{Y}_0 = \int_{-\pi}^{\pi} h_g(\lambda) d\omega(\lambda). \tag{36}$$

Then the interpolation error is

$$\begin{aligned} E|Y_0 - \hat{Y}_0|^2 &= E\left| \int_{-\pi}^{\pi} d\omega(\lambda) - \int_{-\pi}^{\pi} h_g(\lambda) d\omega(\lambda) \right|^2 \\ &= 2\pi \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} p(\lambda)^{-1} d\lambda \right]^{-1}. \end{aligned} \tag{37}$$

To compare interpolation error of (29) with that of (37), we suppose that $\{X_t\}$ is generated by the first order autoregressive process

$$X_t = \theta X_{t-1} + \epsilon_t,$$

where $|\theta| < 1$ and $\epsilon \sim$ i.i.d $N(0, 1-\theta^2)$. For $q = 2$ in (37), the spectral density $p(\lambda)$ of Hermite-transformed process Y_t is given by

$$2 \int_{-\pi}^{\pi} f(\lambda - x) f(x) dx = \frac{1}{\pi} \left[\frac{(1-\theta^2)(1+\theta^2)}{(1-\theta^2 e^{-i\lambda})(1-\theta^2 e^{i\lambda})} \right].$$

Calculating the integral for the above two cases, we note that for $q = 2$, the interpolation error of Hermite-transformed process is equal to that of bias-adjusted square-transformed process.

We next consider the misspecified interpolation problem for Hermite-transformed process with true spectral density $p(\lambda) = q!g^{*q}(\lambda)$. Denoting a conjectured spectral density for $p(\lambda)$ by

$$j(\lambda) = \frac{1}{2\pi} |a(e^{-i\lambda})|^2,$$

it is easy to see that the response function $h_j(\lambda)$ of the interpolator based on $j(\lambda)$ is given by

$$h_j(\lambda) = 1 - j(\lambda)^{-1}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} j(\lambda)^{-1} d\lambda\right\}^{-1}. \tag{38}$$

Based on $j(\lambda)$ we can construct the best linear interpolator of Y_0 by

$$\hat{Y}_0 = \int_{-\pi}^{\pi} h_j(\lambda) d\omega(\lambda). \tag{39}$$

Then the interpolation error is

$$E|Y_0 - \hat{Y}_0|^2 = \left\{\int_{-\pi}^{\pi} \frac{p(\lambda)}{j(\lambda)^2} d\lambda\right\}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} j(\lambda)^{-1} d\lambda\right\}^{-2}. \tag{40}$$

Let $\{X_t\}$ be a stationary Gaussian process with $E(X_t) = 0$, $EX_t^2 = 1$, and spectral density

$$g(\lambda) = \frac{1}{2\pi}|c(e^{-i\lambda})|^2.$$

We next investigate an interpolation problem for Hermite-transformed process, $H_q(X_t)$. Then the response function $h_g(\lambda)$ of the best linear interpolator based on X_t, $t \neq 0$ for X_0 is given by

$$h_g(\lambda) = 1 - g(\lambda)^{-1}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\lambda)^{-1} d\lambda\right\}^{-1}. \tag{41}$$

To interpolate $H_q(X_0)$ we use a naive interpolator $H_q(\hat{X}_0)$, where

$$\hat{X}_0 = \int_{-\pi}^{\pi} h_g(\lambda) dz(\lambda).$$

Here, letting $\tilde{X}_0 = B\hat{X}_0$, we assume $E[(B\hat{X}_0)^2] = 1$. Hence

$$B = \left[1 - 2\pi\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\lambda)^{-1} d\lambda\right\}^{-1}\right]^{-\frac{1}{2}}. \tag{42}$$

Then we can provide the interpolation error of $H_q(\tilde{X}_0)$ for $H_q(X_0)$.

Proposition 7.

$$E[\{H_q(X_0) - H_q(\tilde{X}_0)\}^2] = 2q!\left[1 - \left\{1 - 2\pi\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\lambda)^{-1} d\lambda\right)^{-1}\right\}^{\frac{q}{2}}\right]. \tag{43}$$

Furthermore, we may discuss the misspecified interpolation problem. The response function of the interpolator based on $f(\lambda)$ is given by

$$h_f(\lambda) = 1 - f(\lambda)^{-1}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\lambda)^{-1} d\lambda\right\}^{-1}. \tag{44}$$

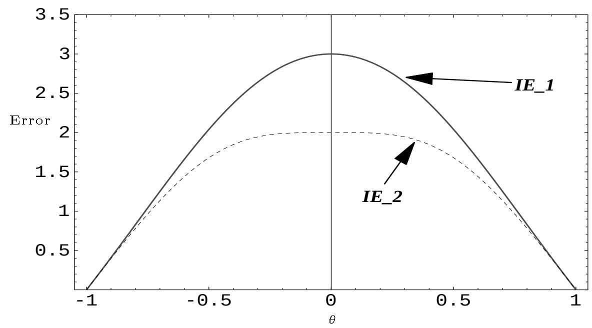

Figure 3. The plot of IE_1 and IE_2 for $g(\lambda)=\frac{1-\theta^2}{2\pi}\frac{1}{|1-\theta e^{-i\lambda}|^2}$, $-1<\theta<1$.

Similarly we have

$$\begin{aligned} B \;=\; & \left[1-2\left\{\int_{-\pi}^{\pi}\frac{g(\lambda)}{f(\lambda)}d\lambda\right\}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\lambda)^{-1}d\lambda\right\}^{-1}\right. \\ & \left.+\left\{\int_{-\pi}^{\pi}\frac{g(\lambda)}{f(\lambda)^2}d\lambda\right\}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\lambda)^{-1}d\lambda\right\}^{-2}\right]^{-\frac{1}{2}}. \end{aligned} \tag{45}$$

Then the following proposition is obtained.

Proposition 8.

$$\begin{aligned} & E[\{H_q(X_0)-H_q(\tilde{X}_0)\}^2] \\ & = 2q!\left[1-B^q\left\{1-\left(\int_{-\pi}^{\pi}\frac{g(\lambda)}{f(\lambda)}d\lambda\right)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\lambda)^{-1}d\lambda\right)^{-1}\right\}^q\right]. \end{aligned} \tag{46}$$

4 Numerical Studies. In this section, we illustrate the previous theoretical results numerically. In Propositions 3 and 4, let

$$g(\lambda)=\frac{1-\theta^2}{2\pi}\frac{1}{|1-\theta e^{-i\lambda}|^2},\quad |\theta|<1.$$

Then, we obtain

$$\begin{aligned} \mathrm{IE}_1 &= \frac{4(1-\theta^2)}{1+\theta^2}-\frac{(1-\theta^2)^2}{(1+\theta^2)^2}, \\ \mathrm{IE}_2 &= \frac{4(1-\theta^2)}{1+\theta^2}-\frac{2(1-\theta^2)^2}{(1+\theta^2)^2}. \end{aligned}$$

In Figure 3 we plotted IE_1 and IE_2 for $-1<\theta<1$, respectively.

Next we examine effect of misspecification of spectra for the interpolation in a transformed process. In Propositions 5 and 6, let

$$\begin{aligned} g(\lambda) &= \frac{1-\theta^2}{2\pi}\frac{1}{|1-\theta e^{-i\lambda}|^2} \quad -1<\theta<1, \qquad \mathrm{AR}(1), \\ f(\lambda) &= \frac{1-\theta^2}{2\pi}|1-\theta e^{-i\lambda}|^2 \quad -1<\theta<1, \qquad \mathrm{MA}(1). \end{aligned}$$

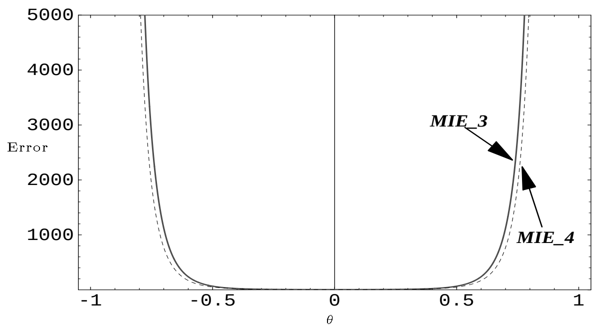

Figure 4. The plot of MIE_3 and MIE_4 for $f(\lambda) = \frac{1-\theta^2}{2\pi}|1-\theta e^{-i\lambda}|^2$, $g(\lambda) = \frac{1-\theta^2}{2\pi}\frac{1}{|1-\theta e^{-i\lambda}|^2}$, $-1 < \theta < 1$.

Then using the residual theorem we get

$$\begin{aligned} \mathrm{MIE}_3 &= \frac{4(1+\theta^2+38\theta^4+52\theta^6+13\theta^8)}{(\theta^2-1)^4}, \\ \mathrm{MIE}_4 &= \frac{4(1+\theta^2+22\theta^4+36\theta^6+9\theta^8)}{(\theta^2-1)^4}. \end{aligned}$$

In Figure 4 we plotted MIE_3 and MIE_4 for $-1 < \theta < 1$, respectively. Then it is seen that if $|\theta| \nearrow 1$, they become very large. This implies that we have to be careful about the misspecification of spectra for the unit root case.

Next we actually interpolate missing data or values of the series that, for some reasons, can not be observed completely or measured accurately.

(1) We consider the following the first order autoregressive process (AR(1))

$$X_t = \theta X_{t-1} + \epsilon_t, \quad |\theta| < 1,$$

where $\epsilon_t \sim$ i.i.d. $N(0,1)$. The true spectral density is given by $g(\lambda) = \frac{1}{2\pi}|1-\theta e^{-i\lambda}|^{-2}$. Then, from (3), it is easily seen that

$$\hat{X}_0 = \frac{\theta}{1+\theta^2}(X_{-1}+X_1).$$

Moreover, denote the interpolation bounds for values of interpolators by σ, 2σ, 3σ, respectively, where from (4),

$$\sigma = \sqrt{2\pi\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\lambda)^{-1}d\lambda\right]^{-1}} = \sqrt{\frac{1}{1+\theta^2}}.$$

We generate $X_1, \ldots, X_{100}$ from the above process. Suppose that X_{25}, X_{50}, and X_{85} are missing.

In Figures 5 and 6 we plotted the values of interpolator for X_{25}, X_{50}, and X_{85}, and also gave interpolation bounds for them in the cases when $\theta = 0.1$ and $\theta = 0.6$, respectively. Figures 5 and 6 show that the values of interpolator are off the interpolation bound, σ, as θ goes to 1.

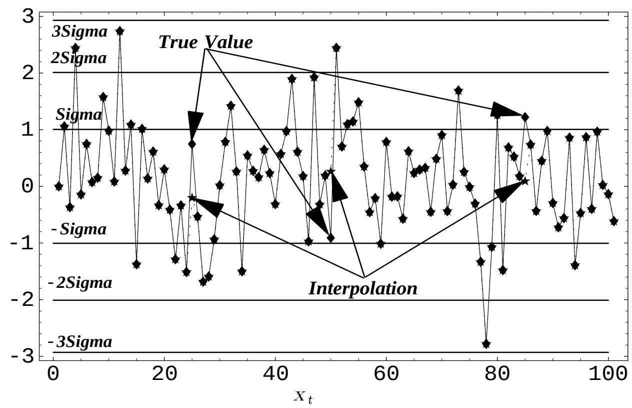

Figure 5. The plot of interpolation for $\theta = 0.1$ in the cases when X_{25}, X_{50}, and X_{85} are missing.

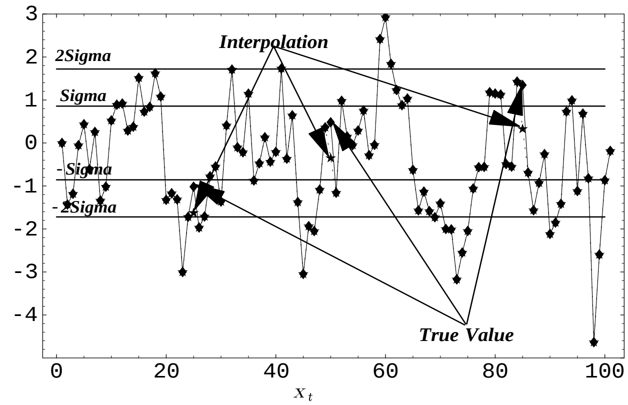

Figure 6. The plot of interpolation for $\theta = 0.6$ in the cases when X_{25}, X_{50}, and X_{85} are missing.

(2) In order to examine effect of misspecification for the interpolation, we now consider the following the first order moving average process (MA(1))

$$X_t = \epsilon_t + \theta\epsilon_{t-1}, \quad |\theta| < 1, \quad (\epsilon_t \sim \text{i.i.d.} N(0.1)),$$

hence the true spectral density is $g(\lambda) = \frac{1}{2\pi}|1 - \theta e^{-i\lambda}|^2$.

Suppose that an interpolator is constructed by the spectral density $f(\lambda)$ of AR(1) process although the true one is $g(\lambda)$. Then the interpolator is

$$\hat{X}_0 = \frac{\theta}{1+\theta^2}(X_{-1} + X_1).$$

We generate $X_1, \dots, X_{100}$ from the above MA(1) process. Suppose that X_{25}, X_{50}, and X_{85} are missing.

In Figures 7 and 8 we plotted the values of interpolator for X_{25}, X_{50}, and X_{85} in the cases when $\theta = 0.1$ and $\theta = 0.9$, respectively. Figures 7 and 8 show that if θ goes to 1, the error between true values and interpolator values is very large. Thus, this implies that we have to be careful about the misspecification of spectra of interpolation.

To illustrate the ideas of the previous section, we investigate the following examples.

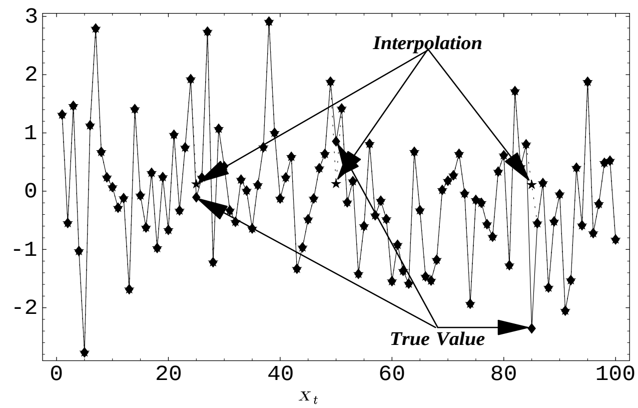

Figure 7. The plot of interpolation for $\theta = 0.1$ in the cases when X_{25}, X_{50}, and X_{85} are missing.

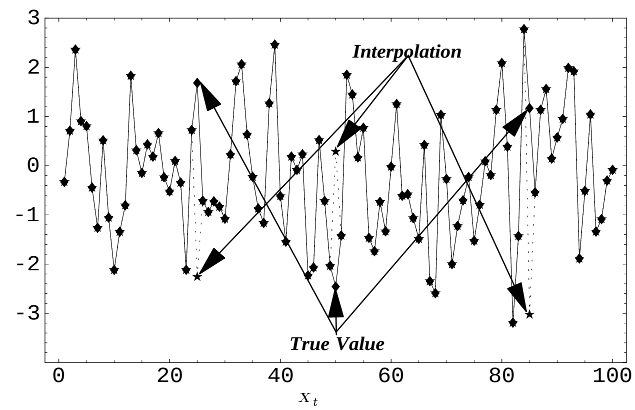

Figure 8. The plot of interpolation for $\theta = 0.9$ in the cases when X_{25}, X_{50}, and X_{85} are missing.

Example : *Monthly accidental death data*

The original data ($N = 72$) of Figure 9 is the number of the monthly accidental deaths in the United States for the period January 1973 to December 1979, as reported in Brockwell and Davis (1991). We use S-plus for Windows to make the whole calculations. To discuss the interpolation and prediction problems from the original data, we use only observations between January 1973 to December 1978 and assume that value of February 1976 is missing. Therefore the number of data is 59, i.e., $N = 59$. If we fit an AR(p) model to these data ($N = 59$) by using AIC criterion, the selected order p of the model is 15 and the estimated coefficients are obtained by Yule-Walker equations. Also it is not difficult to show that the best linear interpolator for AR(p) model is

$$\hat{X}_0 = -\left(\sum_{j=0}^{p} \theta_j^2\right)^{-1} \sum_{l=-p,\, l\neq 0}^{p} \left(\sum_{j=0}^{p-|l|} \theta_j \theta_{j+l}\right) X_l. \tag{47}$$

From (47) the value of February 1976 obtained from the interpolator for the fitted AR(15) model is given in Figure 9. Fitting an AR(p) model to the interpolated data set ($N = 60$) by AIC criterion, the selected order of the model is obtained 13. The predicted values

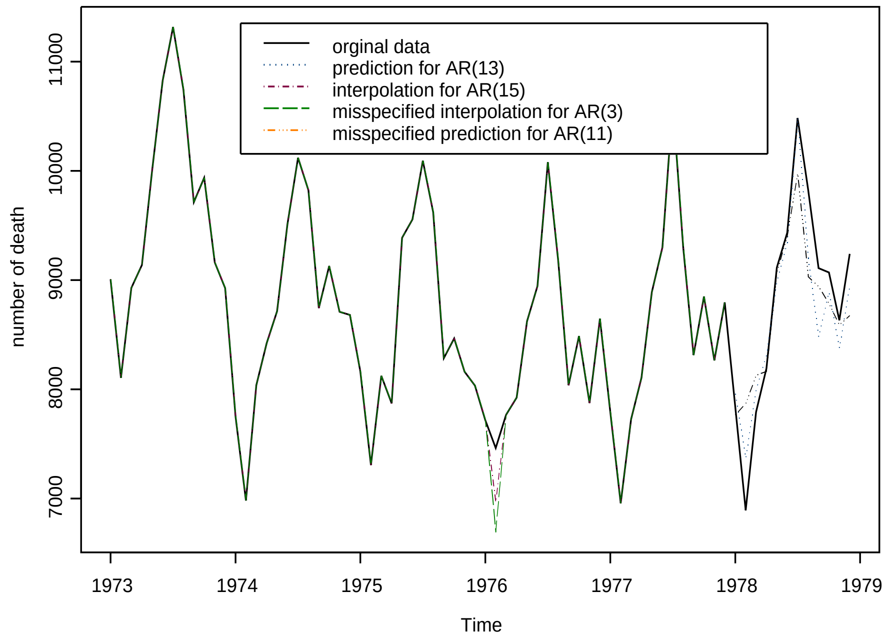

Figure 9. The plot of interpolation and prediction based on true model and incorrectly fitted model for the monthly accidental deaths data in the United States.

from January 1979 to December 1979 obtained by fitting an AR(13) model to the data are displayed in Figure 9. Furthermore, we give the values of the interpolation and the prediction obtained from incorrectly estimated AR(3) model and AR(11) model. Figure 9 also shows that the error of the values of interpolation and prediction in the case of incorrectly estimated order.

5 Proofs.

Proof of Proposition 1. Taking $\overline{E}$ of (8) we obtain

$$\begin{aligned}\overline{E}\{\mathrm{M}(\hat{\theta}_Q)\} &= \mathrm{M}(\underline{\theta}) + \frac{\partial}{\partial\theta'}\mathrm{M}(\underline{\theta})\frac{1}{\sqrt{T}}\overline{E}[\sqrt{T}(\hat{\theta}_Q - \underline{\theta})] \\ &\quad + \frac{1}{2T}\mathrm{tr}\left\{\frac{\partial^2}{\partial\theta\partial\theta'}\mathrm{M}(\underline{\theta})\overline{E}[T(\hat{\theta}_Q - \underline{\theta})(\hat{\theta}_Q - \underline{\theta})']\right\} + \text{lower order.} \qquad \text{(A.1)}\end{aligned}$$

The asymptotic distribution of $\sqrt{T}(\hat{\theta}_Q - \underline{\theta})$ is given by the following lemma.

Lemma 1 (Hosoya-Taniguchi (1982)). (i) $p - \lim_{T\to\infty}\hat{\theta}_Q = \underline{\theta}$,
(ii) $\sqrt{T}(\hat{\theta}_Q - \underline{\theta}) \xrightarrow{L} N(0, M_f^{-1}\tilde{V}M_f^{-1})$ *under* g.

Hence the result follows from (A.1) and Lemma 1.

Proof of Proposition 2. The proof follows from the same arguments as in the proof of Proposition 1.

Proof of Proposition 3. Using (25), (26) and fundamental formula for higher order moments, we obtain

$$
\begin{aligned}
&E[\{X_0^2 - \hat{X}_0^2\}^2] \\
&\quad = E[\{(X_0 + \hat{X}_0)(X_0 - \hat{X}_0)\}^2] \\
&\quad = 2[E\{(X_0 + \hat{X}_0)(X_0 - \hat{X}_0)\}]^2 + E\{(X_0 + \hat{X}_0)^2\}E\{(X_0 - \hat{X}_0)^2\} \\
&\quad = 2\left[E\left\{\int_{-\pi}^{\pi}(1 + h_g(\lambda))dz(\lambda)\int_{-\pi}^{\pi}(1 - h_g(\mu))dz(\mu)\right\}\right]^2 \\
&\qquad + \left[E\left\{\int_{-\pi}^{\pi}(1 + h_g(\lambda))dz(\lambda)\int_{-\pi}^{\pi}(1 + h_g(\mu))dz(\mu)\right\}\right. \\
&\qquad\quad \left.\times E\left\{\int_{-\pi}^{\pi}(1 - h_g(\lambda))dz(\lambda)\int_{-\pi}^{\pi}(1 - h_g(\mu))dz(\mu)\right\}\right] \\
&\quad = 2\left[\int_{-\pi}^{\pi}(1 + h_g(\lambda))(1 - h_g(\lambda))g(\lambda)d\lambda\right]^2 \\
&\qquad + \left\{\int_{-\pi}^{\pi}(1 + h_g(\lambda))^2 g(\lambda)d\lambda\right\}\left\{\int_{-\pi}^{\pi}(1 - h_g(\lambda))^2 g(\lambda)d\lambda\right\} \\
&\quad = 8\pi\left\{\int_{-\pi}^{\pi} g(\lambda)d\lambda\right\}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\lambda)^{-1}d\lambda\right\}^{-1} - 4\pi^2\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\lambda)^{-1}d\lambda\right\}^{-2}.
\end{aligned}
$$

Proof of Proposition 4. Let b be the bias of the predictor given by (28). Then we have

$$
\begin{aligned}
E[\{X_0^2 - 1 - \hat{X}_0^2 + b\}^2] &= E[\{X_0^2 - \hat{X}_0^2 - (1 - b)\}^2] \\
&= E\{X_0^2 - \hat{X}_0^2\}^2 - 2(1 - b)E\{X_0^2 - \hat{X}_0^2\} + (1 - b)^2 \\
&= IE_1 - (1 - b)^2,
\end{aligned}
$$

because $E\{X_0^2 - \hat{X}_0^2\} = 1 - b$.

Proof of Proposition 5. Using (31), we can prove the assertion similarly as in Proposition 3.

Proof of Proposition 6. The proof follows from the same arguments as in the proof of Proposition 4.

Proof of Proposition 7. Since the Hermite polynomials constitute an orthogonal system with respect to the standard normal p.d.f., we see that $H_q(X_0)$ and $H_q(\tilde{X}_0)$ have mean 0 and variance $q!$, respectively. Noting $\tilde{X}_0 = B\hat{X}_0$ and $\int_{-\pi}^{\pi} g(\lambda)d\lambda = 1$, we have

$$
\begin{aligned}
&E[\{H_q(X_0) - H_q(\tilde{X}_0)\}^2] \\
&\quad = E[H_q(X_0)^2] - 2E[H_q(X_0)H_q(\tilde{X}_0)] + E[H_q(\tilde{X}_0)^2] \\
&\quad = q! - 2q!\mathrm{cov}(X_0, \tilde{X}_0)^q + q! \\
&\quad = 2q!\left[1 - \left\{B\,\mathrm{cov}(X_0, \hat{X}_0)\right\}^q\right] \\
&\quad = 2q!\left[1 - \left\{B\int_{-\pi}^{\pi} h_g(\lambda)g(\lambda)d\lambda\right\}\right] \\
&\quad = 2q!\left[1 - \left\{B\int_{-\pi}^{\pi}\left\{g(\lambda) - \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\lambda)^{-1}d\lambda\right)^{-1}\right\}d\lambda\right\}^q\right] \\
&\quad = 2q!(1 - B^{-q}).
\end{aligned}
$$

Proof of Proposition 8. Noting (44) and (45), we have

$$\begin{aligned} &E[\{H_q(X_0) - H_q(\tilde{X}_0)\}^2] \\ &\quad = q! - 2q![B\operatorname{cov}(X_0, \hat{X}_0)]^q + q! \\ &\quad = 2q!\left[1 - B^q\left\{1 - \left(\int_{-\pi}^{\pi} \frac{g(\lambda)}{f(\lambda)} d\lambda\right)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\lambda)^{-1} d\lambda\right)^{-1}\right\}^q\right]. \end{aligned}$$

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