ON ESSENTIALLY DEFINITE, AND SEMIDEFINITE TUPLES OF OPERATORS

Dedicated to Professor Masahiro Nakamura with respect and affection

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Abstract. Let $\mathcal{B}(\mathcal{H})^n$ be the space of n-tuples of operators in $\mathcal{B}(\mathcal{H})$, the algebra of bounded operators on a Hilbert space $\mathcal{H}$. Given a set $\mathcal{G}$ of maps from $\mathcal{B}(\mathcal{H})^n$ into itself, which is appropriate for our situation, a tuple $T \in \mathcal{B}(\mathcal{H})^n$ is said to be $\mathcal{G}$-definite (resp., $\mathcal{G}$-semidefinite) if for any $\phi \in \mathcal{G}$, each term of $\phi(T)$ is zero (resp., positive), and essentially $\mathcal{G}$-definite (resp., essentially $\mathcal{G}$-semidefinite) if for any $\phi \in \mathcal{G}$, the image of each term of $\phi(T)$ by the canonical quotient map of $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H})/\mathcal{C}(\mathcal{H})$, $\mathcal{C}(\mathcal{H})$ the algebra of compact operators on $\mathcal{H}$, is zero (resp., positive). We will show that any essentially $\mathcal{G}$-definite tuple can be decomposed into a direct sum of a $\mathcal{G}$-definite tuple and *irreducible* essentially $\mathcal{G}$-definite, non $\mathcal{G}$-definite tuples, and that the parallel statement is also true for essentially $\mathcal{G}$-semidefinite tuples.

1. Introduction.

Let $\mathcal{H}$ be a separable Hilbert space. $\mathcal{B}(\mathcal{H})$ (resp., $\mathcal{C}(\mathcal{H})$) denotes the algebra of all bounded operators (resp., compact operators) on $\mathcal{H}$, π the quotient map of $\mathcal{B}(\mathcal{H})$ onto the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{C}(\mathcal{H})$. We say that $T \in \mathcal{B}(\mathcal{H})$ is *essentially normal* if it satisfies that

$$\pi(T^*T - TT^*) = O.$$

Behncke [1] proved that any essentially normal operator can be decomposed into a direct sum of a normal operator and irreducible essentially normal non normal operators.

Let $\mathcal{G}$ be a set of polynomials in two noncommuting variables. We say that $T \in \mathcal{B}(\mathcal{H})$ is *$\mathcal{G}$-definite* (resp., *$\mathcal{G}$-semidefinite*) if it satisfies that

$$g(T, T^*) = O \quad (\text{resp., } g(T, T^*) \geq O)$$

for any $g \in \mathcal{G}$. It is observed in Fujii-Kajiwara-Kato-Kubo [4] that any $T \in \mathcal{B}(\mathcal{H})$ has a maximal subspace $\mathcal{M}$ which reduces T such that the restriction $T|_{\mathcal{M}}$ of T on $\mathcal{M}$ is $\mathcal{G}$-definite (resp., $\mathcal{G}$-semidefinite).

So, the question naturally arises: if T is *essentially $\mathcal{G}$-definite* (resp., *$\mathcal{G}$-semidefinite*), namely, T satisfies that

$$\pi(g(T, T^*)) = O \quad (\text{resp., } \pi(g(T, T^*)) \geq O)$$

for any $g \in \mathcal{G}$, does the restriction $T|_{\mathcal{M}^\perp}$ of T on $\mathcal{M}^\perp$ admit a decomposition into a direct sum of irreducible essentially $\mathcal{G}$-definite (resp., $\mathcal{G}$-semidefinite), non $\mathcal{G}$-definite (resp., non $\mathcal{G}$-semidefinite) operators. In the present paper we will concern with this problem and give an affirmative answer, cf. Brown-Fong-Hadwin [2].

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But we rather like to put ourselves in a more general situation instead, dealing with sets of maps beyond polynomial calculi, and, with tuples of operators.

2. Essentially $\mathcal{G}$-definite, and $\mathcal{G}$-semidefinite tuples.

Let $\mathcal{B}(\mathcal{H})^n$ be the set of all n-tuples $T = (T_1, T_2, \ldots, T_n)$ of operators $T_1, T_2, \ldots, T_n \in \mathcal{B}(\mathcal{H})$. It may be considered as a C^*-algebra under the term-wise defined linear operations, multiplication, involution, and the norm

$$||T|| = \max\{||T_1||, ||T_2||, \ldots, ||T_n||\}, \quad T = (T_1, T_2, \ldots, T_n).$$

Let $\mathcal{E}$ be the set of all maps from $\mathcal{B}(\mathcal{H})^n$ into itself. $\mathcal{E}$ is an algebra under the linear operations and multiplication defined in the usual manner. We further introduce to $\mathcal{E}$ the pointwise norm topology. We may consider $\mathcal{E}$ as the product space $\prod_{T \in \mathcal{B}(\mathcal{H})^n} X_T$ of X_T's, where $X_T = \mathcal{B}(\mathcal{H})^n$ for any $T \in \mathcal{B}(\mathcal{H})^n$, then the pointwise norm topology for $\mathcal{E}$ coincides with the Tihonov topology for $\prod_{T \in \mathcal{B}(\mathcal{H})^n} X_T$, where each X_T is equipped with the norm topology introduced above.

$\mathcal{E}$ is a topological algebra, in the sense that $\mathcal{E}$ is a locally convex topological linear space and the multiplication is a continuous map of each of the factors for a fixed second factor.

For given n-tuples $A = (A_1, A_2, \ldots, A_n)$, $B = (B_1, B_2, \ldots, B_n) \in \mathcal{B}(\mathcal{H})^n$, we put

$$\lambda_{A,B}(T) = (A_1T_1B_1, A_2T_2B_2, \ldots, A_nT_nB_n)$$

for $T = (T_1, T_2, \ldots, T_n) \in \mathcal{B}(\mathcal{H})^n$, and for a given n-tuple $p = (p_1, p_2, \ldots, p_n)$ of polynomials in $2n$ noncommuting variables, we put

$$\begin{aligned} &\psi_p(T) \\ &= (p_1(T_1, \ldots, T_n, T_1^*, \ldots, T_n^*), p_2(T_1, \ldots, T_n, T_1^*, \ldots, T_n^*), \ldots, p_n(T_1, \ldots, T_n, T_1^*, \ldots, T_n^*)) \end{aligned}$$

for $T = (T_1, T_2, \ldots, T_n) \in \mathcal{B}(\mathcal{H})^n$.

We say that a subspace (which means closed one throughout) $\mathcal{M}$ of $\mathcal{H}$ reduces a tuple $T = (T_1, T_2, \ldots, T_n) \in \mathcal{B}(\mathcal{H})^n$ if $\mathcal{M}$ reduces each of the terms $T_1, T_2, \ldots, T_n$ of T. In the case, we have the restricted tuple $T|_\mathcal{M} = (T_1|_\mathcal{M}, T_2|_\mathcal{M}, \ldots, T_n|_\mathcal{M})$. If $\phi = \lambda_{A,B}$ and $\mathcal{M}$ reduces tuples A, B, then we have a map $\phi_\mathcal{M} = \lambda_{A|_\mathcal{M}, B|_\mathcal{M}}$ from $\mathcal{B}(\mathcal{M})^n$ into itself; if $\psi = \psi_p$, then we have a map $\psi_\mathcal{M} = \psi_p|_{\mathcal{B}(\mathcal{M})^n}$ from $\mathcal{B}(\mathcal{M})^n$ into itself.

Let $\mathcal{S}$ be a subset of $\mathcal{B}(\mathcal{H})$ which contains the identity operator I on $\mathcal{H}$, and $\mathcal{S}^n$ the set of all n-tuples $T = (T_1, T_2, \ldots, T_n)$ with $T_1, T_2, \ldots, T_n \in \mathcal{S}$. We denote by $\mathcal{E}_\mathcal{S}$ the pointwise norm closed subalgebra generated by the maps $\lambda_{A,B}$, $A, B \in \mathcal{S}^n$, and ψ_p, $p \in \mathcal{P}^n$, where $\mathcal{P}^n$ denotes the set of all n-tuples of polynomials in $2n$ noncommuting variables.

We call the n-tuple $O = (O, O, \ldots, O)$ the *zero* tuple, and say that an n-tuple $T = (T_1, T_2, \ldots, T_n)$ is *positive* if $T_j \geq O$ for $j = 1, 2, \ldots, n$; we say that an n-tuple $T = (T_1, T_2, \ldots, T_n)$ is *essentially zero* (resp., *essentially positive*) if $\pi(T_j) = O$ (resp., $\pi(T_j) \geq O$) for $j = 1, 2, \ldots, n$.

Now let $\mathcal{G}$ be a subset of $\mathcal{E}_\mathcal{S}$. We say that a tuple T is *$\mathcal{G}$-definite* (resp., *$\mathcal{G}$-semidefinite*) if for any $\phi \in \mathcal{G}$ the tuple $\phi(T)$ is zero (resp., positive), and *essentially $\mathcal{G}$-definite* (resp., *essentially $\mathcal{G}$-semidefinite*) if for any $\phi \in \mathcal{G}$ the tuple $\phi(T)$ is essentially zero (resp., essentially positive).

If a subspace $\mathcal{M}$ of $\mathcal{H}$ reduces any operator in $\mathcal{S}$, then $\phi \in \mathcal{G}$ induces the map $\phi_{\mathcal{M}}$ from $\mathcal{B}(\mathcal{M})^n$ into itself by the familiar procedure. We put $\mathcal{G}_{\mathcal{M}} = \{\phi_{\mathcal{M}} : \ \phi \in \mathcal{G}\}$. If $\mathcal{M}$ reduces a tuple T, then it makes sense to ask whether the restriction $T|_{\mathcal{M}}$ is $\mathcal{G}_{\mathcal{M}}$-definite (resp., $\mathcal{G}_{\mathcal{M}}$-semidefinite) or not, and essentially $\mathcal{G}_{\mathcal{M}}$-definite (resp., $\mathcal{G}_{\mathcal{M}}$-semidefinite) or not.

A key step to our main result is the following

Theorem 1. *Let $\mathcal{A}$ be a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ which contains the identity operator on $\mathcal{H}$, and $\mathcal{G}$ a subset of $\mathcal{A}$. If any operator in $\mathcal{G}$ is essentially zero (resp., essentially positive), then there exists an orthogonal family $\{\mathcal{H}_m : m \geq 0\}$ of subspaces of $\mathcal{H}$ which satisfies the following statements*:

(i) $\mathcal{H} = \bigoplus_{m\geq 0} \mathcal{H}_m$, *$\mathcal{H}_m$ reduces $\mathcal{A}$ if $m \geq 0$, and $\mathcal{A}|_{\mathcal{H}_m}$ is irreducible if $m \geq 1$.*

(ii) *$\mathcal{H}_0$ is the maximal subspace of $\mathcal{H}$ which reduces $\mathcal{A}$ such that $T|_{\mathcal{H}_0} = O$ (resp., $T|_{\mathcal{H}_0} \geq O$) for any $T \in \mathcal{G}$.*

Therefore, if $m \geq 1$, $T|_{\mathcal{H}_m}$ is essentially zero (resp., essentially positive) for all $T \in \mathcal{G}$, but $T|_{\mathcal{H}_m} \neq O$ (resp., $T|_{\mathcal{H}_m}$ is not positive) for some $T \in \mathcal{G}$.

Proof. It suffices to prove the case where any operator in $\mathcal{G}$ is essentially positive, because any operator in $\mathcal{G}$ is essentially zero if and only if any operator in $\mathcal{G} \cup -\mathcal{G}$ is essentially positive, where $-\mathcal{G} = \{-T : T \in \mathcal{G}\}$.

Put $\mathcal{J} = \mathcal{A} \cap \mathcal{C}(\mathcal{H})$. Since

$$\pi(T) = |\pi(T)| = \pi(|T|),$$

we have that $T - |T| \in \mathcal{J}$ for any $T \in \mathcal{G}$. We may assume that $T - |T| \neq O$ for some $T \in \mathcal{G}$, and so, $\mathcal{J} \neq \{O\}$. Let $\mathcal{M}$ be the closure of $\mathcal{J}\mathcal{H} = \{Ax : A \in \mathcal{J}, x \in \mathcal{H}\}$, and put $\mathcal{R}_0 = \mathcal{M}^{\perp}$. Then both $\mathcal{M}$ and $\mathcal{R}_0$ reduce $\mathcal{A}$. Since $J|_{\mathcal{R}_0}$ is zero for any $J \in \mathcal{J}$, we have $\mathcal{J}|_{\mathcal{R}_0} = \{O\}$ which implies that

$$T|_{\mathcal{R}_0} = |T||_{\mathcal{R}_0} \geq O$$

for any $T \in \mathcal{G}$.

Consider next the restriction map ρ :

$$\rho(J) = J|_{\mathcal{M}} \quad \text{for} \quad J \in \mathcal{J}.$$

This is a nondegenerate representation of $\mathcal{J}$, and becomes a direct sum of irreducible representations. So, we have an orthogonal family $\{\mathcal{R}_k : k \geq 1\}$ of subspaces of $\mathcal{M}$ such that

$$\mathcal{M} = \bigoplus_{k\geq 1} \mathcal{R}_k,$$

$\mathcal{R}_k$ reduces $\mathcal{J}$, and, $\mathcal{J}|_{\mathcal{R}_k}$ is non zero and irreducible. Therefore $\mathcal{J}|_{\mathcal{R}_k} = \mathcal{C}(\mathcal{R}_k)$, which shows that $\mathcal{R}_k$ reduces $\mathcal{A}$. Hence we conclude that $\mathcal{A}|_{\mathcal{R}_k}$ is irreducible.

On the other hand, it occurs either that

$$T|_{\mathcal{R}_k} = |T||_{\mathcal{R}_k} \text{ for any } T \in \mathcal{G} \tag{1}$$

or that

$$T|_{\mathcal{R}_k} \neq |T||_{\mathcal{R}_k} \text{ for some } T \in \mathcal{G}. \tag{2}$$

Now denote the direct sum of $\mathcal{R}_0$ and all $\mathcal{R}_k$ which satisfy (1) by $\mathcal{H}_0$, and denote $\mathcal{R}_k$ which satisfy (2) by $\mathcal{H}_m$, $m \geq 1$, giving them new letters with new indices. Then, we have an orthogonal family $\{\mathcal{H}_m : m \geq 0\}$ of subspaces of $\mathcal{H}$ which satisfies (i) and (ii) stated in Theorem 1. □

Now we state the aimed

Theorem 2. *Let $\mathcal{G}$ be a subset of $\mathcal{E}_{\mathcal{S}}$. If a tuple $T = (T_1, T_2, \ldots, T_n) \in \mathcal{B}(\mathcal{H})^n$ is essentially $\mathcal{G}$-definite (resp., $\mathcal{G}$-semidefinite), then there exists an orthogonal family $\{\mathcal{H}_m : m \geq 0\}$ of subspaces of $\mathcal{H}$ which satisfies the following statements:*

(i) $\mathcal{H} = \bigoplus_{m \geq 0} \mathcal{H}_m$, *$\mathcal{H}_m$ reduces T and each member of $\mathcal{S}$ if $m \geq 0$, and, there is no nontrivial subspace of $\mathcal{H}_m$ which reduces T and each member of $\mathcal{S}$ if $m \geq 1$.*

(ii) *$\mathcal{H}_0$ is the maximal subspace which reduces T and each member of $\mathcal{S}$ such that $T|_{\mathcal{H}_0}$ is $\mathcal{G}_{\mathcal{H}_0}$-definite (resp., $\mathcal{G}_{\mathcal{H}_0}$-semidefinite).*

Therefore, if $m \geq 1$, $T|_{\mathcal{H}_m}$ is essentially $\mathcal{G}_{\mathcal{H}_m}$-definite (resp., $\mathcal{G}_{\mathcal{H}_m}$-semidefinite), but not $\mathcal{G}_{\mathcal{H}_m}$-definite (resp., $\mathcal{G}_{\mathcal{H}_m}$-semidefinite).

Proof. Let $\mathcal{A}$ be the C^*-subalgebra generated by $T_1, T_2, \ldots, T_n$ and $\mathcal{S}$ and the identity operator I on $\mathcal{H}$, and put

$$\tilde{\mathcal{G}} = \{\tau_j(\phi(T)) : \phi \in \mathcal{G} \text{ and } 1 \leq j \leq n\},$$

where τ_j $(1 \leq j \leq n)$ means the projection:

$$\tau_j(A) = A_j, \quad A = (A_1, A_2, \ldots, A_n).$$

We then apply Theorem 1 to $\mathcal{A}$ and $\tilde{\mathcal{G}}$ to obtain an orthogonal family $\{\mathcal{H}_m : m \geq 0\}$ of subspaces of $\mathcal{H}$ which satisfies (i) and (ii) in Theorem 2.

In fact, $\mathcal{H} = \bigoplus_{m \geq 0} \mathcal{H}_m$, $\mathcal{H}_m$ reduces $\mathcal{A}$ if $m \geq 0$, $\mathcal{A}|_{\mathcal{H}_m}$ is irreducible if $m \geq 1$, and $\mathcal{H}_0$ is the maximal subspace which reduces $\mathcal{A}$ such that, for any $\phi \in \mathcal{G}$,

$$\tau_j(\phi_{\mathcal{H}_0}(T|_{\mathcal{H}_0})) = O \text{ (resp.,} \tau_j(\phi_{\mathcal{H}_0}(T|_{\mathcal{H}_0})) \geq O), \quad 1 \leq j \leq n.$$

So $\mathcal{H}_0$ is the maximal subspace of $\mathcal{H}$ which reduces T and each member of $\mathcal{S}$ such that $T|_{\mathcal{H}_0}$ is $\mathcal{G}_{\mathcal{H}_0}$-definite (resp., $\mathcal{G}_{\mathcal{H}_0}$-semidefinite). □

3. Several examples.

We show that Theorem 2 is not vacuous, giving examples of an operator which is essentially $\mathcal{G}$-definite, but not $\mathcal{G}$-definite, and an operator which is essentially $\mathcal{G}$-semidefinite, but not $\mathcal{G}$-semidefinite. In fact, in Example 1 we have an irreducible, essentially normal, non hyponormal operator, and in Example 2 an essentially subnormal, non essentially normal, non hyponormal operator.

Example 1. The weighted shift T with weights $\lambda_0, \lambda_1, \lambda_2, \ldots$, where

$$\lambda_0 = 1, \quad \lambda_n = \left(1 - \sum_{k=1}^{n} 1/2^k\right)^{1/2} \quad \text{for } \ n \geq 1,$$

on the Hilbert space $\mathcal{H} = l^2$ is an essentially normal, non hyponormal operator.

In fact, $T^*T - TT^*$ turns out to be a diagonal operator with diagonal

$$1, -/2, -1/2^2, -1/2^3, \ldots.$$

So, $T^*T - TT^*$ is compact, but not positive.

Furthermore, since $\lambda_n \neq 0$ for all $n \geq 0$, it follows that T is irreducible.

Example 2. Let $\mathcal{H}$ be a countably infinite direct sum of copies of l^2. Define $X,\ K \in \mathcal{B}(\mathcal{H})$ by the operator matrices

$$X = \begin{pmatrix} O & & & \\ I & O & & \\ & I & \ddots & \\ & & \ddots & \end{pmatrix}, \quad K = \begin{pmatrix} O & & & \\ A & O & & \\ & O & \ddots & \\ & & \ddots & \end{pmatrix},$$

resp., where A is the diagonal operator on l^2 with diagonal $1, 1/2, 1/3, \ldots$. Then the operator $T = X + K$ is an essentially subnormal, non essentially normal, non hyponormal operator.

In fact, since X is subnornal, X satisfies that for any $A_1, A_2, \ldots, A_n (n \geq 1)$ in the C^*-algebra generated by X and I,

$$\sum_{j,\ k=0}^{n} A_j{}^* X^{*k} X^j A_k \geq O,$$

see [3]. So we have

$$\sum_{j,\ k=0}^{n} \pi(A_j)^* \pi(X)^{*k} \pi(X)^j \pi(A_k) = \pi\left(\sum_{j,\ k=0}^{n} A_j{}^* X^{*k} X^j A_k\right) \geq O.$$

But the image by π of the C^*-algebra $C^*(T)$ generated by T and I makes the C^*-algebra $C^*(\pi(T))$ generated by $\pi(T)$ and $\pi(I)$ full, it follows that T is essentially subnormal.

Meanwhile, since $X^*X - XX^*$ is a positive diagonal operator with diagonal $I, O, O, O, \ldots$, we have

$$\pi(T^*T - TT^*) = \pi(X^*X - XX^*) \neq O,$$

which implies that T is non essentially normal. Since $T^*T - TT^*$ is a diagonal operator with diagonal

$$(I + A)^2, I - (I + A)^2, O, O, \ldots,$$

$T^*T - TT^*$ is not positive, and hence T is non hyponormal.

Next we see the following

Example 3. We have pairs $A_1,\ B_1$ and $A_2,\ B_2$ of $n \times n$ self-adjoint matrices, by which each C^*-algebra generated is the algebra of all $n \times n$ matrices, the former of which satisfies $A_1 \geq B_1$, but on the contrary the latter fails to satisfy $A_2 \geq B_2$, e.g., for $n = 2$,

$$A_1 = B_2 = \begin{pmatrix} 3 & 0 \\ 0 & 3/2 \end{pmatrix}, \quad B_1 = A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Let us put

$$\mathcal{H}_0 = C^n \otimes l^2, \quad \mathcal{H}_1 = C^n \quad \text{and} \quad \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1;$$

and

$$T_1 = (A_1 \otimes I) \oplus A_2, \quad T_2 = (B_1 \otimes I) \oplus B_2,$$

I the identity operator on l^2. Then, the inequality $T_1 \geq T_2$ does not hold, but it holds essentially, that is, $T_1 - T_2$ is essentially positive. Furthermore, $\mathcal{H}_0$ and $\mathcal{H}_1$ reduces both T_1 and T_2, $\mathcal{H}_0$ is the maximal subspace which reduces both T_1 and T_2 such that the inequality $T_1|_{\mathcal{H}_0} \geq T_2|_{\mathcal{H}_0}$ holds, and there is no nontrivial subspace of $\mathcal{H}_1$ which reduces both A_2 and B_2.

It, however, may be seen as in the following corollary, that what we observed in Example 3 commonly occurs.

Corollary 3. *Let $T_1,\ T_2 \in \mathcal{B}(\mathcal{H})$ be self-adjoint. If $T_1 \geq T_2$ holds essentially, then there exists an orthogonal family $\{\mathcal{H}_m : m \geq 0\}$ of subspaces of $\mathcal{H}$ which satisfies the following statements*:

(i) $\mathcal{H} = \bigoplus_{m \geq 0} \mathcal{H}_m$, *$\mathcal{H}_m$ reduces both T_1 and T_2 if $m \geq 0$, and, there is no nontrivial subspace of $\mathcal{H}_m$ which reduces both T_1 and T_2 if $m \geq 1$.*

(ii) *$\mathcal{H}_0$ is the maximal subspace which reduces T_1 and T_2 such that $T_1|_{\mathcal{H}_0} \geq T_2|_{\mathcal{H}_0}$ holds.*

Therefore, if $m \geq 1$, $T_1|_{\mathcal{H}_m} \geq T_2|_{\mathcal{H}_m}$ holds essentially, but $T_1|_{\mathcal{H}_m} \geq T_2|_{\mathcal{H}_m}$ does not hold.

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