SOME TYPES OF POSITIVE IMPLICATIVE HYPER BCK-IDEALS

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Abstract. Several types of positive implicative hyper BCK-ideals in hyper BCK-algebras are considered, and then their relations are discussed.

1. Introduction. The study of BCK-algebras was initiated by K. Iséki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Since then a great deal of literature has been produced on the theory of BCK-algebras. In particular, emphasis seems to have been put on the ideal theory of BCK-algebras. The hyperstructure theory (called also multialgebras) was introduced in 1934 by F. Marty [9] at the 8th congress of Scandinavian Mathematiciens. Around the 40's, several authors worked on hypergroups, especially in France and in the United States, but also in Italy, Russia and Japan. Over the following decades, many important results appeared, but above all since the 70's onwards the most luxuriant flourishing of hyperstructures has been seen. Hyperstructures have many applications to several sectors of both pure and applied sciences. In [8], Y. B. Jun et al. applied the hyperstructures to BCK-algebras, and introduced the concept of a hyper BCK-algebra which is a generalization of a BCK-algebra, and investigated some related properties. They also introduced the notion of a hyper BCK-ideal and a weak hyper BCK-ideal, and gave relations between hyper BCK-ideals and weak hyper BCK-ideals. Y. B. Jun et al. [7] gave a condition for a hyper BCK-algebra to be a BCK-algebra, and introduced the notion of a strong hyper BCK-ideal and a reflexive hyper BCK-ideal. They showed that every strong hyper BCK-ideal is a hypersubalgebra, a weak hyper BCK-ideal and a hyper BCK-ideal; and every reflexive hyper BCK-ideal is a strong hyper BCK-ideal. Y. B. Jun and X. L. Xin [6] introduced the concept of weak positive implicative and positive implicative hyper BCK-algebras, and investigated some related properties. They gave a relation between a weak positive implicative hyper BCK-algebra and a positive implicative hyper BCK-algebra. They also introduced the notion of a positive implicative hyper BCK-ideal, and stated its characterizations. In this paper, we display several types of positive implicative hyper BCK-ideals in hyper BCK-algebras. That is, we introduce the notion of $PI(\ll,\subseteq,\subseteq)_{BCK}$-ideals, $PI(\subseteq,\subseteq,\subseteq)_{BCK}$-ideals $PI(\ll,\ll,\subseteq)_{BCK}$-ideals, and $PI(\ll,\ll,\ll)_{BCK}$-ideals, and then we show that (1) Every $PI(\subseteq,\subseteq,\subseteq)_{BCK}$-ideal is a weak hyper BCK-ideal, (2) Every $PI(\ll,\subseteq,\subseteq)_{BCK}$-ideal is a $PI(\subseteq,\subseteq,\subseteq)_{BCK}$-ideal, (3) Every $PI(\ll,\ll,\subseteq)_{BCK}$-ideal is both a $PI(\ll,\subseteq,\subseteq)_{BCK}$-ideal and a $PI(\ll,\ll,\ll)_{BCK}$-ideal, (4) Every $PI(\ll,\subseteq,\subseteq)_{BCK}$-ideal (or $PI(\ll,\ll,\subseteq)_{BCK}$-ideal) is a hyper BCK-ideal, and (5) Every closed $PI(\ll,\ll,\ll)_{BCK}$-ideal is a hyper BCK-ideal. Finally we provide a characterization of a $PI(\subseteq,\subseteq,\subseteq)_{BCK}$-ideal.

2. Preliminaries. We include some elementary aspects of hyper BCK-algebras that are necessary for this paper, and for more details we refer to [8] and [4]. Let H be a non-empty set endowed with a hyper operation "$\circ$", that is, $\circ$ is a function from $H\times H$ to $\mathcal{P}^*(H)=\mathcal{P}(H)\setminus\{\emptyset\}$. For two subsets A and B of H, denote by $A\circ B$ the set $\bigcup_{a\in A,b\in B} a\circ b$.

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By a *hyper BCK-algebra* we mean a nonempty set H endowed with a hyper operation "$\circ$" and a constant 0 satisfying the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,

(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,

(HK3) $x \circ H \ll \{x\}$,

(HK4) $x \ll y$ and $y \ll x$ imply $x = y$,

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$.

In any hyper BCK-algebra H, the following hold (see [8] and [4]):

(P1) $0 \circ 0 = \{0\}$,

(P2) $0 \ll x$,

(P3) $x \ll x$,

(P4) $A \ll A$,

(P5) $A \subseteq B$ implies $A \ll B$,

(P6) $0 \circ x = \{0\}$,

(P7) $0 \circ A = \{0\}$,

(P8) $A \ll \{0\}$ implies $A = \{0\}$,

(P9) $x \in x \circ 0$,

(P10) $x \circ 0 \ll \{y\}$ implies $x \ll y$,

(P11) $y \ll z$ implies $x \circ z \ll x \circ y$,

(P12) $x \circ y = \{0\}$ implies $(x \circ z) \circ (y \circ z) = \{0\}$ and $x \circ z \ll y \circ z$,

(P13) $A \circ \{0\} = \{0\}$ implies $A = \{0\}$,

(P14) If $(x \circ y) \circ z \ll A$, then $a \circ z \ll A$ for all $a \in x \circ y$,

(P15) If $a \circ b \subseteq A$ for all $a, b \in A$, then $0 \in A$,

(P16) $(A \circ B) \circ C = (A \circ C) \circ B$, $A \circ B \ll A$ and $0 \circ A \ll \{0\}$,

(P17) $x \circ 0 = \{x\}$ and $A \circ 0 = A$

for all $x, y, z \in H$ and for all nonempty subsetes A, B and C of H.

Proposition 2.1 (Jun et al. [8, Proposition 3.3]). *In a hyper BCK-algebra H, the condition* (HK3) *is equivalent to the condition:*

(HK3-1) $x \circ y \ll \{x\}$ *for all* $x, y \in H$.

Definition 2.2 (Jun et al. [8, Definition 3.14]). A nonempty subset I of a hyper BCK-algebra H is called a *hyper BCK-ideal* of H if it satisfies the following conditions:

(HId1) $0 \in I$,

(HId2) $x \circ y \ll I$ and $y \in I$ imply $x \in I$

for all $x, y \in H$.

Definition 2.3 (Jun et al. [8, Definition 3.19]). A nonempty subset I of a hyper BCK-algebra H is called a *weak hyper BCK-ideal* of H if it satisfies (HId1) and

(HId3) $x \circ y \subseteq I$ and $y \in I$ imply $x \in I$

for all $x, y \in H$.

Proposition 2.4 (Jun and Xin [4, Propotion 3.7]). Let A be a subset of a hyper BCK-algebra H. If I is a hyper BCK-ideal of H such that $A \ll I$, then A is contained in I.

3. Positive implicative hyper BCK-ideals. In what follows let H denote a hyper BCK-algebra unless otherwise specified.

Definition 3.1 (Jun and Xin [6, Definition 3.8]). A nonempty subset I of H is called a *positive implicative hyper BCK-ideal* of H (we say here that it is of type $(\ll, \subseteq, \subseteq)$, briefly $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal) if it satisfies (HId1) and

(HId4) $(x \circ y) \circ z \ll I$ and $y \circ z \subseteq I$ imply $x \circ z \subseteq I$
for all $x, y, z \in H$.

Definition 3.2. A nonempty subset I of H is called a *positive implicative hyper BCK-ideal* of type $(\subseteq, \subseteq, \subseteq)$ (briefly, $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal) if it satisfies (HId1) and
(HId5) $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$ imply $x \circ z \subseteq I$
for all $x, y, z \in H$.

Definition 3.3. A nonempty subset I of H is called a *positive implicative hyper BCK-ideal* of type $(\ll, \ll, \subseteq)$ (briefly, $PI(\ll, \ll, \subseteq)_{BCK}$-ideal) if it satisfies (HId1) and
(HId6) $(x \circ y) \circ z \ll I$ and $y \circ z \ll I$ imply $x \circ z \subseteq I$
for all $x, y, z \in H$.

Definition 3.4. A nonempty subset I of H is called a *positive implicative hyper BCK-ideal* of type $(\ll, \ll, \ll)$ (briefly, $PI(\ll, \ll, \ll)_{BCK}$-ideal) if it satisfies (HId1) and
(HId7) $(x \circ y) \circ z \ll I$ and $y \circ z \ll I$ imply $x \circ z \ll I$
for all $x, y, z \in H$.

Example 3.5. Let $H = \{0, a, b\}$ be a hyper BCK-algebra with the following Cayley table:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

Then $I_1 = \{0, a\}$ and $I_2 = \{0, b\}$ are $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideals of H.

Theorem 3.6. *Every $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal is a weak hyper BCK-ideal.*

Proof. Let I be a $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal of H and let $x, y \in H$ be such that $x \circ y \subseteq I$ and $y \in I$. Using (P17), we have $(x \circ y) \circ 0 = x \circ y \subseteq I$ and $y \circ 0 = \{y\} \subseteq I$. It follows from (HId5) and (P17) that $\{x\} = x \circ 0 \subseteq I$, that is, $x \in I$. Hence I is a weak hyper BCK-ideal of H. □

The following example shows that the converse of Theorem 3.6 may not be true.

Example 3.7. (1) Let $H = \{0, a, b, c\}$ be a hyper BCK-algebra with the following table:

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0\}$	$\{0\}$
c	$\{c\}$	$\{c\}$	$\{b\}$	$\{0\}$

Then $I = \{0, a\}$ is a (weak) hyper BCK-ideal but not $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal since $(c \circ b) \circ b \subseteq I$ and $b \circ b \subseteq I$ but $c \circ b \not\subseteq I$.

(2) Let $H = \{0, a, b, c\}$ be a hyper BCK-algebra with the following table:

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$	$\{a\}$
b	$\{b\}$	$\{a\}$	$\{0\}$	$\{b\}$
c	$\{c\}$	$\{c\}$	$\{c\}$	$\{0\}$

Then $I = \{0, c\}$ is a (weak) hyper BCK-ideal but not $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal since $(b \circ a) \circ a \subseteq I$ and $a \circ a \subseteq I$ but $b \circ a \nsubseteq I$.

Definition 3.8 (Jun and Xin [6]). A hyper BCK-algebra H is said to be *positive implicative* if it satisfies the equality $(x \circ y) \circ z = (x \circ z) \circ (y \circ z)$ for all $x, y, z \in H$.

Theorem 3.9. *In a positive implicative hyper BCK-algebra, the notion of a weak hyper BCK-ideal and a $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal coincide.*

Proof. It is sufficient to show that every weak hyper BCK-ideal is a $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal. Let I be a weak hyper BCK-ideal of a positive implicative hyper BCK-algebra H and let $x, y, z \in H$ be such that $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$. Since H is positive implicative, we have $(x \circ z) \circ (y \circ z) = (x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$. Hence $a \circ b \subseteq I$ and $b \in I$ for all $a \in x \circ z$ and $b \in y \circ z$. It follows from (HId3) that $a \in I$ so that $x \circ z \subseteq I$. Therefore I is a $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal of H, which completes the proof. □

Theorem 3.10. *Every $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal is a $PI(\subseteq, \subseteq, \subseteq)_{BCK}$-ideal.*

Proof. The proof is straightforward by using (P5). □

Theorem 3.11. *Every $PI(\ll, \ll, \subseteq)_{BCK}$-ideal is both a $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal and a $PI(\ll, \ll, \ll)_{BCK}$-ideal.*

Proof. Let I be a $PI(\ll, \ll, \subseteq)_{BCK}$-ideal of H. Using (P5), we know that I is a $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of H. Now let $x, y, z \in H$ be such that $(x \circ y) \circ z \ll I$ and $y \circ z \ll I$. Then $x \circ z \subseteq I$ and so $x \circ z \ll I$ by (P5). Hence I is a $PI(\ll, \ll, \ll)_{BCK}$-ideal of H. □

Theorem 3.12. *Every $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal (or $PI(\ll, \ll, \subseteq)_{BCK}$-ideal) is a hyper BCK-ideal.*

Proof. Let I be a $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideal of H and let $x, y \in H$ be such that $x \circ y \ll I$ and $y \in I$. Then $(x \circ y) \circ 0 = x \circ y \ll I$ and $y \circ 0 = \{y\} \subseteq I$ by (P17). It follows from (HId4) and (P17) that $\{x\} = x \circ 0 \subseteq I$, i.e., $x \in I$. Hence I is a hyper BCK-ideal of H. Now let I be a $PI(\ll, \ll, \subseteq)_{BCK}$-ideal of H and let $x, y \in H$ be such that $x \circ y \ll I$ and $y \in I$. Then $(x \circ y) \circ 0 = x \circ y \ll I$ and $y \circ 0 = \{y\} \ll I$ by using (P5) and (P17). It follows from (HId6) and (P17) that $\{x\} = x \circ 0 \subseteq I$ so that $x \in I$. Therefore I is a hyper BCK-ideal of H. □

The following example shows that there is a $PI(\ll, \ll, \ll)_{BCK}$-ideal which is not a hyper BCK-ideal.

Example 3.13. In Example 3.5, $\{0, b\}$ is a $PI(\ll, \ll, \ll)_{BCK}$-ideal, but not a hyper BCK-ideal.

Definition 3.14. A nonempty subset I of H is said to be *closed* if, for every $x, y \in H$, $x \ll y$ and $y \in I$ imply $x \in I$.

Note that, in Example 3.5, $\{0, a\}$ is closed, which is a $PI(\ll, \ll, \ll)_{BCK}$-ideal.

Theorem 3.15. *Every closed $PI(\ll, \ll, \ll)_{BCK}$-ideal is a hyper BCK-ideal.*

Proof. Let I be a closed $PI(\ll, \ll, \ll)_{BCK}$-ideal of H and let $x, y \in H$ be such that $x \circ y \ll I$ and $y \in I$. Then $(x \circ y) \circ 0 = x \circ y \ll I$ and $y \circ 0 = \{y\} \ll I$ by (P5) and (P17). It follows from (HId7) and (P17) that $\{x\} = x \circ 0 \ll I$ so that there exists $a \in I$ such that $x \ll a$. Since I is closed, we get $x \in I$. Therefore I is a hyper BCK-ideal of H. □

Corollary 3.16. *Every closed $PI(\ll,\ll,\ll)_{BCK}$-ideal is $PI(\ll,\ll,\subseteq)_{BCK}$-ideal.*

Proof. The proof is by Proposition 2.4 and Theorem 3.15. □

Noticing that every hyper BCK-ideal is a weak hyper BCK-ideal, and using Theorems 3.9 and 3.15, we have the following corollary.

Corollary 3.17. *In a positive implicative hyper BCK-algebra, every closed $PI(\ll,\ll,\ll)_{BCK}$-ideal is a $PI(\subseteq,\subseteq,\subseteq)_{BCK}$-ideal.*

Theorem 3.18. *Let I be a nonempty subset of H. Then I is a $PI(\subseteq,\subseteq,\subseteq)_{BCK}$-ideal of H if and only if the set $I_a := \{x \in H \mid x \circ a \subseteq I\}$, where $a \in H$, is a weak hyper BCK-ideal of H.*

Proof. Assume that I is a $PI(\subseteq,\subseteq,\subseteq)_{BCK}$-ideal of H. Since $0 \circ a = \{0\} \subseteq I$ for all $a \in H$, we have $0 \in I_a$. Let $x, y, a \in H$ be such that $x \circ y \subseteq I_a$ and $y \in I_a$. Then $(x \circ y) \circ a \subseteq I$ and $y \circ a \subseteq I$. It follows from (HId5) that $x \circ a \subseteq I$, that is, $x \in I_a$. Hence I_a is a weak hyper BCK-ideal of H. Conversely suppose that I_a is a weak hyper BCK-ideal of H for $a \in H$. Clearly $0 \in I$. Let $x, y, z \in H$ be such that $x, y, z \in H$ be such that $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$. Then $x \circ y \subseteq I_z$ and $y \in I_z$. Since I_z is a weak hyper BCK-ideal of H, we get $x \in I_z$, i.e., $x \circ z \subseteq I$. Therefore I is a $PI(\subseteq,\subseteq,\subseteq)_{BCK}$-ideal of H. □

Corollary 3.19. (1) *If I is a $PI(\ll,\ll,\subseteq)_{BCK}$-ideal (or $PI(\ll,\subseteq,\subseteq)_{BCK}$-ideal) of H, then the set $I_a := \{x \in H \mid x \circ a \subseteq I\}$, where $a \in H$, is a weak hyper BCK-ideal of H.*

(2) *If I is a closed $PI(\ll,\ll,\ll)_{BCK}$-ideal of a positive implicative hyper BCK-algebra H, then the set $I_a := \{x \in H \mid x \circ a \subseteq I\}$, where $a \in H$, is a weak hyper BCK-ideal of H.*

Proof. The proof is straightforward. □

The following is our question: In a positive implicative hyper BCK-algebra, is a $PI(\subseteq,\subseteq,\subseteq)_{BCK}$-ideal a hyper BCK-ideal? But we have a negative answer as seen in the following example.

Example 3.20. Let $H = \{0, a, b\}$ be a set with the following table:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{b\}$	$\{0, a\}$

Then H is a positive implicative hyper BCK-algebra, and $I = \{0, b\}$ is a $PI(\subseteq,\subseteq,\subseteq)_{BCK}$-ideal, but not a hyper BCK-ideal of H since $a \circ b \ll I$ and $b \in I$ but $a \notin I$.

We pose an open problem: Under which condition(s), is a $PI(\subseteq,\subseteq,\subseteq)_{BCK}$-ideal a hyper BCK-ideal?

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