CONVEXITY OF FUZZY-VALUED MAPS AND MINIMIZATION THEOREMS

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Abstract. In this paper, we first prove some theorems connected with the convexity of fuzzy-valued maps. In the course of our discussion, we shall have a necessary and sufficient condition for fuzzy-valued maps to be convex. Next, we establish some minimization theorems for fuzzy-valued maps on compact topological spaces.

1. Introduction

The theory of fuzzy sets was initiated by Zadeh [12] with a view to dealing mathematically with objects or systems which cannot be characterized precisely. Since then, many kinds of fuzzy concepts have been provided for various scientific disciplines and other ones. Amongst these, the concept of fuzzy numbers was introduced by Dubois and Prade [3] as a natural way of treating indefiniteness (fuzziness) of our judgment about the objects under consideration. This was a fuzzy analogue of the concept of real numbers and led to optimization problems with constraints in terms of fuzzy-valued maps, that is, the mappings whose values are fuzzy numbers (see Section 4):

> C being a subset (describing the constraints by fuzzy numbers) of a linear space and F being a fuzzy-valued map on C, find $x_0 \in C$ such that $F(x_0) \preceq F(x)$ for all $x \in C$,

where $\preceq$ denotes the partial order relation on the set of all fuzzy numbers (see Section 2). Indeed, from a standpoint of linear programming problems, such problems have been discussed by Dubois and Prade [4], Ramík and Římánek [8], Campos and Verdegay [2], Ramík and Rommelfanger [9, 10] and others. On the other hand, Nanda and Kar [6] introduced the concept of the convexity of fuzzy-valued maps on a convex subset of a linear space and gave some characterizations of various types of convex fuzzy-valued maps on a linear space. They also deduced some properties of a minimum solution to the above problems.

In this paper, we study the convexity of fuzzy-valued maps and establish some minimization theorems for fuzzy-valued maps. In Section 2, we give notation and terminology to be employed throughout the present paper. In Section 3, we prove some theorems connected with the convexity of fuzzy-valued maps on a convex subset of a linear space. Then we shall have a necessary and sufficient condition for fuzzy-valued maps to be convex. In Section 4, after a brief introduction of optimization problems with fuzzy constraints, we define the lower semicontinuity of fuzzy-valued maps on a topological space and obtain a sufficient

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condition for them to be lower semicontinuous. Next, we establish some minimization theorems for lower semicontinuous fuzzy-valued maps on a compact convex subset of a linear topological space connected with the existence of solutions to the above mentioned fuzzy optimization problems.

2. Preliminaries

Throughout this paper, all linear spaces are real, and we denote by $\mathbf{R}$ the set of real numbers and by $\mathbf{1}_E$ the characteristic function for an arbitrary set E. We shall also use the letter $\mathbf{R}$ to denote the real line. Let X be a linear topological space and let C, D be subsets of X. Then $\mathrm{cl}C$ denotes the closure of C, and there is defined $C+D = \{c+d : c \in C,\ d \in D\}$ and $\lambda C = \{\lambda c : c \in C\}$ for any $\lambda \in \mathbf{R}$.

Let A be a fuzzy set in $\mathbf{R}$. We denote by A_r the r-level set of A, which is a subset of $\mathbf{R}$ and is defined by $A_r = \{x \in \mathbf{R} : A(x) \geq r\}$ for every $r \in (0,1]$ and $A_0 = \mathrm{cl}\{x \in \mathbf{R} : A(x) > 0\}$, respectively. Then A is said to be convex (respectively closed) if for every $r \in (0,1]$, A_r is a convex (respectively closed) subset of $\mathbf{R}$.

Let A, B be fuzzy sets in $\mathbf{R}$. By Zadeh's extension principle [13], we define the addition $\oplus$ to yield a fuzzy set $A \oplus B$ in $\mathbf{R}$ by

$$(A \oplus B)(z) = \sup_{z=x+y,x,y\in\mathbf{R}} \min(A(x), B(y))$$

for all $z \in \mathbf{R}$. Also, we define the multiplication $\odot$ to yield a fuzzy set $A \odot B$ in $\mathbf{R}$ by

$$(A \odot B)(z) = \sup_{z=xy,x,y\in\mathbf{R}} \min(A(x), B(y))$$

for all $z \in \mathbf{R}$. Then $[A \oplus B]_r$ and $[A \odot B]_r$ denote the r-level sets of $A \oplus B$ and $A \odot B$ for every $r \in [0,1]$, respectively. Note that, if both A and B are convex, then $A \oplus B$ is convex; see [7].

Let A be a fuzzy set in $\mathbf{R}$. A is said to be a fuzzy number [3, 5] if it satisfies the following conditions:

(i) A is convex;
(ii) there exists a unique real number $m \in \mathbf{R}$ such that $A(m) = 1$;
(iii) A_0 is a bounded subset of $\mathbf{R}$.

Then the symbol $\mathcal{F}$ denotes the set of fuzzy numbers. Note that for any $A \in \mathcal{F}$, A_0 is a compact convex subset of $\mathbf{R}$ and that for every $A, B \in \mathcal{F}$, $A \oplus B$ belongs to $\mathcal{F}$.

Let $A, B \in \mathcal{F}$. We define an order relation $\preceq$ on $\mathcal{F}$ [8] as follows:

$$A \preceq B \text{ if and only if } \sup A_r \leq \sup B_r \text{ and } \inf A_r \leq \inf B_r \text{ for all } r \in [0,1].$$

Note that the order relation $\preceq$ satisfies the axioms of a partial order relation. Let $A \in \mathcal{F}$ and let $\lambda \in \mathbf{R}$. For the sake of convenience, we shall write $A \preceq \lambda$ instead of $A \preceq \mathbf{1}_{\{\lambda\}}$. Similarly, we shall write $A \oplus \lambda$ (respectively λA, $A\lambda$) for $A \oplus \mathbf{1}_{\{\lambda\}}$ (respectively $\mathbf{1}_{\{\lambda\}} \odot A$, $A \odot \mathbf{1}_{\{\lambda\}}$). Then we observe that if $\lambda \neq 0$, then $(\lambda A)(z) = A(\frac{z}{\lambda})$ for all $z \in \mathbf{R}$ and that $(0A)(z) = \mathbf{1}_{\{0\}}(z)$ for all $z \in \mathbf{R}$. Consequently, it follows that $[\lambda A]_r = \lambda A_r$ for all $r \in [0,1]$ and hence, that $\lambda A \in \mathcal{F}$.

Let C be a convex subset of a linear space and let f be a real-valued function on C. f is said to be convex if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$ for any $x, y \in C$ and any $\lambda \in (0,1)$. f is called concave if $-f$ is convex. Moreover, f is said to be quasi-concave if for every $c \in \mathbf{R}$, $\{x \in C : f(x) \geq c\}$ is a convex subset of C. Let C, I be nonempty sets and let φ be a real-valued function on $C \times I$. Then φ is called concavelike in its second variable if for any $y_1, y_2 \in I$ and any $\lambda \in (0,1)$, there exists $y_0 \in I$ such that $\varphi(x, y_0) \geq \lambda\varphi(x, y_1) + (1-\lambda)\varphi(x, y_2)$ for all $x \in C$. We know the following minimax theorem; see, for instance, [11].

Let C be a compact convex subset of a linear topological space, let I be a nonempty set and let φ be a real-valued function on $C \times I$ satisfying the following conditions:

(i) For each $y \in I$, the function $x \mapsto \varphi(x, y)$ is lower semicontinuous and convex;
(ii) φ is concavelike in its second variable.

Then the following holds:

$$\sup_{y \in I} \min_{x \in C} \varphi(x, y) = \min_{x \in C} \sup_{y \in I} \varphi(x, y).$$

3. Convexity of fuzzy-valued maps

Let X be a nonempty set. A mapping $F : X \to \mathcal{F}$ defined on X is called a fuzzy-valued map on X if for every $x \in X$, $F(x)$ belongs to $\mathcal{F}$. Let C be a convex subset of a linear space and let F be a fuzzy-valued map on C. F is said to be convex [6] if for every $x, y \in C$ and every $\lambda \in (0, 1)$,

$$F(\lambda x + (1 - \lambda)y) \preceq \lambda F(x) \oplus (1 - \lambda)F(y).$$

In this section, we study the convexity of fuzzy-valued maps on a linear space. We first have the following two lemmas.

Lemma 3.1. *For every $A \in \mathcal{F}$, we have*

$$\lim_{r \to +0} \sup A_r = \sup A_0 \text{ and } \lim_{r \to +0} \inf A_r = \inf A_0.$$

Proof. It is easy to see that for every $r_1, r_2 \in (0, 1]$ with $r_1 > r_2$, $A_{r_1} \subset A_{r_2} \subset A_0$. So, we have that $\sup A_{r_1} \le \sup A_{r_2} \le \sup A_0$. Therefore, since A_0 is a nonempty bounded subset of $\mathbf{R}$, $\lim_{r \to +0} \sup A_r = \sup_{r>0} \sup A_r$ exists. Put $\eta = \lim_{r \to +0} \sup A_r$ and $\xi = \sup A_0$. Then, by virtue of the above, it is clear that that $\eta \le \xi$. Conversely, let us take any $x \in \mathbf{R}$ with $A(x) > 0$. Then, by choosing $r_0 > 0$ such that $A(x) > r_0 > 0$, we observe that x belongs to A_{r_0} and hence, that $x \le \sup A_{r_0} \le \sup_{r>0} \sup A_r = \eta$. Consequently, since $x \in \mathbf{R}$ with $A(x) > 0$ is arbitrary, we deduce that for every $y \in A_0$, $y \le \eta$. This implies that $\xi \le \eta$. Thus, we have proved that $\xi = \eta$. Similarly, we show that $\lim_{r \to +0} \inf A_r = \inf A_0$. □

Lemma 3.2. *For every $A \in \mathcal{F}$ and every $r \in (0, 1]$, we have*

$$\lim_{\delta \to r-0} A_\delta = \inf_{\delta<r} \sup A_\delta = \sup A_r \text{ and } \lim_{\delta \to r-0} A_\delta = \sup_{\delta<r} \inf A_\delta = \inf A_r.$$

Proof. Let $r \in (0, 1]$ be fixed arbitrarily and take any $\delta > 0$ with $\delta < r$. Since $A_r \subset A_\delta$, we see at once that $\lim_{\delta \to r-0} A_\delta = \inf_{\delta<r} \sup A_\delta$ exists. Set $\xi = \sup A_r$ and $\eta = \inf_{\delta<r} \sup A_\delta$. Then, by the above observation, the inequality $\xi \le \eta$ evidently follows. In order to prove that the inverse inequality holds, let us assume that $\xi < \eta$. Then there exists $x_0 \in \mathbf{R}$ such that $\xi < x_0 < \eta$. We infer that

$$\inf A_\delta \le \inf A_r \le \sup A_r = \xi < x_0 < \eta = \inf_{\delta<r} \sup A_\delta \le \sup A_\delta$$

and hence, by convexity of a subset A_δ of $\mathbf{R}$, that x_0 belongs to A_δ. Since $\delta < r$ is arbitrary, this implies that $x_0 \in \bigcap_{\delta<r} A_\delta = A_r$. Consequently, we have that $x_0 \le \sup A_r = \xi < x_0$. This is a contradiction. So, we deduce that $\xi = \eta$. By the same way, we prove that $\lim_{\delta \to r-0} \inf A_\delta = \sup_{\delta<r} \inf A_\delta = \inf A_r$. □

By Lemmas 3.1and 3.2, we obtain the following lemma.

Lemma 3.3. *For every $A, B \in \mathcal{F}$ and every $r \in [0, 1]$, we have*

$$\sup[A \oplus B]_r = \sup A_r + \sup B_r \text{ and } \inf[A \oplus B]_r = \inf A_r + \inf B_r.$$

PROOF. Let $A, B \in \mathcal{F}$. We know that $A \oplus B$ belongs to $\mathcal{F}$. Let us fix an arbitrary $r \in (0,1]$. Then the inequality $\sup[A \oplus B]_r \geq \sup A_r + \sup B_r$ is obvious. Indeed, if $z \in A_r + B_r$, then there exist $x_0 \in A_r$ and $y_0 \in B_r$ such that $z = x_0 + y_0$. Since $A(x_0) \geq r$ and $B(y_0) \geq r$, we infer that

$$\begin{aligned} (A \oplus B)(z) &= \sup_{z=x+y, x,y\in\mathbf{R}} \min(A(x), B(y)) \\ &\geq \min(A(x_0), B(y_0)) \geq r \end{aligned}$$

and hence, that $z \in [A \oplus B]_r$. This implies that $[A \oplus B]_r \supset A_r + B_r$ and consequently, that $\sup[A \oplus B]_r \geq \sup(A_r + B_r) = \sup A_r + \sup B_r$.

We next claim that the inverse inequality holds. Let $z \in [A \oplus B]_r$ and choose any $\delta > 0$ with $\delta < r$. Since $(A \oplus B)(z) > \delta$, there exist $x_0, y_0 \in \mathbf{R}$ such that $\min(A(x_0), B(y_0)) \geq \delta$ and $z = x_0 + y_0$. This implies that $z \in A_\delta + B_\delta$ and consequently, that $[A \oplus B]_r \subset A_\delta + B_\delta$. Hence, we have that $\sup[A \oplus B]_r \leq \sup A_\delta + \sup B_\delta$. Therefore, since $\delta > 0$ with $\delta < r$ is arbitrary, it follows from Lemma 3.2 that $\sup[A \oplus B]_r \leq \sup A_r + \sup B_r$.

Thus, we deduce that $\sup[A \oplus B]_r = \sup A_r + \sup B_r$ for all $r \in (0,1]$. Further, by applying Lemma 3.1 to both sides of the above equality, we have that $\sup[A \oplus B]_0 = \sup A_0 + \sup B_0$. By the same method, we prove that $\inf[A \oplus B]_r = \inf A_r + \inf B_r$ for all $r \in [0,1]$. Hence, we have proved the lemma. □

Let F be a fuzzy-valued map on a nonempty set X. $[F(x)]_r$ denotes the r-level set of $F(x) \in \mathcal{F}$ for every $x \in X$ and every $r \in [0,1]$.

We now prove the following theorem.

Theorem 3.1. *Let C be a convex subset of a linear space and let F be a convex fuzzy-valued map on C. Then, for any $r \in [0,1]$, the real-valued function f_r on C defined by $f_r(x) = \sup[F(x)]_r$ for every $x \in C$ is convex.*

PROOF. Let $x, y \in C$, let $\lambda \in (0,1)$ and let $r \in [0,1]$. We know that $[\lambda F(x)]_r = \lambda[F(x)]_r$ and that $\lambda F(x) \in \mathcal{F}$. Since F is convex, it is obvious that

$$\begin{aligned} f_r(\lambda x + (1-\lambda)y) &= \sup[F(\lambda x + (1-\lambda)y)]_r \\ &\leq \sup[\lambda F(x) \oplus (1-\lambda)F(y)]_r. \end{aligned}$$

Moreover, we infer by Lemma 3.3 that

$$\begin{aligned} \sup[\lambda F(x) \oplus (1-\lambda)F(y)]_r &= \sup[\lambda F(x)]_r + \sup[(1-\lambda)F(y)]_r \\ &= \sup \lambda[F(x)]_r + \sup(1-\lambda)[F(y)]_r \\ &= \lambda \sup[F(x)]_r + (1-\lambda)\sup[F(y)]_r \\ &= \lambda f_r(x) + (1-\lambda) f_r(y). \end{aligned}$$

Hence, we deduce that for any $r \in [0,1]$,

$$f_r(\lambda x + (1-\lambda)y) \leq \lambda f_r(x) + (1-\lambda) f_r(y).$$

This completes the proof. □

By the same way, we prove the following theorem.

Theorem 3.2. *Let C be a convex subset of a linear space and let F be a convex fuzzy-valued map on C. Then, for any $r \in [0,1]$, the real-valued function g_r on C defined by $g_r(x) = \inf[F(x)]_r$ for every $x \in C$ is convex.*

Let $A, B \in \mathcal{F}$ and let us assume that both A and B are closed, that is, for every $r \in [0,1]$, each of the r-level sets A_r, B_r is a closed subset of $\mathbf{R}$. Then we know that $[A \oplus B]_r = A_r + B_r$ for all $r \in [0,1]$, so that, the proof of Theorems 3.1 and 3.2 is straightforward; see [7]. However, if both A and B are not closed, we can not use this fact.

Indeed, the following example implies that if A is not closed, then there exists $r \in [0,1]$ such that $[A \oplus B]_r \neq A_r + B_r$.

Example 3.1. Define $A \in \mathcal{F}$ by

$$A(x) = \begin{cases} 0, & \text{if } x < -1, \\ 1 - |x|, & \text{if } -1 \leq x < \frac{1}{2}, \\ 0, & \text{if } \frac{1}{2} \leq x, \end{cases}$$

and define $B \in \mathcal{F}$ by

$$B(x) = \begin{cases} 0, & \text{if } x < -\frac{3}{2}, \\ 1 - |x+1|, & \text{if } -\frac{3}{2} \leq x \leq 0, \\ 0, & \text{if } 0 < x. \end{cases}$$

Then we observe that $(A \oplus B)(0) = \sup_{x \in \mathbf{R}} \min(A(x), B(-x)) = \frac{1}{2}$ and consequently, that $0 \in [A \oplus B]_{\frac{1}{2}}$. On the other hand, since $A_{\frac{1}{2}} = [-\frac{1}{2}, \frac{1}{2})$ and $B_{\frac{1}{2}} = [-\frac{3}{2}, -\frac{1}{2}]$, it follows that $0 \notin A_{\frac{1}{2}} + B_{\frac{1}{2}} = [-2, 0)$. Therefore, we have that $[A \oplus B]_{\frac{1}{2}} \neq A_{\frac{1}{2}} + B_{\frac{1}{2}}$.

Let F be a fuzzy-valued map on a nonempty set X and let $r \in [0,1]$. Throughout the rest of the present paper, f_r^F and g_r^F denote the real-valued functions on X defined by $f_r^F(x) = \sup[F(x)]_r$ for every $x \in X$ and by $g_r^F(x) = \inf[F(x)]_r$ for every $x \in X$, respectively.

Using Lemma 3.3, we also have the following theorem.

Theorem 3.3. *Let C be a convex subset of a linear space and let F be a fuzzy-valued map on C. Assume that for every $r \in [0,1]$, both f_r^F and g_r^F are convex. Then F is convex.*

Proof. Let $x, y \in C$, let $\lambda \in (0,1)$ and let $r \in [0,1]$. By Lemma 3.3, we infer that

$$\begin{aligned} \sup[\lambda F(x) \oplus (1-\lambda)F(y)]_r &= \sup[\lambda F(x)]_r + \sup[(1-\lambda)F(y)]_r \\ &= \lambda \sup[F(x)]_r + (1-\lambda)\sup[F(y)]_r \\ &= \lambda f_r^F(x) + (1-\lambda) f_r^F(y). \end{aligned}$$

Therefore, it follows from hypothesis that

$$\begin{aligned} \sup[F(\lambda x + (1-\lambda)y)]_r &= f_r^F(\lambda x + (1-\lambda)y) \\ &\leq \lambda f_r^F(x) + (1-\lambda) f_r^F(y) \\ &= \sup[\lambda F(x) \oplus (1-\lambda)F(y)]_r. \end{aligned}$$

Similarly, we deduce that

$$\inf[F(\lambda x + (1-\lambda)y)]_r \leq \inf[\lambda F(x) \oplus (1-\lambda)F(y)]_r.$$

Hence, we have proved the theorem. □

We remark that Theorems 3.1, 3.2 and 3.3 give the characterization of the convexity of a fuzzy-valued map F in the sense that F is convex if and only if for every $r \in [0,1]$, both f_r^F and g_r^F are convex.

For further comprehension, let us employ the concept of L-R fuzzy number [8].

Let S be a function from $\mathbf{R}$ to $(-\infty, 1]$. S is called a shape function [5] if it satisfies the following conditions:

(i) S is quasi-concave;
(ii) $S(x) = 1$ if and only if $x = 0$;
(iii) the set $\{x \in \mathbf{R} : S(x) > 0\}$ is a bounded subset of $\mathbf{R}$;
(iv) $S(x) = S(-x)$ for all $x \in \mathbf{R}$.

Then, for any shape function S and any $r \in (0,1]$, S_r and S_0 denote the subsets $\{x \in \mathbf{R} : S(x) \geq r\}$ and $\mathrm{cl}\{x \in \mathbf{R} : S(x) > 0\}$ of $\mathbf{R}$, respectively. Moreover, for every $r \in [0,1]$, we put $k_r^S = \sup S_r \in [0, \infty)$.

Let S, T be shape functions, let $m \in \mathbf{R}$ and let $\alpha, \beta \geq 0$. Then the L-R fuzzy number μ is a fuzzy number defined by the relation

$$\mu(x) = \begin{cases} \max(S(\frac{x-m}{\alpha}), 0), & \text{if } x \leq m, \\ \max(T(\frac{x-m}{\beta}), 0), & \text{if } x \geq m. \end{cases}$$

Then we shall say that μ is generated by the shape functions S and T. Further, we shall denote the L-R fuzzy number μ by

$$\mu = (m, \alpha, \beta)_{L_S R_T}$$

in the form of a parametric representation.

We should mention that the above definition of μ includes the cases where $\alpha = 0$ and $\beta > 0$ for instance. In these cases, we define μ by

$$\mu(x) = \begin{cases} 0, & \text{if } x < m, \\ \max(T(\frac{x-m}{\beta}), 0), & \text{if } x \geq m. \end{cases}$$

Further, in the case where $\alpha, \beta = 0$, μ is defined by $\mu = \mathbf{1}_{\{m\}}$.

As an immediate consequence of Theorem 3.3, we obtain the following theorem regarding the convexity of fuzzy-valued maps whose range consist of L-R fuzzy numbers generated by the same shape functions.

Theorem 3.4. *Let C be a convex subset of a linear space, let m be a real-valued function on C, let $\alpha, \beta : C \to [0, \infty)$ be functions and let $S, T : \mathbf{R} \to (-\infty, 1]$ be shape functions. Let F be a fuzzy-valued map on C such that for every $x \in C$, $F(x)$ is an L-R fuzzy number denoted by*

$$F(x) = (m(x), \alpha(x), \beta(x))_{L_S R_T}$$

in the form of a parametric representation. Assume that m and β are convex and that α is concave. Then F is convex.

Proof. Without loss of generality, we may assume that $S(x) \geq 0$ and $T(x) \geq 0$ for all $x \in \mathbf{R}$. Let $x \in C$ and let $r \in [0,1]$. We observe that

$$[F(x)]_r = \Big(m(x) + \alpha(x) S_r\Big) \bigcup \Big(m(x) + \beta(x) T_r\Big)$$

and therefore, that

$$f_r^F(x) = \sup[F(x)]_r = m(x) + \sup T_r \beta(x) = m(x) + k_r^T \beta(x).$$

Similarly, we have that $g_r^F(x) = \inf[F(x)]_r = m(x) + \inf S_r \alpha(x)$. Further, we infer by the definition of a shape function that $k_r^S + \inf S_r = \sup S_r + \inf S_r = 0$ and consequently, that $g_r^F(x) = m(x) - k_r^S \alpha(x)$. Hence, since $k_r^S, k_r^T \geq 0$, we deduce from the assumptions of m, α and β that both f_r^F and g_r^F are convex. Thus, by Theorem 3.3, we have proved that F is convex. □

We next present the following result relating to the comparison of two L-R fuzzy numbers of the same type, which was essentially stated in [5, 8]. For the sake of completeness, we give the proof.

Proposition 3.1 ([5, 8]). *Let $m, n \in \mathbf{R}$, let $\alpha, \beta, \gamma, \delta \geq 0$ and let S, T be shape functions. Then, for two L-R fuzzy numbers A and B denoted respectively by $A = (m, \alpha, \beta)_{L_S R_T}$ and $B = (n, \gamma, \delta)_{L_S R_T}$ in the form of a parametric representation, $A \preceq B$ if and only if $\sup A_1 \leq \sup B_1$, $\sup A_0 \leq \sup B_0$ and $\inf A_0 \leq \inf B_0$.*

PROOF. Since the "only if part" is obvious by the definition of the order relation $\preceq$, it suffices to prove the "if part". Let $r \in [0,1]$. We observe that $\sup A_r = m + k_r^T \beta$ and $\sup B_r = n + k_r^T \delta$. Therefore, since $\sup A_1 = m$ and $\sup B_1 = n$, we infer that

$$\begin{aligned} k_0^T(\sup A_r - \sup B_r) &= k_0^T(m + k_r^T\beta - (n + k_r^T\delta)) \\ &= k_0^T(m-n) + k_r^T k_0^T(\beta - \delta) \\ &= k_0^T(\sup A_1 - \sup B_1) + k_r^T(\sup A_0 - \sup B_0 - (\sup A_1 - \sup B_1)) \\ &= (k_0^T - k_r^T)(\sup A_1 - \sup B_1) + k_r^T(\sup A_0 - \sup B_0). \end{aligned}$$

It is obvious that $k_0^T \geq k_r^T \geq 0$, so that, we have that $\sup A_r \leq \sup B_r$. By the same way, we deduce that $\inf A_r \leq \inf B_r$. This completes the proof. □

Applying Lemma 3.3 and Proposition 3.1, we simply obtain the following result, which was essentially presented in [5].

Proposition 3.2 ([5]). *Let C be a convex subset of a linear space, let m be a real-valued function on C, let $\alpha, \beta : C \to [0,\infty)$ be functions and let $S, T : \mathbf{R} \to (-\infty, 1]$ be shape functions. Let F be a fuzzy-valued map on C such that for every $x \in C$, $F(x)$ is an L-R fuzzy number denoted by*

$$F(x) = (m(x), \alpha(x), \beta(x))_{L_S R_T}$$

in the form of a parametric representation. Then, F is convex if and only if f_1^F, f_0^F and g_0^F are convex.

PROOF. Let $x, y \in C$ and let $\lambda \in (0,1)$. By Lemma 3.3 and Proposition 3.1, we infer that $F(\lambda x + (1-\lambda)y) \preceq \lambda F(x) \oplus (1-\lambda)F(y)$ if and only if $f_1^F(\lambda x + (1-\lambda)y) \leq \lambda f_1^F(x) + (1-\lambda) f_1^F(y)$, $f_0^F(\lambda x + (1-\lambda)y) \leq \lambda f_0^F(x) + (1-\lambda) f_0^F(y)$ and $g_0^F(\lambda x + (1-\lambda)y) \leq \lambda g_0^F(x) + (1-\lambda) g_0^F(y)$. This completes the proof. □

4. Minimization theorems for fuzzy-valued maps

In this section, we establish minimization theorems for fuzzy-valued maps on a compact topological space. First, let us begin with a brief introduction of optimization problems with fuzzy constraints, which were discussed in the previous papers (see [2, 4, 8] for instance).

Let $X = \mathbf{R}^n$, let $E = \mathbf{R}_+^n = [0,\infty)^n \subset X$, let $c_1, c_2, \ldots, c_n \in \mathbf{R}$ and let f be a real-valued function on E defined by

$$f(x) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

for every $x = (x_1, x_2, \ldots, x_n) \in E$. Let $a_{ij} \in \mathbf{R}$ for every $i = 1, 2, \ldots, m$ and every $j = 1, 2, \ldots, n$, let $g_1, g_2, \ldots, g_m$ be real-valued functions on E defined respectively by

$$g_i(x) = a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n$$

for every $x = (x_1, x_2, \ldots, x_n) \in E$ and every $i = 1, 2, \ldots, m$ and let

$$E_0 = \{x \in E : g_1(x) \leq b_1, g_2(x) \leq b_2, \ldots, g_m(x) \leq b_m\},$$

where $b_1, b_2, \ldots, b_m \in \mathbf{R}$. As a usual linear programming problem, we know the following:

Find $x_0 \in E_0$ such that $f(x_0) \leq f(x)$ for all $x \in E_0$.

However, in actual problems, the coefficients of the constraint functions $g_1, g_2, \ldots, g_m$ are often fuzzy because of, for instance, vagueness or impreciseness of our judgment (estimation, evaluation or measurement) about the data. The concept of fuzzy numbers might be applied to these cases.

Let S, T be shape functions and let A_{ij} be L-R fuzzy numbers denoted by $A_{ij} = (a_{ij}, \alpha_{ij}, \beta_{ij})_{L_S R_T}$ in the form of a parametric representation for every $i = 1, 2, \ldots, m$ and every $j = 1, 2, \ldots, n$, where $\alpha_{ij}, \beta_{ij} \geq 0$. Then we define fuzzy-valued maps $G_1, G_2, \ldots, G_m$ on E respectively by

$$G_i(x) = A_{i1}x_1 \oplus A_{i2}x_2 \oplus \cdots \oplus A_{in}x_n$$

for every $x = (x_1, x_2, \ldots, x_n) \in E$ and every $i = 1, 2, \ldots, m$ and let

$$E_1 = \{x \in E : G_1(x) \preceq b_1, G_2(x) \preceq b_2, \ldots, G_m(x) \preceq b_m\}.$$

Then, as a contingent plan, the following fuzzy optimization problem can be stated:

Find $x_0 \in E_1$ such that $f(x_0) \leq f(x)$ for all $x \in E_1$.

In the above, the objective function f is nonfuzzy, whereas the constraint functions are fuzzy. So, it could be appropriate to consider that the objective function is also fuzzy, namely, a fuzzy-valued map on E. In that situation, we might for instance employ a fuzzy-valued map F on E instead of f defined by

$$F(x) = C_1x_1 \oplus C_2x_2 \oplus \cdots \oplus C_nx_n$$

for every $x = (x_1, x_2, \ldots, x_n) \in E$, where $C_1, C_2, \ldots, C_n$ are L-R fuzzy numbers denoted respectively by

$$C_i = (c_i, \gamma_i, \delta_i)_{L_S R_T}$$

in the form of a parametric representation for every $i = 1, 2, \ldots, n$, where $\gamma_i, \delta_i \geq 0$. Consequently, the following minimization problem has been presented:

Find $x_0 \in E_1$ such that $F(x_0) \preceq F(x)$ for all $x \in E_1$.

It may be said from the above discussion that these problems are fuzzy analogues of linear programming problems. Naturally, the following optimization problem in terms of fuzzy-valued maps arises as a fuzzy analogue of convex problems; see also [6, 9, 10]:

C being a convex subset of a linear space and F being a convex fuzzy-valued map on C, find $x_0 \in C$ such that $F(x_0) \preceq F(x)$ for all $x \in C$.

In the rest of the section, we prove some theorems in connection with the existence of solutions to the above problem.

First of all, let us give a definition of the lower semicontinuity of fuzzy-valued maps on a topological space.

Let F be a fuzzy-valued map on a topological space X. F is said to be lower semicontinuous on X if for every $r \in [0, 1]$, both f_r^F and g_r^F are lower semicontinuous on X.

Next, concerning the concept of L-R fuzzy numbers, we prove the following theorem, which gives a sufficient condition for fuzzy-valued maps on a topological space to be lower semicontinuous.

Theorem 4.1. *Let C be a nonempty subset of a topological space, let m be a real-valued function on C, let $\alpha, \beta : C \to [0, \infty)$ be functions and let $S, T : \mathbf{R} \to (-\infty, 1]$ be shape functions. Let F be a fuzzy-valued map on C such that for every $x \in C$, $F(x)$ is an L-R fuzzy number denoted by*

$$F(x) = (m(x), \alpha(x), \beta(x))_{L_S R_T}$$

in the form of a parametric representation. Assume that m and β are lower semicontinuous on C and that α is upper semicontinuous on C. Then F is lower semicontinuous on C.

PROOF. Let $x \in C$ and let $r \in [0,1]$. Since $f_r^F(x) = m(x) + k_r^T\beta(x)$, we infer by the assumptions of m, β that f_r is lower semicontinuous on C. Similarly, by the equation $g_r^F(x) = m(x) - k_r^S\alpha(x)$, it follows from the assumption of α that g_r is lower semicontinuous on C. This completes the proof. □

Moreover, we need the following definition and notation. Let X, I be nonempty sets and let $\{h_i : i \in I\}$ be a family of real-valued functions X. $x_0 \in X$ is said to be a common minimizer of $\{h_i : i \in I\}$ in X if $h_i(x_0) \leq h_i(x)$ for every $x \in X$ and every $i \in I$. Let F be a fuzzy-valued map on X. We denote by $\mathcal{P}^F$ the family $\{f_r^F, g_r^F : r \in [0,1]\}$ of real-valued functions on X. Then, for any $x_0 \in X$, it is easy to check that $F(x_0) \preceq F(x)$ for every $x \in X$ if and only if x_0 is a common minimizer of $\mathcal{P}^F$ in X.

Now, we establish the following minimization theorem.

Theorem 4.2. *Let F be a lower semicontinuous fuzzy-valued map on a compact topological space X. Suppose that every finite subfamily of $\mathcal{P}^F$ has a common minimizer in X. Then there exists $x_0 \in X$ such that $F(x_0) \preceq F(x)$ for every $x \in X$.*

PROOF. For every $r \in [0,1]$, let $C_r = \{x \in X : f_r^F(x) = \inf_{y \in X} f_r^F(y)\}$ and let $D_r = \{x \in X : g_r^F(x) = \inf_{y \in X} g_r^F(y)\}$. We observe that both C_r and D_r are nonempty closed subsets of X. Further, we infer by hypothesis that the family $\{C_r \cap D_r : r \in [0,1]\}$ has finite intersection property. Therefore, we deduce that $\bigcap_{r \in [0,1]} (C_r \cap D_r) \neq \emptyset$. This implies that there exists $x_0 \in X$ such that for every $x \in X$ and every $r \in [0,1]$, $f_r^F(x_0) \leq f_r^F(x)$ and $g_r^F(x_0) \leq g_r^F(x)$, that is, x_0 is a common minimizer of $\mathcal{P}^F$ in X. Hence, we have $x_0 \in X$ such that $F(x_0) \preceq F(x)$ for every $x \in X$. This completes the proof. □

Further, we obtain the following minimization theorem for lower semicontinuous and convex fuzzy-valued maps on a compact convex subset of a linear topological space.

Theorem 4.3. *Let C be a compact convex subset of a linear topological space and let F be a lower semicontinuous and convex fuzzy-valued map on C. Let φ be a real-valued function on $C \times \mathcal{P}^F$ defined by $\varphi(x,h) = h(x) - \min_{u \in C} h(u)$ for every $(x,h) \in C \times \mathcal{P}^F$. Suppose that φ is concavelike in its second variable. Then there exists $x_0 \in C$ such that $F(x_0) \preceq F(x)$ for all $x \in C$.*

PROOF. By the hypothesis of F, we deduce from Theorems 3.1 and 3.2 that for every $h \in \mathcal{P}^F$, the function $x \mapsto \varphi(x,h)$ is convex. Therefore, we infer that φ satisfies the assumptions of minimax theorem. Hence, we have that

$$\sup_{h \in \mathcal{P}^F} \min_{x \in C} \varphi(x,h) = \min_{x \in C} \sup_{h \in \mathcal{P}^F} \varphi(x,h)$$

and consequently, that $\min_{x \in C} \sup_{h \in \mathcal{P}^F} \varphi(x,h) = 0$. This implies that there exists $x_0 \in C$ such that $\varphi(x_0, h) \leq 0$ for all $h \in \mathcal{P}^F$, that is, $h(x_0) \leq \min_{u \in C} h(u)$ for all $h \in \mathcal{P}^F$. This completes the proof. □

Moreover, we prove the following two theorems relating to the concept of L-R fuzzy numbers.

Let F be a fuzzy-valued map on a nonempty set X. The symbol $\mathcal{P}_0^F$ denotes the family $\{f_1^F, f_0^F, g_0^F\}$ of real-valued functions on X.

Theorem 4.4. *Let C be a nonempty subset of a topological space, let m be a real-valued function on C, let $\alpha, \beta : C \to [0, \infty)$ be functions and let $S, T : \mathbf{R} \to (-\infty, 1]$ be shape functions. Let F be a fuzzy-valued map on C such that for every $x \in C$, $F(x)$ is an L-R*

fuzzy number denoted by

$$F(x) = (m(x), \alpha(x), \beta(x))_{L_S R_T}$$

in the form of a parametric representation. Then there exists $x_0 \in X$ *such that* $F(x_0) \preceq F(x)$ *for all* $x \in X$ *if and only if there exists* $x_0 \in X$ *such that* x_0 *is a common minimizer of* $\mathcal{P}_0^F$ *in* X.

Proof. Let $x_0 \in X$. By Proposition 3.1, it is easy to verify that $F(x_0) \preceq F(x)$ for all $x \in X$ if and only if for every $x \in X$, $f_1^F(x_0) \le f_1^F(x)$, $f_0^F(x_0) \le f_0^F(x)$ and $g_0^F(x_0) \le g_0^F(x)$, that is, x_0 is a common minimizer of $\mathcal{P}_0^F$ in X. This completes the proof. □

Theorem 4.5. *Let* C *be a compact convex subset of a linear topological space, let* F *be a convex fuzzy-valued map on* C *such that for every* $x \in C$, $F(x)$ *is an* L-R *fuzzy number generated respectively by the same shape functions. Suppose that for every* $h \in \mathcal{P}_0^F$, h *is lower semicontinuous on* C *and that the real-valued function* φ_0 *on* $C \times \mathcal{P}_0^F$ *defined by* $\varphi_0(x,h) = h(x) - \min_{u \in C} h(u)$ *for every* $(x,h) \in C \times \mathcal{P}_0^F$ *is concavelike in its second variable. Then there exists* $x_0 \in C$ *such that* $F(x_0) \preceq F(x)$ *for all* $x \in C$.

Proof. Applying minimax theorem, we have that

$$\sup_{h \in \mathcal{P}_0^F} \min_{x \in C} \varphi_0(x,h) = \min_{x \in C} \sup_{h \in \mathcal{P}_0^F} \varphi_0(x,h).$$

Consequently, we deduce that there exists $x_0 \in C$ such that $\varphi_0(x_0, h) \le 0$ for all $h \in \mathcal{P}_0^F$, that is, $h(x_0) \le \min_{u \in C} h(u)$ for all $h \in \mathcal{P}_0^F$. Hence, by Theorem 4.4, the statement ensues. □

By the above discussion, we remark that, when we impose on a fuzzy-valued map F being "L-R fuzzy-valued", we can confine ourselves to the family $\mathcal{P}_0^F$.

Finally, let us provide an example of fuzzy-valued maps satisfying the assumptions in Theorem 4.5.

Example 4.1. Let C be a compact convex subset of a linear topological space, let $m : C \to \mathbf{R}$ be a lower semicontinuous and convex function, let $\alpha, \beta \ge 0$ and let $S, T : \mathbf{R} \to (-\infty, 1]$ be shape functions. By virtue of Theorems 3.4 and 4.1, let us define a lower semicontinuous and convex fuzzy-valued map F on C by the relation

$$F(x) = (m(x), \alpha, \beta)_{L_S R_T}$$

for every $x \in C$. Then we infer that for every $r \in [0,1]$, $f_r^F(x) = m(x) + k_r^T \beta$ and $g_r^F(x) = m(x) - k_r^S \alpha$ and therefore, that φ_0 is concavelike in its second variable.

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