RENEWAL THEOREMS IN THE PRESENCE OF ROOTS OF THE CHARACTERISTIC EQUATION

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Abstract. Let $\{X_i\}$ be a sequence of independent identically distributed random variables with $EX_1 > 0$, and let $\{S_k\}$ be the sequence of the partial sums. We obtain asymptotic expansions for the renewal measure $\sum_{k=0}^{\infty} P(S_k \in \cdot)$, taking into account the influence of the roots of the characteristic equation $1 - E\exp(sX_1) = 0$ which lie in the strip of analyticity of the Laplace transform $E\exp(sX_1)$. The exact asymptotic behaviour of the remainder terms is established. We also give submultiplicative estimates for the remainders.

1. Introduction

Let $\{X_i\}$ be a sequence of independent identically distributed random variables with common non-arithmetic distribution F and expectation $EX_1 > 0$. Denote by $\{S_k\}$ the sequence of the partial sums: $S_k = \sum_{i=1}^{k} X_i$, $k \geq 1$, $S_0 = 0$. Let $n \geq 1$ be an arbitrary integer. We consider generalized renewal measures of the following form:

$$\Phi_n(A) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{n \cdot (n+k-1)!}{k!} P(S_k \in A), \qquad A \in \mathcal{B}, \tag{1}$$

where $\mathcal{B}$ is the σ-algebra of all Borel subsets of the real line $\mathbf{R}$. Put $x^- = \max(0, -x)$. The measure Φ_n is σ-finite $\Leftrightarrow \mathbf{E}(X_1^-)^n < \infty$ [28, Proposition]. The measure $H(A) \stackrel{\text{def}}{=} \Phi_1(A) = \sum_{k=0}^{\infty} P(S_k \in A)$ is the usual renewal measure on the whole line generated by F. When the underlying distribution F is concentrated on $[0,\infty)$, the measures Φ_n are closely related to higher renewal moments (see [32]).

In [32], we investigated the exact asymptotic behaviour of Φ_n when there are no non-zero roots of the *characteristic equation*

$$1 - E\exp(sX_1) = 0. \tag{2}$$

This paper concerns asymptotic expansions for Φ_n (and, in particular, for the renewal measure H) which take into account the influence of the roots of (2) lying in a non-degenerate strip of analyticity of the Laplace transform $E\exp(sX_1)$. Such expansions ((9) below) both for the renewal measure H and for Φ_n have been considered by many authors under various assumptions (see [29] and the references therein).

The plan of the paper is as follows. In Section 2 we give necessary definitions and cite auxiliary results. In Section 3 we study the *exact* asymptotic behaviour of the remainder terms $\mathcal{R}_n$ of the expansions for Φ_n by comparing $\mathcal{R}_n$ with a distribution G of the class $\mathcal{S}(\gamma)$, $\gamma > 0$ (Definition 1). Namely, suppose we know $\lim_{x\to\infty} F((x,\infty))/G((x,\infty))$. Then our goal will consist in determining $\lim_{x\to\infty} \mathcal{R}_n((x,\infty))/G((x,\infty))$ (Theorem 5). The

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knowledge of the exact asymptotic behaviour of $\mathcal{R}_n((x,\infty))$ will allow us to obtain expansions for the generalized renewal function $\Phi_n((-\infty,x])$ (in particular, for the renewal function $H((-\infty,x]))$ and for $\Phi_n([0,x])$ with exact asymptotic behaviour of the remainder term (Corollary 2). A new feature of this paper is the discussion of *converse statements*, i.e. taking $\lim_{x\to\infty}\mathcal{R}_n((x,\infty))/G((x,\infty))$ as starting point, we prove the existence of $\lim_{x\to\infty}F((x,\infty))/G((x,\infty))$ and establish the connection between these limits. The last section is dedicated to *integral* estimates for the remainder terms $\mathcal{R}_n$ in expansions of the form (9). In [29], the following estimate was obtained: $\int_0^\infty \exp(rx)\,|\mathcal{R}_n|(dx) < \infty$, where $r>0$ and $|\mathcal{R}_n|$ is the total variation of $\mathcal{R}_n$. Theorems 3 and 4 about Laplace transforms allow us to generalize the results of [29] in two directions. First, some roots of (2) may lie on the boundary of the strip of analyticity of $E\exp(sX_1)$. Second, estimates of the total variation of the remainder term are given in the form of integrals with *submultiplicative* weight functions (Definition 2) satisfying natural monotonicity conditions. Exponential functions previously used in [29] and in other papers for estimating the remainder terms are a particular case of the submultiplicative functions.

2. Preliminary results

Definition 1. A probability distribution G concentrated on $[0,\infty)$ *belongs to the class* $\mathcal{S}(\gamma)$, $\gamma\ge 0$, if

(a) $G((x,\infty))>0 \quad \forall x\ge 0$,

(b) $\displaystyle\lim_{x\to\infty}\frac{G((x+y,\infty))}{G((x,\infty))}=e^{-\gamma y} \quad \forall y\in\mathbf{R}$,

(c) $\displaystyle\lim_{x\to\infty}\frac{G*G((x,\infty))}{G((x,\infty))}=2\int_0^\infty e^{\gamma x}\,G(dx)<\infty$.

Just to convey an idea of what the $\mathcal{S}(\gamma)$-distributions are like, we give two very simple examples of $G\in\mathcal{S}(\gamma)$ [6]: (i) G is absolutely continuous with density $g(x)\sim x^{-b}e^{-\gamma x}$, $b>1$, and (ii) G is absolutely continuous with density $g(x)\sim e^{-ax^\alpha}e^{-\gamma x}$, $a>0$, $0<\alpha<1$.

The class $\mathcal{S}=\mathcal{S}(0)$ (later called the class of *subexponential* distributions) was introduced by Chistyakov [5], while the classes $\mathcal{S}(\gamma)$ for positive γ were first considered by Chover, Ney, and Wainger [6, 7]. The importance of such distributions has widely been illustrated by the fact that in many cases the exact asymptotic behaviour of probabilistic quantities of interest can be expressed in terms of the distributions of $\mathcal{S}(\gamma)$. There is a rather extensive literature concerning both the properties of $\mathcal{S}(\gamma)$-distributions themselves and their use in various areas of probability theory (branching processes, queueing theory, infinite divisibility, etc.); see, e.g. Athreya and Ney [2], Teugels [36], Veraverbeke [38], Embrechts, Goldie and Veraverbeke [13], Embrechts and Goldie [11, 12], Pitman [22], Embrechts and Veraverbeke [14], Cline [8, 9], Frenk [16], Sgibnev [27, 28, 31], Klüppelberg [20], Bertoin and Doney [4], Jelenković and Lazar[19], Alsmeyer and Sgibnev [1].

Remark 1. It is worth noting that the relation $F\in\mathcal{S}(\gamma)$ with $\gamma>0$ is *not* equivalent to $F_\gamma\in\mathcal{S}(0)$, where $F_\gamma(dx)\overset{\text{def}}{=}e^{\gamma x}F(dx)\big/\int_0^\infty e^{\gamma x}F(dx)$. Namely, while $F\in\mathcal{S}(\gamma)\Rightarrow F_\gamma\in\mathcal{S}(0)$, the converse does not hold in general [12, Theorem 3.1].

Definition 2. A function $\varphi(x)$, $x\in\mathbf{R}$ $(\mathbf{R}_+)$, is called *submultiplicative* if $\varphi(x)$ is a finite, positive, Borel measurable function with the following properties:

$$\varphi(0)=1,\qquad \varphi(x+y)\le\varphi(x)\,\varphi(y)\quad\text{for all } x,y\in\mathbf{R}\ (\mathbf{R}_+).$$

It is well known [18, Section 7.6] that

$$(3)\qquad -\infty < r_-(\varphi) \stackrel{\text{def}}{=} \lim_{x\to-\infty}\frac{\log\varphi(x)}{x} = \sup_{x<0}\frac{\log\varphi(x)}{x} \le \inf_{x>0}\frac{\log\varphi(x)}{x} = \lim_{x\to\infty}\frac{\log\varphi(x)}{x} \stackrel{\text{def}}{=} r_+(\varphi) < \infty.$$

Here are some examples of such functions on $\mathbf{R}_+$: $\varphi(x) = (1+x)^r$, $r > 0$; $\varphi(x) = \exp(cx^\alpha)$ with $c > 0$ and $\alpha \in (0,1)$; $\varphi(x) = \exp(\gamma x)$ for γ real. In the first two cases $r_+(\varphi) = 0$ while in the last case $r_+(\varphi) = \gamma$. Putting $\varphi(x) \equiv 1$ for $x < 0$ in the above examples, we obtain submultiplicative functions defined on the whole line $\mathbf{R}$. The product of a finite number of submultiplicative functions is again a submultiplicative function. If $R(x)$, $x \in \mathbf{R}_+$, is a positive, ultimately non-decreasing regularly varying function at infinity with a non-negative exponent α (i.e. $R(tx)/R(x) \to t^\alpha$ for $t > 0$ as $x \to \infty$ [15, Section VIII.8]), then there exist a non-decreasing submultiplicative function $\varphi(x)$ and a point $x_0 \in (0,\infty)$ such that $c_1R(x) \le \varphi(x) \le c_2R(x)$ for all $x \ge x_0$, where c_1 and c_2 are positive constants [30, Proposition].

Let $\nu(A)$, $A \in \mathcal{B}$, be a complex-valued σ-finite measure. Denote by $|\nu|(A)$ the total variation of the measure ν on the set A: $|\nu|(A) = \sup\sum_j |\nu(A_j)|$, where the supremum is taken over all countable partitions of the set A into disjoint subsets $A_j \in \mathcal{B}$. Consider the collection $S(\varphi)$ of all complex-valued measures ν defined on $\mathcal{B}$ and such that $\|\nu\|_\varphi \stackrel{\text{def}}{=} \int_{\mathbf{R}} \varphi(x)\,|\nu|(dx) < \infty$. The collection $S(\varphi)$ is a Banach algebra with norm $\|\cdot\|_\varphi$ by the usual operations of addition and scalar multiplication of measures, the product of two elements ν and κ of $S(\varphi)$ is defined as their convolution $\nu * \kappa$ [18, Section 4.16]. The unit element of $S(\varphi)$ is the Dirac measure δ, i.e. the atomic measure of unit mass at the origin.

Denote by $\hat\nu(s)$ the Laplace transform of a measure $\nu \in S(\varphi)$: $\hat\nu(s) \stackrel{\text{def}}{=} \int_{\mathbf{R}} \exp(sx)\,\nu(dx)$. Relation (3) implies that $\hat\nu(s)$ converges absolutely with respect to $|\nu|$ for all s in the strip $\Pi(\varphi) \stackrel{\text{def}}{=} \{r_-(\varphi) \le \Re s \le r_+(\varphi)\}$, i.e. $\int_{\mathbf{R}} \exp(\Re s x)\,|\nu|(dx) < \infty$.

Fix $G \in \mathcal{S}(\gamma)$, $\gamma > 0$. Set $\tau(x) = G((x,\infty))$, $x \ge 0$, and

$$Q(\nu) \stackrel{\text{def}}{=} \sup_{x\ge 0}\frac{|\nu|((x,\infty))}{\tau(x)} < \infty.$$

Let $\gamma' \in [0,\gamma)$. Consider the following collections of complex-valued σ-finite measures [25]: $S(\gamma',\gamma) \stackrel{\text{def}}{=} S(\varphi)$ with $\varphi(x) = \max(e^{\gamma' x}, e^{\gamma x})$ and

$$\mathfrak{S}(\gamma',\tau) \stackrel{\text{def}}{=} \{\nu \in S(\gamma',\gamma) : Q(\nu) < \infty\},$$

$$\mathfrak{So}(\gamma',\tau) \stackrel{\text{def}}{=} \left\{\nu \in \mathfrak{S}(\gamma',\tau) : \lim_{x\to\infty}\frac{|\nu|((x,\infty))}{\tau(x)} = 0\right\},$$

$$\mathfrak{Sl}(\gamma',\tau) \stackrel{\text{def}}{=} \left\{\nu \in S(\gamma',\gamma) : Q(\nu) < \infty,\ \exists \lim_{x\to\infty}\frac{\nu((x,\infty))}{\tau(x)} \stackrel{\text{def}}{=} \mathfrak{l}(\nu) \in \mathbf{C}\right\}$$

The collection $\mathfrak{S}(\gamma',\tau)$ is a Banach algebra with a norm $\|\cdot\|$, equivalent to the norm

$$\|\nu\|'_\tau = \int_{\mathbf{R}} \max(e^{\gamma' x}, e^{\gamma x})\,|\nu|(dx) + Q(\nu),$$

and the collections $\mathfrak{So}(\gamma',\tau)$ and $\mathfrak{Sl}(\gamma',\tau)$ are Banach subalgebras of $\mathfrak{S}(\gamma',\tau)$. By ν^{k*} we shall denote the k-fold convolution of the measure ν: $\nu^{1*} \stackrel{\text{def}}{=} \nu$, $\nu^{k*} \stackrel{\text{def}}{=} \nu * \nu^{(k-1)*}$, $\nu^{0*} \stackrel{\text{def}}{=} \delta$.

If ν, $\mu \in \mathfrak{Sl}(\gamma', \tau)$, then (see [25])

$$\mathfrak{l}(\nu * \mu) = \mathfrak{l}(\nu)\hat{\mu}(\gamma) + \mathfrak{l}(\mu)\hat{\nu}(\gamma). \tag{4}$$

For brevity we will denote $\mathfrak{Sl}(0, \tau)$ by $\mathfrak{Sl}(\tau)$.

Remark 2. If a measure ν is finite, then $\nu \in \mathfrak{Sl}(\tau) \Leftrightarrow \nu \in \mathfrak{Sl}(\gamma', \tau)$.

We agree that *from now on the parameters* γ *and* γ' *satisfy the inequalities* $0 \le \gamma' < \gamma$. The following theorem and lemma have been proved in [34].

Theorem 1. *Let* G *be an arbitrary* $\mathcal{S}(\gamma)$*-distribution with* $\gamma > 0$*. Set* $\tau(x) = G((x, \infty))$, $x \ge 0$*. Suppose* $\nu \in \mathfrak{Sl}(\gamma', \tau)$ *and* $\gamma' \le \Re\beta < \gamma$*. If* $\Re\beta = \gamma'$*, then assume additionally that* $\int_{-\infty}^{0} |x| e^{\gamma' x}\, |\nu|(dx) < \infty$*. Then the function*

$$\frac{\hat{\nu}(s) - \hat{\nu}(\beta)}{s - \beta}, \qquad \gamma' \le \Re s \le \gamma,$$

is the Laplace transform of the measure $\kappa \in \mathfrak{Sl}(\gamma', \tau)$ *with the density*

$$v(x; \beta) \stackrel{\text{def}}{=} \begin{cases} -\int_{-\infty}^{x} e^{\beta(y-x)}\, \nu(dy) & \text{for } x < 0, \\ \int_{x}^{\infty} e^{\beta(y-x)}\, \nu(dy) & \text{for } x \ge 0; \end{cases} \tag{5}$$

moreover, $\mathfrak{l}(\kappa) = \mathfrak{l}(\nu)/(\gamma - \beta)$*.*

In connection with Theorem 1 we introduce the following notation. Let $\beta \in \mathbf{C}$, and let ν be a σ-finite measure such that the measure $\int_A \exp(\beta x)\, \nu(dx)$, $A \in \mathcal{B}$, is finite. Denote

$$T(\beta)\nu(A) \stackrel{\text{def}}{=} \int_A v(x; \beta)\, dx, \qquad A \in \mathcal{B},$$

where the function $v(x; \beta)$, $x \in \mathbf{R}$, is given by (5). If $\int_{\mathbf{R}} |x| e^{\Re\beta x}\, |\nu|(dx) < \infty$, then the Laplace transform of the measure $T(\beta)\nu$ is of the following form:

$$[T(\beta)\nu]^{\wedge}(s) = \frac{\hat{\nu}(s) - \hat{\nu}(\beta)}{s - \beta}, \qquad \Re s = \Re\beta;$$

at $s = \beta$ we set $[T(\beta)\nu]^{\wedge}(\beta) \stackrel{\text{def}}{=} \int_{\mathbf{R}} x e^{\beta x}\, \nu(dx)$. Denote for brevity $T \stackrel{\text{def}}{=} T(\beta)$ if $\beta = 0$.

Lemma 1. *Let* $\nu \in S(\gamma', \gamma)$ *and* $\int_{\mathbf{R}} |x| e^{\gamma x}\, |\nu|(dx) < \infty$*. If* $T(\gamma)\nu \in \mathfrak{Sl}(\gamma', \tau)$ *and the restriction of* ν *to* $[0, \infty)$ *is a non-negative (or non-positive) measure, then* $\nu \in \mathfrak{Sl}(\gamma', \tau)$*; moreover,* $\mathfrak{l}(\nu) = 0$*.*

We shall need the following result on the values of an analytic function at elements of the algebras $\mathfrak{Sl}(\gamma', \tau)$, $\mathfrak{S}(\gamma', \tau)$, $\mathfrak{So}(\gamma', \tau)$ (see [25, Theorem 3 and Remark 2]).

Theorem 2. *Let* $f(z)$ *be an analytic function in a domain* $\mathcal{D} \subset \mathbf{C}$ *containing the spectrum* $\sigma(\nu)$ *of an element* $\nu \in S(\gamma', \gamma)$*, and let* $f(\nu) \in S(\gamma', \gamma)$ *be the value of* $f(z)$ *at* $\nu \in S(\gamma', \gamma)$*. If* $\nu \in \mathfrak{Sl}(\gamma', \tau)$*, then* $f(\nu) \in \mathfrak{Sl}(\gamma', \tau)$ *and the following equality holds:* $\mathfrak{l}[f(\nu)] = f'[\hat{\nu}(\gamma)] \cdot \mathfrak{l}(\nu)$*. If* $\nu \in \mathfrak{S}(\gamma', \tau)$ $(\mathfrak{So}(\gamma', \tau))$*, then* $f(\nu) \in \mathfrak{S}(\gamma', \tau)$ $(\mathfrak{So}(\gamma', \tau))$*.*

Proofs of the following two theorems can be found in [33].

Theorem 3. *Let* $\varphi(x)$, $x \in \mathbf{R}$*, be a submultiplicative function such that* $r_-(\varphi) < r_+(\varphi)$*. Suppose the function* $\varphi(x)/\exp[r_+(\varphi)x]$, $x \ge 0$*, is non-decreasing and* $\varphi(x)/\exp[r_-(\varphi)x]$, $x \le 0$*, is non-increasing. Assume* $\nu \in S(\varphi)$ *and let* β *be an interior point of* $\Pi(\varphi)$*. Then* $\hat{\kappa}(s) \stackrel{\text{def}}{=} [\hat{\nu}(s) - \hat{\nu}(\beta)]/(s - \beta)$, $s \in \Pi(\varphi)$*, is the Laplace transform of a measure* $\kappa \in S(\varphi)$*.*

If β lies on the boundary of $\Pi(\varphi)$, the situation becomes more involved. Nevertheless, the following theorem holds (for the sake of definiteness we consider the case $\Re\beta = r_+(\varphi)$).

Theorem 4. *Let $\varphi(x)$, $x \in \mathbf{R}$, be a submultiplicative function. Suppose that the function $\varphi(x)/\exp[r_+(\varphi)x]$, $x \geq 0$, is non-decreasing and $\varphi(x)/\exp[r_-(\varphi)x]$, $x \leq 0$, is non-increasing. Assume that*

$$\int_0^\infty (1+x)\varphi(x)\,|\nu|(dx) < \infty \quad or \quad \int_{\mathbf{R}} (1+|x|)\varphi(x)\,|\nu|(dx) < \infty, \tag{6}$$

depending on whether $r_-(\varphi) < r_+(\varphi)$ or $r_-(\varphi) = r_+(\varphi)$. Let $\Re\beta = r_+(\varphi)$. Then $\hat{\kappa}(s) \stackrel{\text{def}}{=} [\hat{\nu}(s) - \hat{\nu}(\beta)]/(s-\beta)$, $s \in \Pi(\varphi)$, is the Laplace transform of a measure $\kappa \in S(\varphi)$.

It will be convenient to have at our disposal a specific case of Theorem 4. It corresponds to $\varphi(x) \stackrel{\text{def}}{=} \exp(\gamma' x)$, $x < 0$, and $\varphi(x) \stackrel{\text{def}}{=} (1+x)^{k-1}\exp(\gamma x)$, $x \geq 0$.

Corollary 1. *Let $\nu \in S(\gamma', \gamma)$ and $\int_{\mathbf{R}} |x|^k e^{\gamma x}\,|\nu|(dx) < \infty$ for $k \geq 1$. Then $T(\gamma)\nu \in S(\gamma', \gamma)$ and $\int_{\mathbf{R}} |x|^{k-1} e^{\gamma x}\,|T(\gamma)\nu|(dx) < \infty$.*

The absolutely continuous component of an arbitrary distribution F will be denoted by F_c and its singular component, by F_s: $F_s = F - F_c$.

3. Exact asymptotic behaviour

Let $\gamma > 0$ and $\hat{F}(\gamma) < \infty$. Suppose that the set $\mathcal{Z}$ of the roots of the characteristic equation $1 - \hat{F}(s) = 0$ which lie in the strip $\{0 < \Re s \leq \gamma\}$ is finite. We do not exclude the case $\mathcal{Z} = \emptyset$. Denote the elements of $\mathcal{Z}$ by $s_1, s_2, \ldots, s_l$, and the multiplicity of s_j by m_j; this means that $1 - \hat{F}(s) = (s - s_j)^{m_j} F_j(s)$, where $F_j(s_j) \neq 0$. If $s \in \mathcal{Z}$, then $\overline{s} \in \mathcal{Z}$ and $\overline{s}$ has the same multiplicity as s.

Proposition 1. *For every $G \in \mathcal{S}(\gamma)$ with $\gamma > 0$, there exists $F \in \mathcal{S}(\gamma)$ such that $\mathcal{Z} \neq \emptyset$ and $F((x,\infty)) \sim G((x,\infty))$ as $x \to \infty$. In other words: as far as the tail behaviour is concerned, the subclass of all $F \in \mathcal{S}(\gamma)$ with non-empty $\mathcal{Z}$ is as rich as the class $\mathcal{S}(\gamma)$ itself.*

Proof. Let $G \in \mathcal{S}(\gamma)$, $\gamma > 0$. Suppose that we have found an absolutely continuous probability distribution F_0 on $[0,\infty)$ such that $\hat{F}_0(\gamma) < \infty$ and the set $\mathcal{Z}_0$ corresponding to F_0 is non-empty. Let $p > 0$ be a sufficiently small number to be chosen later. Take $B > A > 0$ such that

$$\int_A^\infty e^{\gamma x}\,F_0(dx) = p, \qquad \int_B^\infty e^{\gamma x}\,G(dx) < p \quad \text{and} \quad F_0((A,\infty)) \geq G((B,\infty)).$$

Put $F \stackrel{\text{def}}{=} F_0|_{[0,A]} + G|_{(B,\infty)} + c\delta$, where $c \stackrel{\text{def}}{=} F_0((A,\infty)) - G((B,\infty)) < p$. Then F is a probability distribution with $\hat{F}(s) = \int_0^A e^{sx}\,F_0(dx) + \int_B^\infty e^{sx}\,G(dx) + c$. Since $F((x,\infty)) = G((x,\infty))$ $\forall x \geq B$, we have $F \in \mathcal{S}(\gamma)$ [12, Theorem 2.7].

Now take a simple contour Γ lying entirely in $\{\Re s < \gamma\}$ such that a non-empty subset of $\mathcal{Z}_0$ is inside Γ and $\mathcal{Z}_0 \cap \Gamma = \emptyset$. Let $\Delta \stackrel{\text{def}}{=} \min_{s\in\Gamma} |1 - \hat{F}_0(s)| > 0$. Put $p = \Delta/3$. Then, for $s \in \Gamma$,

$$|\hat{F}(s) - \hat{F}_0(s)| \leq \int_A^\infty e^{\gamma x}\,F_0(dx) + \int_B^\infty e^{\gamma x}\,G(dx) + c < \Delta.$$

By Rouché's theorem [37, Section 3.4], the functions $1 - \hat{F}(s)$ and $1 - \hat{F}_0(s)$ have the same number of zeros inside Γ. Hence the set $\mathcal{Z}$ corresponding to F is non-empty. Finally, it remains to produce an F_0 with the above properties. This is easily done by modifying an example due to V. A. Topchii [21]. The function

$$u_0(\lambda) = \frac{\lambda[\lambda^2 - 2(1-\varepsilon)\lambda + 1 - \varepsilon]^2}{(\lambda - 1)^5}$$

has zeros at $\lambda = 1-\varepsilon \pm i\sqrt{\varepsilon(1-\varepsilon)}$, and, for small ε (e.g. $\varepsilon = 0.05$), the function $1-u_0(\lambda)$, $\Re\lambda < 1$, is the Laplace transform of a non-negative absolutely continuous measure [21]. If we scale the argument of $1-u_0(\lambda)$, we obtain the Laplace transform $\hat{F}_0(s)$ of the desired F_0: $\hat{F}_0(s) \stackrel{\text{def}}{=} 1-u_0(as)$, $\Re s < 1/a$, where $a = (1-\varepsilon/2)/\gamma$. $\square$

Put

$$\hat{\Phi}_n(s) \stackrel{\text{def}}{=} \frac{n!}{[1-\hat{F}(s)]^n}, \qquad s \in \{0 \le \Re s \le \gamma\} \setminus (\mathcal{Z} \cup \{0\}).$$

Let $s_j \in \mathcal{Z}$ and $\int_0^\infty (1+x)^{(n+1)m_j} e^{\Re s_j x} F(dx) < \infty$. Define the coefficients $B_{jk}^{(n)}$, $k = 1, \ldots, nm_j$, by the asymptotic expansion

$$\hat{\Phi}_n(s) = \sum_{k=1}^{nm_j} (-1)^k \frac{B_{jk}^{(n)}}{(s-s_j)^k} + o\left(\frac{1}{s-s_j}\right) \qquad \text{as } s \to s_j. \tag{7}$$

It is clear that, for each fixed k, the coefficient $B_{jk}^{(n)}$ can be expressed explicitly in terms of the moments $\int_{\mathbf{R}} x^p e^{s_j x} F(dx)$ or, which is equivalent, in terms of the derivatives $[\hat{F}(s)]^{(p)}_{s=s_j}$. For instance, $m_j = 1 \Rightarrow B_{jn}^{(n)} = n!/[\hat{F}'(s_j)]^n$. However, since we consider the general case of arbitrary multiplicities m_j, there seems to be no acceptable formula for expressing the $B_{jk}^{(n)}$ for *all* k in terms of the above moments, so relation (7) may be regarded as an appropriate way of defining $B_{jk}^{(n)}$. Let $E|X_1|^{m+1} < \infty$. Let the coefficients $\gamma_k^{(n)}$, $k = n-m$, $n-m+1$, $\ldots$, n, be defined by

$$\hat{\Phi}_n(s) = \sum_{k=n-m}^{n} (-1)^k \frac{\gamma_k^{(n)}}{s^k} + o\left(\frac{1}{s^{n-m}}\right) \qquad \text{as } s \to 0. \tag{8}$$

Relation (8) will be used for the values $m = n-1$ and $m = n$. Denote by $\mathcal{E}_j$ the complex-valued measure with density $\mathbf{1}_{(0,\infty)}(x)\exp(-s_j x)$ ($\mathbf{1}_A(x)$ is the indicator of A). The Laplace transform of $\mathcal{E}_j$ is equal to $1/(s_j - s)$, $\Re(s-s_j) < 0$. Denote by L the restriction of Lebesgue measure to $(0,\infty)$.

Theorem 5. *Let $\{X_i\}_{i=1}^\infty$ be a sequence of independent identically distributed random variables with distribution F. Suppose $0 < \mu \stackrel{\text{def}}{=} EX_1 < \infty$, $E(X_1^-)^n < \infty$ and $\hat{F}(\gamma) < \infty$ for $\gamma > 0$. Let Φ_n be defined by (1), where $n > 0$ is an integer.*

Suppose $(F^{m*})^\wedge_s(\gamma) < 1$ for some integer $m \ge 1$ and $\hat{F}(s) \ne 1$ for $\Re s = \gamma$. Let s_j be the roots of the equation $1-\hat{F}(s) = 0$ lying in the strip $\{0 < \Re s \le \gamma\}$ and having multiplicities m_j, $j = 1, \ldots, l$. If $F \in \mathfrak{S}\mathfrak{l}(\tau)$, then the following representation holds:

$$\Phi_n = \sum_{k=1}^{n} \gamma_k^{(n)} L^{k*} + \sum_{j=1}^{l}\sum_{k=1}^{nm_j} B_{jk}^{(n)} \mathcal{E}_j^{k*} + \mathcal{R}_n, \tag{9}$$

where $\mathcal{R}_n^+ \stackrel{\text{def}}{=} \mathcal{R}_n|_{[0,\infty)} \in \mathfrak{S}\mathfrak{l}(\tau)$; moreover,

$$\mathfrak{l}(\mathcal{R}_n^+) = \lim_{x\to\infty} \frac{\mathcal{R}_n((x,\infty))}{\tau(x)} = \frac{n \cdot n!\, \mathfrak{l}(F)}{[1-\hat{F}(\gamma)]^{n+1}}. \tag{10}$$

Conversely, if (9) holds for $n = 1$ with $B_{jm_j}^{(1)} \ne 0$, $j = 1, \ldots, l$, and $\mathcal{R}_1^+ \in \mathfrak{S}\mathfrak{l}(\tau)$, then $F \in \mathfrak{S}\mathfrak{l}(\tau)$, $(F^{m*})^\wedge_s(\gamma) < 1$ for some $m \ge 1$ and $\hat{F}(s) \ne 1$ for $\Re s = \gamma$; moreover, the s_j are the roots of $1-\hat{F}(s) = 0$ lying in the strip $\{0 < \Re s \le \gamma\}$ and having multiplicities m_j.

Remark 3. For every $s_i \in \mathcal{Z}$, there exists $s_j \in \mathcal{Z}$ such that $\overline{s_j} = s_i$ and $m_j = m_i$; moreover, $B_{ik}^{(n)} = \overline{B_{jk}^{(n)}}$ and $\mathcal{E}_i = \overline{\mathcal{E}_j}$. Therefore, the double sum in (9) is a *real-valued* measure.

Proof of Theorem 5. The conditions $(F^{m*})_s^{\wedge}(\gamma) < 1$ and $\hat{F}(s) \neq 1$ for $\Re s = \gamma$ imply that the set $\mathcal{Z}$ is finite [17]. Choose $r > \gamma$. Put $p = \sum_{j=1}^{l} m_j + 1$ and

$$v(s) \stackrel{\text{def}}{=} \frac{[1 - \hat{F}(s)](s-r)^p}{s \prod_{j=1}^{l}(s-s_j)^{m_j}}, \qquad 0 \leq \Re s \leq \gamma,$$

defining the values of $v(s)$ at points $s \in \mathcal{Z} \cup \{0\}$ by continuity. We show that $v(s)$ is the Laplace transform $\hat{V}(s)$ of a real-valued measure $V \in \mathfrak{Sl}(\tau)$. By representing a rational function as a sum of partial fractions, we have

$$v(s) = [1 - \hat{F}(s)] \left[1 + \frac{c}{s} + \sum_{j=1}^{l} \sum_{k=1}^{m_j} \frac{C_{jk}}{(s-s_j)^k} \right],$$

where c, C_{jk} are constants. Consider the ratio $[\hat{F}(s) - 1]/(s-s_j)^k$ for $k \leq m_j$. By Theorem 1, this expression is the Laplace transform of the measure $T(s_j)^k F$, which belongs to $\mathfrak{Sl}(\tau)$. Similarly, $[\hat{F}(s) - 1]/s$ is the Laplace transform of $T(F) \in \mathfrak{Sl}(\tau)$. Hence $v(s) = \hat{V}(s)$, where $V \in \mathfrak{Sl}(\tau)$. By Theorem 1,

$$\mathfrak{l}(V) = -\mathfrak{l}(F) \left[1 + \frac{c}{\gamma} + \sum_{j=1}^{l} \sum_{k=1}^{m_j} \frac{C_{jk}}{(\gamma - s_j)^k} \right] = \frac{-\mathfrak{l}(F)(\gamma - r)^p}{\gamma \prod_{j=1}^{l}(\gamma - s_j)^{m_j}}.$$

As shown in the proof of Lemma 2 in [32], there exists an inverse $V^{-1} \in S(0, \gamma)$. Notice that if $E(X_1^-)^n < \infty$, we have $\int_{-\infty}^{0} |x|^{n-1} F(x)\, dx < \infty$ (which implies $\int_{-\infty}^{0} |x|^{n-1} |V|(dx) < \infty$) and $\int_{-\infty}^{0} |x|^{n-1} |V^{-1}|(dx) < \infty$. If $E(X_1)^{n+1} < \infty$, then $\int_{-\infty}^{0} |x|^{n} |V^{-1}|(dx) < \infty$. By Theorem 2 with $f(z) = 1/z$, we have $V^{-1} \in \mathfrak{Sl}(\tau)$ and

$$\mathfrak{l}(V^{-1}) = \frac{\mathfrak{l}(F)\gamma \prod_{j=1}^{l}(\gamma - s_j)^{m_j}}{[1 - \hat{F}(\gamma)]^2 (\gamma - r)^p}.$$

Put $W \stackrel{\text{def}}{=} (V^{-1})^{n*}$. If we set in Theorem 2 $f(z) = z^n$, then we obtain

$$\mathfrak{l}(W) = \frac{n\mathfrak{l}(F)[\gamma \prod_{j=1}^{l}(\gamma - s_j)^{m_j}]^n}{[1 - \hat{F}(\gamma)]^{n+1} (\gamma - r)^{pn}}.$$

We now show that for $s \in \{0 \leq \Re s \leq \gamma\} \setminus (\mathcal{Z} \cup \{0\})$, the following equality holds:

$$\hat{\Phi}_n(s) = \sum_{k=1}^{n} (-1)^k \frac{\gamma_k^{(n)}}{s^k} + \sum_{j=1}^{l} \sum_{k=1}^{n m_j} (-1)^k \frac{B_{jk}^{(n)}}{(s-s_j)^k} + \hat{Q}(s) + \frac{\hat{r}_n(s) - \hat{r}_n(0)}{s}, \tag{11}$$

where $\hat{Q}(s)$ and $\hat{r}_n(s)$ are the Laplace transform of some Q and r_n such that $Q \in \mathfrak{Sl}(\tau)$ and r_n is a finite measure concentrated on $(-\infty, 0)$. We have $\hat{W}(s) = 1/v(s)^n \stackrel{\text{def}}{=} w(s)$ and

$$\hat{\Phi}_n(s) = w(s) \frac{n!\,(s-r)^{np}}{s^n \prod_{j=1}^{l}(s-s_j)^{n m_j}} = w(s) \left[n! + \sum_{k=1}^{n} \frac{a_k}{s^k} + \sum_{j=1}^{l} \sum_{k=1}^{n m_j} \frac{d_{jk}}{(s-s_j)^k} \right].$$

Next,

$$\frac{w(s)}{(s-s_j)^k} = \frac{w(s_j)}{(s-s_j)^k} + \frac{w(s) - w(s_j)}{(s-s_j)^k} = \sum_{i=0}^{k-1} \frac{w_{i,j}(s_j)}{(s-s_j)^{k-i}} + w_{k,j}(s),$$

where $w_{0,j}(s) \stackrel{\text{def}}{=} w(s)$, $w_{p,j}(s) \stackrel{\text{def}}{=} [w_{p-1,j}(s) - w_{p-1,j}(s_j)]/(s - s_j)$, $p = 1, \dots, k$. By Theorem 1, $w_{p,j}(s)$ is the Laplace transform of $W_{p,j} \stackrel{\text{def}}{=} T(s_j)^p W \in \mathfrak{S}\mathfrak{l}(\tau)$ and $\mathfrak{l}(W_{p,j}) = \mathfrak{l}(W)/(\gamma - s_j)^p$. By the uniqueness of the expansion (7),

$$w(s) \sum_{j=1}^{l} \sum_{k=1}^{nm_j} \frac{d_{jk}}{(s-s_j)^k} = \sum_{j=1}^{l} \sum_{k=1}^{nm_j} (-1)^k \frac{B_{jk}^{(n)}}{(s-s_j)^k} + \hat{q}_n(s),$$

where $\hat{q}_n(s)$ is the Laplace transform of some measure $q_n \in \mathfrak{S}\mathfrak{l}(\tau)$. Similarly, by the uniqueness of the expansion (8),

$$w(s) \sum_{k=1}^{n} \frac{a_k}{s^k} = \sum_{k=1}^{n} (-1)^k \frac{\gamma_k^{(n)}}{s^k} + \sum_{k=1}^{n} a_k w_k(s),$$

where $w_0(s) \stackrel{\text{def}}{=} w(s)$, $w_i(s) \stackrel{\text{def}}{=} [w_{i-1}(s) - w_{i-1}(0)]/s$, $i = 1, \dots, n$. Applying successively Theorem 1 and Corollary 1, we conclude that $w_k(s)$ is the Laplace transform of the measure $W_k \stackrel{\text{def}}{=} T^k W \in \mathfrak{S}\mathfrak{l}(\tau)$, $\mathfrak{l}(W_k) = \mathfrak{l}(W)/\gamma^k$ and $\int_{-\infty}^{0} |x|^{n-k-1} |W_k|(dx) < \infty$ for $k = 1, \dots, n-1$. It is clear that $W_n|_{[0,\infty)} = T\left(W_{n-1}|_{[0,\infty)}\right) \in \mathfrak{S}\mathfrak{l}(\tau)$. However, $W_n|_{(-\infty,0)}$ is, generally speaking, a σ-*finite* measure. (If $E(X^-)^{n+1} < \infty$, then $W_n|_{(-\infty,0)}$ is a finite measure and W_n is an element of $\mathfrak{S}\mathfrak{l}(\tau)$.) Put $Q \stackrel{\text{def}}{=} n!\,W + q_n + \sum_{k=1}^{n-1} a_k W_k + a_n W_n|_{[0,\infty)}$, $r_n \stackrel{\text{def}}{=} a_n W_{n-1}|_{(-\infty,0)}$ and $\mathcal{R}_n \stackrel{\text{def}}{=} Q + T(r_n)$. This proves (11). Taking into account the values of the functional $\mathfrak{l}$ at $W_{p,j}$ and W_k, we have

$$\lim_{x\to\infty} \frac{\mathcal{R}_n((x,\infty))}{\tau(x)} = \mathfrak{l}(Q) = \mathfrak{l}(W) \left[n! + \sum_{k=1}^{n} \frac{a_k}{\gamma^k} + \sum_{j=1}^{l} \sum_{k=1}^{nm_j} \frac{d_{jk}}{(\gamma - s_j)^k} \right]$$

$$= \mathfrak{l}(W) \frac{n!\,(\gamma - r)^{pn}}{[\gamma \prod_{j=1}^{l} (\gamma - s_j)^{m_j}]^n} = \frac{n \cdot n!\,\mathfrak{l}(F)}{[1 - \hat{F}(\gamma)]^{n+1}}.$$

If $X_i \geq 0$ a.s., then relation (11) is also true for $\Re s < 0$; moreover, $\hat{\Phi}_n(s)$, $\Re s < 0$, is the Laplace transform of Φ_n. Therefore, if we go over from the Laplace transforms in (11) to the corresponding measures, then we obtain (9) with the desired asymptotic behaviour of the remainder. In case the X_i are *real-valued*, the transition from (11) to (9) is carried out by means of the theory of generalized functions, as in the proof of Theorem 4 in [32].

We now prove the converse statement. Suppose that (9) holds for $n = 1$ and $\mathcal{R}_1^+ \in \mathfrak{S}\mathfrak{l}(\tau)$. First, we note that the condition $(F^{m*})_s^{\wedge}(\gamma) < 1$ is necessary for $\int_0^\infty e^{\gamma x} |\mathcal{R}_1|(dx) < \infty$ [32, Remark 2]. Further, $\mathcal{R}_1^- \stackrel{\text{def}}{=} \mathcal{R}_1|_{(-\infty,0)} = \nu - T(\nu)$, where ν is a finite measure on $(-\infty, 0)$. Actually, F^{m*} has a non-null absolutely continuous component. Therefore, if we take the function $v(s) \stackrel{\text{def}}{=} [1 - \hat{F}(s)](s-1)/s$, $\Re s = 0$, as starting point and repeat the arguments of the proof of Theorem 4 of [32] in the context of the Banach algebra of finite measures, then we obtain $1/[1 - \hat{F}(s)] = -1/(\mu s) + w(s) - w_1(s)$, where $w(s)$ is the Laplace transform of a finite measure W and $w_1(s) = [w(s) - w(0)]/s$. If we go over to measures (as was done in [32, the proof of Theorem 4]), then we get $\Phi_1 = L/\mu + W - T(W)$. Consequently, $\mathcal{R}_1^- = \Phi_1|_{(-\infty,0)} = \nu - T(\nu)$, where $\nu = W|_{(-\infty,0)}$ is a finite measure.

Our next goal is to establish a relation of the form (11), taking (9) for $n = 1$ as starting point.

Lemma 2. *For all imaginary $s \neq 0$,*

$$\frac{1}{1-\hat{F}(s)} = -\frac{1}{\mu s} + \sum_{j=1}^{l}\sum_{k=1}^{m_j}(-1)^k \frac{B_{jk}^{(1)}}{(s-s_j)^k} + \hat{\mathcal{R}}_1^+(s) + \hat{\nu}(s) - \frac{\hat{\nu}(s)-\hat{\nu}(0)}{s}. \tag{12}$$

Proof of Lemma 2. Denote by $\mathcal{S}_1$ the space of rapidly decreasing functions in $\mathbf{R}$ and by $\mathcal{S}_1'$ the dual space (the space of tempered distributions) [26, Chapter 7]. The measures which appear in (9) belong to $\mathcal{S}_1'$ [26, Section 7.12]. Denote by $\mathcal{F}(u)$ the Fourier transform of $u \in \mathcal{S}_1'$: $\mathcal{F}(u)(\psi) \overset{\text{def}}{=} u(\mathcal{F}(\psi))$, $\psi \in \mathcal{S}_1$, where

$$\mathcal{F}(\psi)(t) \overset{\text{def}}{=} (2\pi)^{-1/2}\int_{\mathbf{R}} \psi(x)\exp(-itx)\,dx, \qquad t \in \mathbf{R}.$$

Let ν be a σ-finite measure defining an element in $\mathcal{S}_1'$. For arbitrary $a \in \mathbf{R}$, we set $\nu_a(A) = \nu(A-a)$, $A \in \mathcal{B}$. Define the element $\Delta a\nu \in \mathcal{S}_1'$ by $\Delta a\nu \overset{\text{def}}{=} \nu - \nu_a$. Then $\mathcal{F}(\Delta a\nu) = [1-\exp(-iat)]\mathcal{F}(\nu)$. If ν and κ are any two measures which define tempered distributions and for which the convolution $\nu * \kappa$ makes sense, then obviously $\Delta a(\nu * \kappa) = \nu * (\Delta a\kappa) = (\Delta a\nu) * \kappa$. It is also clear that ΔaL is Lebesgue measure on the interval $[0,a]$ and, therefore, the tempered distribution $\mathcal{F}(\Delta aL)$ can be identified with the function $[1-\exp(-iat)]/[it(2\pi)^{1/2}]$, $t \in \mathbf{R}$.

Apply successively the operator Δa and the Fourier transform to both sides of (9) with $n=1$. For an arbitrary element $\psi \in \mathcal{S}_1$, we have

$$\mathcal{F}(\Delta a\Phi_1)(\psi) = \frac{1}{\mu}\mathcal{F}(\Delta aL)(\psi) + \sum_{j=1}^{l}\sum_{k=1}^{m_j} B_{jk}^{(1)}\mathcal{F}(\Delta a\mathcal{E}_j^{k*})(\psi) + \mathcal{F}(\Delta a\mathcal{R}_1^+)(\psi) + \mathcal{F}(\Delta a\mathcal{R}_1^-)(\psi).$$

By Lemma 4 of [32], the left-hand side equals

$$(2\pi)^{-1/2}\int_{\mathbf{R}}[1-\exp(-iat)][1-\hat{F}(-it)]^{-1}\psi(t)\,dt.$$

Further,

$$\mathcal{F}(\Delta a\mathcal{E}_j^{k*})(\psi) = (2\pi)^{-1/2}\int_{\mathbf{R}}[1-\exp(-iat)](it+s_j)^{-k}\psi(t)\,dt,$$
$$\mathcal{F}(\Delta a\mathcal{R}_1^+)(\psi) = (2\pi)^{-1/2}\int_{\mathbf{R}}[1-\exp(-iat)]\hat{\mathcal{R}}_1^+(-it)\psi(t)\,dt,$$
$$\mathcal{F}(\Delta a\mathcal{R}_1^-)(\psi) = (2\pi)^{-1/2}\int_{\mathbf{R}}[1-\exp(-iat)]\hat{\mathcal{R}}_1^-(-it)\psi(t)\,dt$$

(the last equality being a consequence of Lemma 3 in [32]; here $\hat{\mathcal{R}}_1^-(s) \overset{\text{def}}{=} \hat{\nu}(s) - [\hat{\nu}(s) - \hat{\nu}(0)]/s$). Thus, equality (12) holds on the line $\{\Re s = 0\}$ almost everywhere with respect to Lebesgue measure. In view of the continuity of all the functions involved, the equality holds for all s such that $\Re s = 0$ and $s \neq 0$. Lemma 2 is proved. □

Lemma 3. *The points $s_1, \ldots, s_l$ are roots of the characteristic equation with multiplicities $m_1, \ldots, m_l$, respectively, and $\hat{F}(s) \neq 1$ for $\Re s = \gamma$.*

Proof of Lemma 3. We show that equality (12) holds in the whole strip $\{0 \leq \Re s \leq \gamma\}$ except for the points $s_1, \ldots, s_l$ and the origin.

Definition 3. Let $f(z)$ be a meromorphic function defined in a domain G with rectifiable boundary Γ. An *angular boundary value* of $f(z_0)$ at point $z_0 \in \Gamma$ is called a value to which $f(z)$ tends when $z \to z_0 \in \Gamma$ along all non-tangential paths [23, Chapter IV, Section 4.4].

Both sides of (12) — we denote them by $f(s)$ and $g(s)$ — are meromorphic functions in the strip $\{0 < \Re s < \gamma\}$. The set $\{\Re s = 0\} \setminus \{0\}$ is a part of the boundary which has positive Lebesgue measure. The functions $f(s)$ and $g(s)$ are continuous and take the same values on $\{\Re s = 0\} \setminus \{0\}$. Thus, the functions have the same angular boundary values on $\{\Re s = 0\} \setminus \{0\}$ [23, Chapter IV, Section 4.4]. The Lusin-Privalov theorem [23, Chapter IV, Section 2.5] states that if two functions $f_1(z)$ and $f_2(z)$, meromorphic in the unit disk, have the same angular boundary values on a set of positive Lebesgue measure, then $f_1(z) \equiv f_2(z)$. By the conformal mapping theorem, the Lusin-Privalov theorem also holds for the rectangle $\{0 < \Re s < \gamma, |\Im s| < m\}$ with $m > 0$. Hence $f(s) \equiv g(s)$ in $\{0 < \Re s < \gamma, |\Im s| < m\}$. Letting $m \to \infty$, we obtain $f(s) \equiv g(s)$ in the strip $\{0 < \Re s < \gamma\}$. It follows that the function $1/[1-\hat{F}(s)]$ has poles at the points s_j (and only at these points) with multiplicities m_j or, which is the same, the points s_j are the roots of the characteristic equation with multiplicities m_j, $j = 1, \ldots, l$. The function $g(s)$ is defined and continuous in the whole strip $\{0 \leq \Re s \leq \gamma\}$ except for the points $s_1, \ldots, s_l$ and the origin. Moreover, $g(s)$ is bounded on $\{\Re s = \gamma\}$. Hence $\hat{F}(s) \neq 1$ for $\Re s = \gamma$. Lemma 3 is proved. $\square$

We now show that $F \in \mathfrak{Sl}(\tau)$. Choose $\beta > \gamma$ and multiply both sides of (12) by $q(s) \stackrel{\text{def}}{=} s\prod_{j=1}^{l}(s-s_j)^{m_j}/(s-\beta)^p$ with $p = \sum_{j=1}^{l} m_j + 1$. We obtain

$$
\begin{aligned}
(13) \quad w(s) \stackrel{\text{def}}{=} \frac{s\prod_{j=1}^{l}(s-s_j)^{m_j}}{[1-\hat{F}(s)](s-\beta)^p} &= \frac{P(s)}{(s-\beta)^p} + \hat{\mathcal{R}}_1^+(s)q(s) \\
&\quad + \hat{\nu}(s)q(s) - [\hat{\nu}(s) - \hat{\nu}(0)]\frac{\prod_{j=1}^{l}(s-s_j)^{m_j}}{(s-\beta)^p} \\
&\stackrel{\text{def}}{=} a_1(s) + a_2(s) + a_3(s) - a_4(s),
\end{aligned}
$$

where $P(s)$ is a polynomial of degree $p-1$. If we represent $a_1(s)$ as a sum of partial fractions, then we see that $a_1(s)$ is the Laplace transform of a measure of the form $\sum_{k=1}^{p} \alpha_k \mathcal{E}_\beta^{k*}$ belonging to $\mathfrak{Sl}(\tau)$. Actually, property (b) of Definition 1 implies $\lim_{x\to\infty} e^{-\beta x}/\tau(x) = 0$. It follows that $\mathcal{E}_\beta \in \mathfrak{Sl}(\tau)$ and $\mathfrak{l}(\mathcal{E}_\beta) = 0$. Similarly, the functions $a_i(s)$, $i = 2, 3, 4$, are the Laplace transforms of measures in $\mathfrak{Sl}(\tau)$. Thus, $w(s)$ is the Laplace transform of a measure $W \in \mathfrak{Sl}(\tau)$.

Choose $\gamma' \in (0, \min_{1\leq j\leq l} \Re s_j)$. We show that W has an inverse in $\mathfrak{Sl}(\gamma', \tau)$. We have

$$
\int_{-\infty}^{0} \exp(\gamma' x)\, |\mathcal{R}_1|(dx) = \int_{-\infty}^{0} \exp(\gamma' x)\, \Phi_1(dx) < \infty
$$

[32, Remark 2]. Therefore, $\mathcal{R}_1^+ \in \mathfrak{Sl}(\tau) \Rightarrow \mathcal{R}_1 \in \mathfrak{Sl}(\gamma', \tau)$. The space $\mathcal{M}$ of maximal ideals of the Banach algebra $S(\gamma', \gamma)$ is split into two sets: $\mathcal{M}_1$ is the set of maximal ideals not containing the collection of all absolutely continuous measures in $S(\gamma', \gamma)$, and $\mathcal{M}_2 = \mathcal{M} \setminus \mathcal{M}_1$. If $M \in \mathcal{M}_1$, then the homomorphism $S(\gamma', \gamma) \to \mathbf{C}$ generated by M is of the form $\nu \to \hat{\nu}(s_0)$, where s_0 is a complex number such that $\gamma' \leq \Re s_0 \leq \gamma$. In this case, $M = \{\nu \in S(\gamma', \gamma) : \hat{\nu}(s_0) = 0\}$ [18, Chapter IV, Section 4]. We recall that each $M \in \mathcal{M}$ induces a homomorphism $S(\gamma', \gamma) \to \mathbf{C}$ with the kernel M. We denote by $\nu(M)$ the value of the homomorphism at $\nu \in S(\gamma', \gamma)$. If $M \in \mathcal{M}_2$, then $\nu(M) = 0$ for every absolutely continuous measure $\nu \in S(\gamma', \gamma)$. If $M \in \mathcal{M}_1$, then $W(M) = w(s_0)$ for some $s_0 \in \{\gamma' \leq \Re s \leq \gamma\}$, and hence $W(M) \neq 0$. Let $M \in \mathcal{M}_2$. As shown in [32, the proof of Lemma 2], $|F(M)| < 1$. It follows from (13) that

$$
W(M) = (\mathcal{R}_1)_s(M) = \sum_{m=0}^{\infty} (F^{m*})_s(M) = \sum_{m=0}^{\infty} (F^{m*})(M) = \frac{1}{1-F(M)} \neq 0.
$$

Thus, $W(M) \neq 0$ for every $M \in \mathcal{M}$. In other words, $W \notin M$ for every $M \in \mathcal{M}$. Since each maximal ideal M of $\mathfrak{Sl}(\gamma', \tau)$ is representable in the form $M = M_1 \cap \mathfrak{Sl}(\gamma', \tau)$, where M_1 is a maximal ideal of $S(\gamma', \gamma)$ (see [25, Theorem 2, Remark 2]), we conclude that W belongs to no maximal ideal of $\mathfrak{Sl}(\gamma', \tau)$. This means that there exists an inverse $V \overset{\text{def}}{=} W^{-1} \in \mathfrak{Sl}(\gamma', \tau)$ with

$$\hat{V}(s) = \frac{[1 - \hat{F}(s)](s - r)^p}{s \prod_{j=1}^{l} (s - s_j)^{m_j}}.$$

We have $1 - \hat{F}(s) = \hat{V}(s) q(s)$. If we represent $q(s)$ as a sum of partial fractions, then we obtain $1 - \hat{F}(s) = \hat{V}(s)[1 + \sum_{k=1}^{p} (-1)^k \beta_k / (s - r)^k]$. Since $V * \mathcal{E}_r^{k*} \in \mathfrak{Sl}(\gamma', \tau)$, we get $\delta - F \in \mathfrak{Sl}(\gamma', \tau)$. Finally, $F \in \mathfrak{Sl}(\tau)$ since F is a finite measure (see Remark 2). This completes the proof of Theorem 5. $\square$

The following generalization of Theorems 5 and 6 of [32] holds (the constant $\Gamma_0^{(n)}$ is the same as in [32, Theorem 6]).

Corollary 2. *Under the hypotheses of Theorem* 5*,*

$$\Phi_n([0, x]) = \Gamma_0^{(n)} + \sum_{k=1}^{n} \gamma_k^{(n)} \frac{x^k}{k!} - \sum_{j=1}^{l} \sum_{k=1}^{n m_j} B_{jk}^{(n)} \mathcal{E}_j^{k*}((x, \infty)) - \mathcal{R}_n((x, \infty)),$$

and if, additionally, $E(X_1^-)^{n+1} < \infty$*, then*

$$\Phi_n((-\infty, x]) = \sum_{k=0}^{n} \gamma_k^{(n)} \frac{x^k}{k!} - \sum_{j=1}^{l} \sum_{k=1}^{n m_j} B_{jk}^{(n)} \mathcal{E}_j^{k*}((x, \infty)) - \mathcal{R}_n((x, \infty)),$$

where $\gamma_0^{(n)}$ *is defined by* (8)*. In both cases,* $\mathcal{R}_n((x, \infty))$ *satisfies* (10)*.*

The proof of Corollary 2 almost coincides with the proofs of Theorems 5 and 6 of [32] and is therefore omitted.

Instead of studying the influence of the roots of $1 - \hat{F}(s) = 0$ on the asymptotic behaviour of Φ_n on the negative half-axis, it is convenient to consider the equivalent problem of studying the asymptotic behaviour of Φ_n on $[0, \infty)$ for random walks drifting to $-\infty$. This will allow us to avoid introducing new notation.

So let $S_k \to -\infty$ as $k \to \infty$ with probability one. Denote $x^+ = \max(0, x)$. It follows from [35, Corollary 5.2] that if $E(X_1^+)^{n+1} < \infty$, then $\Phi_n((t, \infty))$ is finite for all t, even if $EX_1 = -\infty$.

If $\mathcal{Z} \neq \emptyset$, then among the elements of $\mathcal{Z}$ there exists only one real root, say $q = s_1$, with multiplicity one. The terms of (14) corresponding to this root will yield the main contribution to the asymptotic behaviour of Φ_n. In the theorem below we do not exclude the possibility $\mathcal{Z} = \emptyset$. In this case we shall put $l = 0$ and interpret a sum of the form $\sum_{j=1}^{l}$ of the empty set of summands to be equal to zero; similarly, a product of the form $\prod_{j=1}^{l}$ of the empty set of factors will be regarded as equal to one.

Theorem 6. *Let* $\{X_i\}_{i=1}^{\infty}$ *be a sequence of independent identically distributed random variables with a non-arithmetic distribution* F *and let* $\{S_k\}_{k=0}^{\infty}$ *be a random walk such that with probability one* $S_k \to -\infty$ *as* $k \to \infty$*. Let* $\hat{F}(\gamma) < \infty$*,* $\gamma > 0$*, and let* Φ_n *be defined by* (1)*.*

Suppose $(F^{m*})_s^{\wedge}(\gamma) < 1$ *for some* $m \geq 1$ *and* $\hat{F}(s) \neq 1$ *for* $\Re s = \gamma$*. Let* s_j *be the roots of* $1 - \hat{F}(s) = 0$ *lying in* $\{0 < \Re s \leq \gamma\}$ *and having multiplicities* m_j*,* $j = 1, \ldots, l$*. If*

$F \in \mathfrak{Sl}(\tau)$, then the following representation holds:

$$\Phi_n = \sum_{j=1}^{l} \sum_{k=1}^{nm_j} B_{jk}^{(n)} \mathcal{E}_j^{k*} + \mathcal{R}_n, \tag{14}$$

where the coefficients $B_{jk}^{(n)}$ are defined by (7) and $\mathcal{R}_n \in \mathfrak{Sl}(\gamma', \tau)$ for some $\gamma' \in (0, \gamma)$; moreover,

$$\mathfrak{l}(\mathcal{R}_n) = \lim_{x\to\infty} \frac{\mathcal{R}_n((x,\infty))}{\tau(x)} = \frac{n \cdot n!}{[1-\hat{F}(\gamma)]^{n+1}} \cdot \lim_{x\to\infty} \frac{F((x,\infty))}{\tau(x)}. \tag{15}$$

Conversely, if (14) holds for $n = 1$ with $B_{jm_j}^{(1)} \neq 0$, $j = 1, \dots, l$, and $\mathcal{R}_1 \in \mathfrak{Sl}(\gamma', \tau)$, then $F \in \mathfrak{Sl}(\tau)$, $(F^{m*})_s^\wedge(\gamma) < 1$ for some $m \geq 1$ and $\hat{F}(s) \neq 1$ for $\Re s = \gamma$; moreover, the s_j are the roots of $1 - \hat{F}(s) = 0$ lying in $\{0 < \Re s < \gamma\}$ and having multiplicities m_j.

Proof. The proof of Theorem 6 is quite similar to that of Theorem 5, so we give only some hints. The set $\mathcal{Z}$ is finite. Choose $\gamma' > 0$ in such a way that $\mathcal{Z} \subset \{\gamma' < \Re s < \gamma\}$. Let $r > \gamma$. Put $p = \sum_{j=1}^{l} m_j$ and

$$v(s) \stackrel{\text{def}}{=} \frac{[1-\hat{F}(s)](s-r)^p}{\prod_{j=1}^{l}(s-s_j)^{m_j}}, \qquad \gamma' \leq \Re s \leq \gamma.$$

The function $v(s)$ is the Laplace transform of a measure $V \in \mathfrak{Sl}(\gamma', \tau)$, which has an inverse $V^{-1} \in \mathfrak{Sl}(\gamma', \tau)$. Set $W \stackrel{\text{def}}{=} (V^{-1})^{n*}$. We have

$$\hat{\Phi}_n(s) = \hat{W}(s) \frac{n!\,(s-r)^{np}}{[\prod_{j=1}^{l}(s-s_j)^{m_j}]^n} = \hat{W}(s) \left[n! + \sum_{j=1}^{l} \sum_{k=1}^{nm_j} \frac{d_{jk}}{(s-s_j)^k} \right].$$

If we perform the familiar calculations, then

$$\hat{\Phi}_n(s) = \sum_{j=1}^{l} \sum_{k=1}^{nm_j} \frac{(-1)^k B_{jk}^{(n)}}{(s-s_j)^k} + \hat{\mathcal{R}}_n(s), \tag{16}$$

where $\hat{\mathcal{R}}_n(s)$ is the Laplace transform of $\mathcal{R}_n \stackrel{\text{def}}{=} n!\,W + \sum_{j=1}^{l} \sum_{k=1}^{nm_j} d_{kj} W_{k,j}$. The measures $W_{k,j}$ are elements of $\mathfrak{Sl}(\gamma', \tau)$. Finally, we have $\mathcal{R}_n \in \mathfrak{Sl}(\gamma', \tau)$ and relation (15) holds. The measures $\mathcal{E}_j$ have Laplace transforms defined for $\Re s < q$. If we go over in (16) from Laplace transforms to measures, then we arrive at the desired expansion (14). (Note that this transition is carried out without invoking the theory of generalized functions.) $\square$

Corollary 3. *Under the hypotheses of Theorem 6,*

$$\Phi_n((x,\infty)) = \sum_{j=1}^{l} \sum_{k=1}^{nm_j} B_{jk}^{(n)} \mathcal{E}_j^{k*}((x,\infty)) + \mathcal{R}_n((x,\infty)),$$

where $\mathcal{R}_n((x,\infty))$ satisfies (15).

Remark 4. The assertion of Corollary 3 in the particular case $n = 1$, $\mathcal{Z} = \emptyset$ and $-\infty < EX_1 < 0$ coincides with assertion II of Theorem 8 of [3] for non-arithmetic distributions. (Note that in this case $\mathcal{Z} = \emptyset \Rightarrow \hat{F}(\gamma) < 1$, i.e. the condition $(F^{m*})_s^\wedge(\gamma) < 1$ is automatically fulfilled.)

Suppose the random walk $\{S_k\}$ drifts to $-\infty$ and $\hat{F}(\gamma) < 1$ for $\gamma > 0$. Then the tails of F and $H = \Phi_1$ have, in essence, the same asymptotic behaviour.

Corollary 4. *Under the hypotheses of Theorem 6, if* $\hat{F}(\gamma) < 1$, *then* $F \in \mathfrak{Sl}(\tau) \Leftrightarrow H \in \mathfrak{Sl}(\gamma', \tau)$, *and in both cases*

$$\lim_{x\to\infty} \frac{H((x,\infty))}{\tau(x)} = \frac{1}{[1-\hat{F}(\gamma)]^2} \cdot \lim_{x\to\infty} \frac{F((x,\infty))}{\tau(x)}.$$

We now consider the asymptotic properties of the measure Φ_n in the "critical" case, i.e. when $\hat{F}(\gamma) = 1$. In this case $\mathcal{Z} = \{\gamma\}$. Recall that the coefficients $B_{1k}^{(n)}$ are defined by (7) for the root $s_1 = \gamma$. The measure $\mathcal{E}_1$ will be denoted here by $\mathcal{E}_\gamma$.

Theorem 7. *Let* $\{X_i\}_{i=1}^\infty$ *be a sequence of independent identically distributed random variables with a common non-arithmetic distribution* F *and let* $\{S_k\}_{k=0}^\infty$ *be a random walk such that with probability one* $S_k \to -\infty$ *as* $k \to \infty$. *Let* $\hat{F}(\gamma) = 1$ *for* $\gamma > 0$. *Suppose* $\int_{\mathbf{R}} |x|^{n+1} e^{\gamma x} F(dx) < \infty$ *and* $(F^{m*})_s^\wedge(\gamma) < 1$ *for some* $m \geq 1$. *If* $T(\gamma)^{n+1}F \in \mathfrak{Sl}(\tau)$, *then the following representation holds*:

$$\Phi_n = \sum_{k=1}^n B_{1k}^{(n)} \mathcal{E}_\gamma^{k*} + \mathcal{R}_n, \tag{17}$$

where $\mathcal{R}_n \in \mathfrak{Sl}(\gamma', \tau)$ *for all* $\gamma' \in (0,\gamma)$ *and*

$$\mathfrak{l}(\mathcal{R}_n) = \frac{(-1)^{n+1} n \cdot n!\, \mathfrak{l}[T(\gamma)^{n+1}F]}{[\hat{F}'(\gamma)]^{n+1}}; \tag{18}$$

in other words: $\lim_{x\to\infty} \mathcal{R}_n((x,\infty))/\tau(x)$ *equals*

$$\frac{(-1)^{n+1} n \cdot n!}{[\hat{F}'(\gamma)]^{n+1}} \cdot \lim_{x\to\infty} \frac{\int_x^\infty e^{-\gamma y_{n+1}} \int_{y_{n+1}}^\infty \cdots \int_{y_2}^\infty \int_{y_1}^\infty e^{\gamma z} F(dz) dy_1 dy_2 \cdots dy_{n+1}}{\tau(x)}.$$

Conversely, let $\int_{\mathbf{R}} x^2 e^{\gamma x} F(dx) < \infty$ *for* $\gamma > 0$. *If* (17) *holds for* $n = 1$ *with* $b \stackrel{\text{def}}{=} B_{11}^{(1)} \neq 0$ *and* $\mathcal{R}_1 \in \mathfrak{Sl}(\gamma', \tau)$ *for some* $\gamma' \in (0,\gamma)$, *then* $\hat{F}(\gamma) = 1$, $T(\gamma)^2 F \in \mathfrak{Sl}(\tau)$, $(F^{m*})_s^\wedge(\gamma) < 1$ *for some* $m \geq 1$, $b = 1/\hat{F}'(\gamma)$ *and equality* (18) *holds.*

Proof. Choose $r > \gamma$ and $\gamma' \in (0,\gamma)$. Put

$$v(s) \stackrel{\text{def}}{=} \frac{[1-\hat{F}(s)](s-r)}{s-\gamma}, \qquad \gamma' \leq \Re s \leq \gamma,$$

the value $v(\gamma)$ being defined by continuity. By Lemma 1, $v(s)$ is the Laplace transform of $V \stackrel{\text{def}}{=} \delta - F + (r-\gamma)T(\gamma)F \in \mathfrak{Sl}(\gamma', \tau)$ with $\mathfrak{l}(V) = 0$. By Corollary 1, $V \in S(\varphi)$, where $\varphi(x) = (1+x)^n e^{\gamma x}$ for $x \geq 0$ and $\varphi(x) = e^{\gamma' x}$ for $x < 0$. Clearly, $S(\varphi) \subset S(\gamma', \gamma)$. As in the proof of Lemma 2 in [32], we conclude that there exists an inverse $V^{-1} \in S(\varphi)$. By Theorem 2, $V^{-1} \in \mathfrak{Sl}(\gamma', \tau)$ with $\mathfrak{l}(V^{-1}) = 0$. Put $W = (V^{-1})^{n*}$. We have $W \in \mathfrak{Sl}(\gamma', \tau)$ and $\mathfrak{l}(W) = 0$. Moreover, $W \in S(\varphi)$. This allows us to apply n times the operator $T(\gamma)$ to W. We write

$$\frac{n!}{[1-\hat{F}(s)]^n} = n!\,\hat{W}(s)\left(\frac{s-r}{s-\gamma}\right)^n = n!\left\{\hat{W}(s) + \sum_{k=1}^n B_k \frac{\hat{W}(s)}{(s-\gamma)^k}\right\}. \tag{19}$$

Let $w_0(s) \stackrel{\text{def}}{=} \hat{W}(s)$, $w_k(s) \stackrel{\text{def}}{=} [T(\gamma)^k W]^\wedge(s)$, $k = 1, \ldots, n$. Similarly, $v_k(s) \stackrel{\text{def}}{=} [T(\gamma)^k V]^\wedge(s)$ and $f_k(s) \stackrel{\text{def}}{=} [T(\gamma)^k F]^\wedge(s)$. We show that $W_k \stackrel{\text{def}}{=} T(\gamma)^k W \in \mathfrak{Sl}(\gamma', \tau)$, $k = 1, \ldots, n$. First, we note that, by Lemma 1, F and $T(\gamma)^k F$, $k = 1, \ldots, n$, are elements of $\mathfrak{Sl}(\gamma', \tau)$ with zero values of the functional $\mathfrak{l}$. Thus, $V_k \stackrel{\text{def}}{=} T(\gamma)^k V = (r-\gamma)T(\gamma)^{k+1}F - T(\gamma)^k F \in \mathfrak{S}\mathfrak{l}(\gamma',\tau)$ $k = 1, \dots, n$, and $\mathfrak{l}(V_k) = 0$, $k = 1, \dots, n-1$. Further,

$$\begin{aligned} w_1(s) = \frac{w_0(s) - w_0(\gamma)}{s-\gamma} &= \frac{1}{s-\gamma}\left[\frac{1}{v_0(s)^n} - \frac{1}{v_0(\gamma)^n}\right] \\ &= -w_0(s)w_0(\gamma)\frac{v_0(s)-v_0(\gamma)}{s-\gamma}\sum_{k=0}^{n-1} v_0(s)^k v_0(\gamma)^{n-k-1} \\ &= w_0(s)w_0(\gamma)[f_1(s) - (r-\gamma)f_2(s)]\sum_{k=0}^{n-1} v_0(s)^k v_0(\gamma)^{n-k-1} \\ &= f_2(s)r_1(s) + q_1(s), \end{aligned}$$

where $r_1(s)$ is a linear combination of products whose factors are $w_0(s)$ and powers of $v_0(s)$, and $q_1(s)$ is a linear combination of products whose factors are $w_0(s)$, $f_1(s)$ and powers of $v_0(s)$. Hence $W_1 \in \mathfrak{S}\mathfrak{l}(\gamma',\tau)$ and, in case $n > 2$, $\mathfrak{l}(W_1) = 0$ by (4). Another iteration yields $w_2(s) = f_3(s)r_1(\gamma) + q_2(s)$, where $q_2(s)$ is a linear combination of products whose factors are $w_i(s)$, $v_i(s)$, $i = 0, 1$, and $f_2(s)$. Consequently, $W_2 \in \mathfrak{S}\mathfrak{l}(\gamma',\tau)$ and, in case $n > 3$, $\mathfrak{l}(W_2) = 0$ by (4). After n steps, we obtain $w_n(s) = f_{n+1}(s)r_1(\gamma) + q_n(s)$, where $q_n(s)$ is a linear combination of products whose factors are $w_i(s)$, $v_i(s)$, $f_i(s)$, $i = 0, \dots, n-1$, and $f_n(s)$. Thus, by (4), $q_n(s)$ is the Laplace transform of some $K_n \in \mathfrak{S}\mathfrak{l}(\gamma',\tau)$ such that $\mathfrak{l}(K_n) = 0$. We have $W_n \in \mathfrak{S}\mathfrak{l}(\gamma',\tau)$ and $\mathfrak{l}(W_n) = r_1(\gamma)\mathfrak{l}[T(\gamma)^{n+1}F]$. Finally, if we transform (19) in the familiar way and note that $B_n = (\gamma - r)^n$ and $r_1(\gamma) = -n(r-\gamma)^{-n}[\hat{F}'(\gamma)]^{-n-1}$, then we will obtain

$$\begin{aligned} \frac{n!}{[1-\hat{F}(s)]^n} = \sum_{k=1}^{n}(-1)^k \frac{B_{1k}^{(n)}}{(s-\gamma)^k} &+ \frac{(-1)^{n+1}n\cdot n!}{[\hat{F}'(\gamma)]^{n+1}} f_{n+1}(s) \\ &+ n!\left\{\sum_{k=1}^{n-1} B_k w_k(s) + B_n q_n(s)\right\}, \qquad \gamma' \le \Re s < \gamma, \end{aligned}$$

where the expression in braces is the Laplace transform of a measure in $\mathfrak{S}\mathfrak{l}(\gamma',\tau)$ with zero value of the functional $\mathfrak{l}$. It remains to set

$$\mathcal{R}_n \stackrel{\text{def}}{=} \frac{(-1)^{n+1}n\cdot n! T(\gamma)^{n+1}F}{[\hat{F}'(\gamma)]^{n+1}} + n!\left\{\sum_{k=1}^{n-1} B_k W_k + B_n K_n\right\}$$

and to go over from Laplace transforms to measures.

We now prove the converse statement. Suppose that (17) holds for $n = 1$ and $\mathcal{R}_1 \in \mathfrak{S}\mathfrak{l}(\gamma',\tau)$. If we go over in (17) from measures to Laplace transforms, then

$$\frac{1}{1-\hat{F}(s)} = -\frac{b}{s-\gamma} + \hat{\mathcal{R}}_1(s), \qquad \gamma' \le \Re s < \gamma. \tag{20}$$

Letting $s \to \gamma$, we see that $\hat{F}(s) \to 1$, i.e. $\hat{F}(\gamma) = 1$. Multiply both sides of (20) by $(s-\gamma)/(s-\beta)$, where $\beta > \gamma$. Then

$$w(s) \stackrel{\text{def}}{=} \frac{s-\gamma}{[1-\hat{F}(s)](s-\beta)} = -\frac{b}{s-\beta} + \frac{\hat{\mathcal{R}}_1(s)(s-\gamma)}{s-\beta}.$$

The right-hand side is the Laplace transform of the measure

$$W \stackrel{\text{def}}{=} b\mathcal{E}_\beta + \mathcal{R}_1 + (\beta-\gamma)\mathcal{R}_1 * \mathcal{E}_\beta,$$

which is an element of $\mathfrak{Sl}(\gamma', \tau)$ since $\beta > \gamma$ (see the proof of Theorem 5). Further, $T(\gamma)W \in \mathfrak{Sl}(\gamma', \tau)$. Actually,

$$\frac{w(s) - w(\gamma)}{s - \gamma} = -\frac{b}{\beta - \gamma} \cdot \frac{1}{s - \beta} + \frac{\hat{\mathcal{R}}_1(s)}{s - \beta}.$$

The left-hand side is the Laplace transform of $T(\gamma)W$ and the right-hand side is the Laplace transform of $b\mathcal{E}_\beta/(\beta - \gamma) - \mathcal{R}_1 * \mathcal{E}_\beta \in \mathfrak{Sl}(\gamma', \tau)$. We now show that $V \stackrel{\text{def}}{=} \delta - F + T(\gamma)F \in \mathfrak{Sl}(\gamma', \tau)$. The relation $\mathcal{R}_1 \in \mathfrak{Sl}(\gamma', \tau)$ implies that $(F^{m*})_s^\wedge(\gamma) < 1$ for some $m \geq 1$ [32, Remark 2]. Hence there exists an inverse $V^{-1} \in S(\gamma', \gamma)$ (see the beginning of the proof of the theorem) with Laplace transform equal to $w(s)$, i.e. $V^{-1} = W$. By Theorem 2 with $f(z) = 1/z$, the element W is invertible in $\mathfrak{Sl}(\gamma', \tau)$. Consequently, $V = W^{-1} \in \mathfrak{Sl}(\gamma', \tau)$. We show that $T(\gamma)V \in \mathfrak{Sl}(\gamma', \tau)$. We have

$$\frac{v(s) - v(\gamma)}{s - \gamma} = -v(s)v(\gamma)\frac{w(s) - w(\gamma)}{s - \gamma}.$$

Hence

$$U_\beta \stackrel{\text{def}}{=} (\beta - \gamma)T(\gamma)^2 F - T(\gamma)F = T(\gamma)V = -v(\gamma)V * T(\gamma)W \in \mathfrak{Sl}(\gamma', \tau).$$

Moreover, $U_{\beta+1} \in \mathfrak{Sl}(\gamma', \tau)$ since $\beta > \gamma$ was chosen arbitrarily. Therefore, $T(\gamma)^2 F = U_{\beta+1} - U_\beta \in \mathfrak{Sl}(\gamma', \tau)$. This completes the proof of Theorem 7. □

Remark 5. Assertions similar to Theorem 1 and Lemma 1 are also valid for the Banach algebras $\mathfrak{So}(\gamma', \tau)$ and $\mathfrak{S}(\gamma', \tau)$. So if we replace throughout the algebra $\mathfrak{Sl}(\gamma', \tau)$ by $\mathfrak{So}(\gamma', \tau)$ (or by $\mathfrak{S}(\gamma', \tau)$), then we will obtain expansions for Φ_n with $|\mathcal{R}|_n((x, \infty)) = o(\tau(x))$ (or $O(\tau(x))$) as $x \to \infty$.

4. Submultiplicative results

Let $\{X_i\}$ a sequence of independent identically distributed random variables with a common non-arithmetic distribution F and positive expectation. Let $r > 0$ and $\hat{F}(r) < \infty$. Suppose the set $\mathcal{Z}$ of the roots of the characteristic equation $1 - \hat{F}(s) = 0$ lying in the strip $\{0 < \Re s \leq r\}$ is finite. Denote the elements of the set $\mathcal{Z}$ by $s_1, s_2, \dots, s_l$, and the multiplicity of the root s_j by m_j.

Theorem 8. *Let $\{X_i\}_{i=1}^\infty$ be a sequence of independent identically distributed random variables with distribution F. Suppose $0 < \mu = EX_1 < \infty$, $E(X_1^-)^n < \infty$ and let Φ_n be defined by* (1). *Let $\varphi(x)$, $x \in \mathbf{R}$, be a submultiplicative function such that $\varphi(x) = 1$ for $x < 0$, $r \stackrel{\text{def}}{=} r_+(\varphi) > 0$ and the function $\varphi(x)/\exp(rx)$, $x \geq 0$, is non-decreasing. Suppose $\hat{F}(r) < \infty$.*

Assume that $(F^{m*})_s^\wedge(r) < 1$ for some $m \geq 1$. Let s_j be the roots of $1 - \hat{F}(s) = 0$ lying in $\{0 < \Re s \leq r\}$ with multiplicities m_j, $j = 1, \dots, l$. Denote by N the maximal multiplicity of the roots lying on $\{\Re s = r\}$ ($N = 0$ means that there are no such roots on this line). If

$$\int_0^\infty (1 + x)^{(n+1)N} \varphi(x) F(dx) < \infty, \tag{21}$$

then Φ_n admits representation (9), *where $\int_0^\infty \varphi(x)\,|\mathcal{R}_n|(dx) < \infty$.*

Proof. We shall use the following system of submultiplicative functions: $\varphi_{k,m}(x) \stackrel{\text{def}}{=} (1 + x)^m \varphi(x)$ for $x \geq 0$ and $\varphi_{k,m}(x) \stackrel{\text{def}}{=} (1 + |x|)^k$ for $x < 0$. Clearly, $r_+(\varphi_{k,m}) = r$ and

$r_-(\varphi_{k,\,m}) = 0$. The hypotheses $E(X_1^-)^n < \infty$ and (21) mean that $F \in S(\varphi_{n,\,(n+1)N})$. Put $p = \sum_{j=1}^{l} m_j + 1$ and

$$v(s) \stackrel{\text{def}}{=} \frac{[1 - \hat{F}(s)](s - r - 1)^p}{s \prod_{j=1}^{l}(s - s_j)^{m_j}}, \qquad 0 \le \Re s \le r;$$

the values of $v(s)$ at $s = 0$ and $s = s_j$, $j = 1, \dots, l$, are defined by continuity. Our first step is to show that $v(s)$ is the Laplace transform of some $V \in S(\varphi_{n-1,\,nN})$. If we represent a rational function as a sum of partial fractions, then

$$v(s) = [1 - \hat{F}(s)] \left[1 + \frac{a}{s} + \sum_{j=1}^{l} \sum_{k=1}^{m_j} \frac{C_{jk}}{(s - s_j)^k} \right], \tag{22}$$

where a, C_{jk} are constants. Consider the expression $[\hat{F}(s) - 1]/(s - s_j)^k$ for $k \le m_j$. By Theorem 3 or by Theorem 4 (depending on whether $\Re s_j < r$ or $\Re s_j = r$), the expression is the Laplace transform of the measure $T(s_j)^k F$ belonging to $S(\varphi_{n,\,(n+1)N})$ or $S(\varphi_{n,\,(n+1)N-k})$, and a fortiori $T(s_j)^k F \in S(\varphi_{n,\,nN})$.

Next, by Theorem 4 (more precisely, by a symmetric assertion for the left-side boundary $\{\Re s = 0\}$), the function $[\hat{F}(s) - 1]/s$ is the Laplace transform of $T(F) \in S(\varphi_{n-1,\,(n+1)N})$. Thus, $v(s) = \hat{V}(s)$, where $V \in S(\varphi_{n-1,\,nN})$. By arguing just in the same way as in the proof of Lemma 2 in [32], we establish that the element V is invertible in $S(\varphi_{n-1,\,nN})$. Set $W \stackrel{\text{def}}{=} (V^{-1})^{n*}$. Proceeding as in the proof of Theorem 5, we verify the validity of equality (11) for $s \in \{0 \le \Re s \le r\} \setminus (\mathcal{Z} \cup \{0\})$ and, by the same, the validity of representation (9). The only difference consists in that we use, instead of Theorem 1, Theorems 3 and 4, depending on whether $\Re s_j < r$ or $\Re s_j = r$ (see the calculations above with (22) as starting point). As a result, we have $W_{p,j} \in S(\varphi_{n-1,\,nN})$ or $W_{p,j} \in S(\varphi_{n-1,\,nN-p})$, depending on whether $\Re s_j < r$ or $\Re s_j = r$. Next, applying an analogue of Theorem 4 for the left-side boundary $\{\Re s = 0\}$ of the strip $\{0 \le \Re s \le r\}$, we have $W_k \in S(\varphi_{n-k-1,\,nN})$ for $k = 1, \dots, n-1$, and also $W_n|_{(0,\,\infty)} = T\left(W_{n-1}|_{(0,\,\infty)}\right) \in S(\varphi_{0,\,nN})$. Therefore, $\mathcal{R}_n|_{(0,\,\infty)} \in S(\varphi_{0,0}) = S(\varphi)$. The proof of the theorem is complete. □

Corollary 5. *Under the hypotheses of Theorem 8 for $n = 1$, the renewal measure $H = \Phi_1$ admits the representation*

$$H = \frac{L}{\mu} + \sum_{j=1}^{l} \sum_{k=1}^{m_j} B_{jk}^{(1)} \mathcal{E}_j^{k*} + \mathcal{R}_1,$$

where $\int_0^\infty \varphi(x)\,|\mathcal{R}_1|(dx) < \infty$.

Corollary 6. *Under the hypotheses of Theorem 8, the equalities of Corollary 2 hold true with the following estimate for the remainder term:*

$$|\mathcal{R}_n((x,\infty))| \le |\mathcal{R}_n|((x,\infty)) = o(1/\varphi(x)) \qquad \text{as } x \to \infty.$$

Instead of studying the submultiplicative behaviour of the measure Φ_n on the negative half-axis, it is convenient to consider the equivalent problem of investigating asymptotic properties of Φ_n on $[0,\infty)$, as was done in the preceding section.

Let $S_k \to -\infty$ as $k \to \infty$ with probability one. In the theorem below we do not exclude the possibility $\mathcal{Z} = \emptyset$. In case $\mathcal{Z} \neq \emptyset$, there is among the elements of $\mathcal{Z}$ a unique real root, say $q = s_1$, with multiplicity one.

Theorem 9. *Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of independent identically distributed random variables with a common non-arithmetic distribution F and let $\{S_k\}_{k=0}^{\infty}$ be a random walk such that with probability one $S_k \to -\infty$ as $k \to \infty$, and let Φ_n be defined by* (1).

Let $\varphi(x)$, $x \in \mathbf{R}$, be a submultiplicative function such that $\varphi(x) = 1$ for $x < 0$, $r \stackrel{\text{def}}{=} r_+(\varphi) > 0$ and the function $\varphi(x)/\exp(rx)$, $x \geq 0$, is non-decreasing. Suppose $\hat{F}(r) < \infty$.

Assume that $(F^{m*})^{\wedge}_s(r) < 1$ for some $m \geq 1$. Let s_j be the roots of $1 - \hat{F}(s) = 0$ lying in $\{0 < \Re s \leq r\}$ with multiplicities m_j, $j = 1, \ldots, l$. Denote by N the maximal multiplicity of the roots which lie on $\{\Re s = r\}$ and suppose that condition (21) *is fulfilled. Then Φ_n admits representation* (14), *where $\int_0^{\infty} \varphi(x)\, |\mathcal{R}_n|(dx) < \infty$.*

Proof. Consider the submultiplicative functions $\psi_m(x) \stackrel{\text{def}}{=} (1+x)^m \varphi(x)$ for $x \geq 0$ and $\psi_m(x) \stackrel{\text{def}}{=} \exp(r'x)$ for $x < 0$, where the number r' is arbitrarily chosen in the interval $(0, q)$. If $\mathcal{Z} = \emptyset$, then r' is arbitrarily chosen in $(0, r)$. It is clear that $r_-(\psi_m) = r'$ and $r_+(\psi_m) = r$. Condition (21) means that $F \in S(\psi_{(n+1)N})$. Put $p = \sum_{j=1}^{l} m_j$ and

$$v(s) \stackrel{\text{def}}{=} \frac{[1 - \hat{F}(s)](s - r - 1)^p}{\prod_{j=1}^{l} (s - s_j)^{m_j}}.$$

Just as in the proof of the preceding theorem, we establish that $v(s)$ is the Laplace transform of a real-valued measure $V \in S(\psi_{nN})$. Set $W \stackrel{\text{def}}{=} (V^{-1})^{n*}$ and $w(s) \stackrel{\text{def}}{=} \hat{W}(s)$. We write

$$\hat{\Phi}_n(s) = \frac{w(s) n!\, (s - r - 1)^{np}}{\prod_{j=1}^{l} (s - s_j)^{n m_j}},$$

and proceed as in the proof of Theorem 5. We obtain

$$\hat{\Phi}_n(s) = \sum_{j=1}^{l} \sum_{k=1}^{n m_j} \frac{(-1)^k B_{jk}^{(n)}}{(s - s_j)^k} + \hat{\mathcal{R}}_n(s), \qquad s \in \{r' \leq \Re s \leq r\} \setminus \mathcal{Z},$$

where $\mathcal{R}_n \in S(\psi_0)$. To complete the proof, it remains to go over from Laplace transforms to the corresponding measures. $\square$

Remark 6. If $N = 0$, then Theorems 8 and 9 admit converse statements, similar to those of Theorems 5 and 6. The proofs of the converse statements for $S(\varphi)$ are almost the same as for $\mathfrak{Sl}(\gamma', \tau)$. The insignificant changes consist in the following: (i) instead of Theorem 1 we use Theorem 3 and (ii) we note that, for $\beta > r_+(\varphi)$, $\mathcal{E}_\beta \in S(\varphi)$, which is a consequence of (3).

In case $\mathcal{Z} = \{r\}$, Theorem 9 admits a converse statement, similar to that of Theorem 7.

Theorem 10. *Suppose that with probability one $S_k \to -\infty$ as $k \to \infty$. Let $\varphi(x)$, $x \in \mathbf{R}$, be a submultiplicative function such that $\varphi(x) = 1$ for $x < 0$, $r \stackrel{\text{def}}{=} r_+(\varphi) > 0$ and the function $\varphi(x)/\exp(rx)$, $x \geq 0$, is non-decreasing. Assume that $\int_{\mathbf{R}} x^2 e^{rx}\, F(dx) < \infty$ and* (17) *holds for $n = l = 1$, where $s_1 = r$, $B_{11}^{(1)}$ is replaced by $b \neq 0$ and $\int_0^{\infty} \varphi(x)\, |\mathcal{R}_1|(dx) < \infty$. Then $\hat{F}(r) = 1$, $T(r)^2 F \in S(\varphi)$ and $(F^{m*})^{\wedge}_s(r) < 1$ for some $m \geq 1$.*

Proof. Consider the submultiplicative function $\psi(x) \stackrel{\text{def}}{=} \varphi(x)$ for $x \geq 0$ and $\psi(x) \stackrel{\text{def}}{=} \exp(\gamma' x)$ for $x < 0$, where $\gamma' \in (0, r)$. We have $r_-(\psi) = \gamma'$ and $r_+(\psi) = r$. If a measure ν is finite, then clearly $\nu \in S(\varphi) \Leftrightarrow \nu \in S(\psi)$. Further, $\mathcal{R}_1 \in S(\psi)$ since $\mathcal{R}_1|_{(-\infty,0)} = \Phi_1|_{(-\infty,0)}$ and, as shown in [32, Remark 2], $\int_{-\infty}^{0} \exp(\gamma' x)\, \Phi_1(dx) < \infty$ $\forall \gamma' > 0$. To complete the proof, it now remains to repeat the arguments of the proof of the converse statement of Theorem 7, where γ must be replaced by r and $\mathfrak{Sl}(\gamma', \tau)$ by $S(\psi)$. One thing to which we should pay special attention is the invertibility of W in the Banach algebra $S(\psi)$. This is established as follows. As was shown in the proof of the converse statement of Theorem 7, the element $V \in S(\gamma', r)$ is invertible in $S(\gamma', r)$ and $V^{-1} = W$, whence $W \in S(\psi)$ is invertible in $S(\gamma', r)$. But then W^{-1} must belong to $S(\psi)$ since each maximal ideal M of $S(\psi)$ is of the form $M = S(\psi) \cap M_1$, where M_1 is a maximal ideal of $S(\gamma', r)$ (this follows from the theorem on the structure of the homomorphisms of $S(\varphi)$ onto $\mathbf{C}$ [24, Theorem 1]). As far as the relation $\mathcal{E}_\beta \in S(\psi)$ is concerned, the reader is referred to Remark 6. $\square$

Corollary 7. *Let $\mathcal{Z} \neq \emptyset$. Then, under the hypotheses of Theorem 9,*

$$\Phi_n((x,\infty)) = e^{-qx} \sum_{k=1}^{n} B_{1k}^{(n)} \sum_{p=0}^{k-1} \frac{|x|^p}{p!\,|q|^{k-p}} + \sum_{j=2}^{l} \sum_{k=1}^{nm_j} B_{jk}^{(n)} \mathcal{E}_j^{k*}((x,\infty)) + \mathcal{R}_n((x,\infty)),$$

where $|\mathcal{R}_n((x,\infty))| \leq |\mathcal{R}_n|((x,\infty)) = o(1/\varphi(x))$ as $x \to \infty$.

If the random walk $\{S_k\}$ drifts to $-\infty$ and $\hat{F}(r_+(\varphi)) < 1$, then the underlying distribution F and the renewal measure $H = \Phi_1$ have the same submultiplicative moments on $[0,\infty)$ (see Theorem 9 and Remark 6). In this case, the condition $(F^{m*})_s^\wedge(\gamma) < 1$ is automatically fulfilled.

Corollary 8. *Suppose $\hat{F}(r) < 1$. Then, under the hypotheses of Theorem 9,*

$$\int_0^\infty \varphi(x)\, F(dx) < \infty \Longleftrightarrow \int_0^\infty \varphi(x)\, H(dx) < \infty.$$

Remark 7. The preceding theory cannot be applied in full extent to exponential distributions or to their mixtures since they are not in $\mathcal{S}(\gamma)$. But in this case, the corresponding renewal measures can be evaluated in explicit form because their Laplace transforms are rational functions [10, Section 4.3].

Remark 8. In all the theorems and corollaries of the present paper, the underlying distribution was assumed to be *non-arithmetic*. However, the whole theory carries over almost word for word to the discrete case by considering similar Banach algebras of measures concentrated on the lattice of integers and replacing the measures L and $\mathcal{E}_j$ by their discrete counterparts; moreover, there is no need, in the arithmetic case, of the condition $(F^{m*})_s^\wedge(\gamma) < 1$.

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