Applications of recursive continued fractions and extended transfer principle

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Abstract

Recursive continued fractions are useful when effectivizing some mathematical properties, since they can be identified with computable irrational numbers.

The aim of this treatise is three-fold. First, the computability structure on the continuum is re-defined by taking the computable irrational sequences as the basis of computable real sequences. Second, a couple of theorems in [khinchin] are effectivized with the help of recursive continued fractions. Third, the 'extended transfer principle' (ETP) is proposed. It is an extension of Pour-El's 'transfer principle'(TP) in [pourel], and our effectivized mathematical results are organized in the light of ETP. ETP is a principle that delineates the structure of a proof of the effective version of a classical theorem with the following feature: it is comprised of tracing faithfully the classical proof, while, in so doing, effectively evaluating certain bounds at appropriate spots, and invoking a general 'side-lemma' if necessary.

Mathematically, we implement effectivizations of Liouville's theorem, the periodicity of quadratic numbers and 'Kuz'min's result', which leads to an effective answer to a Gauss's problem, as applications of recursive continued fractions. It will be verified that either TP or ETP applies to those instances. This circumstance gives likelihood to Pour-El's claim that TP is versatile in variety of approximation problems. As a 'side-lemma', the computability of an inductively defined function sequence is proved, which is applied to Kuz'min's case.