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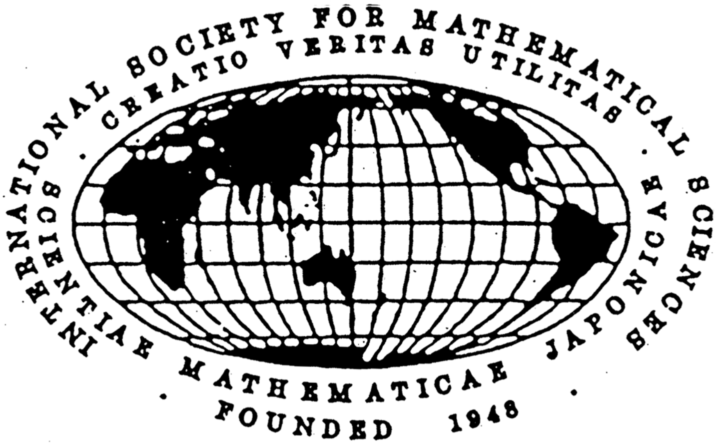

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# OPTIMAL FACILITY LOCATION PROBLEM UNDER POSSIBILITY CHANCE CONSTRAINT CONDITIONS AND BARRIERS

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## Abstract

We consider the following problem: 1) There are demand points and possible construction sites in an urban area with some barriers. We adopt rectilinear distance. 2) We construct two facilities, one is welcome facility and the other obnoxious facility We call welcome facility as A and obnoxious facility as B. Two facilities A and B can be constructed at the same site or constructed separately, that is, at two different sites. We assume that each construction cost of A and B is a random variable with fuzzy mean respectively and construction cost of both facilities simultaneously as a same site is also random variable with fuzzy mean. These are distributed according to normal distributions with fuzzy means. 3) The probability that total construction cost becomes below budget *f* should not be less than the fixed probability level. $\alpha$ and further the possibility that this chance constraint holds should be not less than the fixed level $\beta$. Under this possibility chance constraint *f* should be minimized. 4) We consider three criteria, (a) maximum distance from the construction site of A to all demand points to be minimized, (b) minimum distance from the construction site of B to all demand points to be maximized, (c) budget to be minimized. Since usually there exists no site optimizing three criteria at a time, we seek non-dominated solution after definition of non-domination. Finally, we conclude results and discuss further research problems.

**1 Introduction** There are huge amount of papers regarding facility location problem after Weber has published his paper [8] (so called Weber problem). Hamacher et al. ( [2]) tried to classify these papers by introducing similar codes to classify queueing and scheduling models . For rectilinear distance, we should refer to [1] as a classic but successful model and an efficient algorithm due to geometrical approach. Further for a discrete location problem, refer to review paper [7]. In this paper we consider multi-facility case as one possibility based on rectilinear distance. That is, two types of facilities, welcome facility , the other obnoxious one are constructed. We call welcome facility as A and obnoxious facility as B. Two facilities A and B can be constructed at the same site or constructed separately at two different sites. Construction costs of A and B are random variables with fuzzy means. Section 2 formulates the facility location problem with the tri-criteria und above prominent features. Section 3 proposes a solution procedure to seek non-dominat solutions after the definition of non-domination. Finally, section 4 summarizes the resul and discusses further research problem.

**2 Problem formulation** We consider the following problem:

(1) There are m demand points : $D_i = (a_i, b_i)$ for $i = 1, 2, ..., m$ and $r$ possible constructic sites, $FP_j$ for $j = 1, 2, ..., n$ in an urban area $X = \{(x, y)|0 \leq x \leq p_0.0 \leq y \leq q_0\}$ with son rectangular barriers

$\mathbf{B_k} = \{(x, y)|B_k^1 < x < B_k^2, B_k^3 < y < B_k^4\}, k = 1, 2, ..., s$ Facilities A and B can l constructed in these blocks . Barrier means we cannot pass it inside and so in some case v must make a detour. We adopt rectilinear distance which is used often in an urban are That is, rectilinear distance between points $P = (a, b)$ and $Q = (c, d)$ is $|a - c| + |b - d|$.

(2) We construct two facilities, one is welcome facility (that is, maximum distance to demar points should be minimized), the other is obnoxious one (that is, minimum distance demand points should be maximized). We call welcome one as A and obnoxious one B and two facilities A, B can be constructed at the same site or constructed separatel that is, at two different sites. For each possible construction site $FP_j$, we assume that ea construction cost of A, B is a random variable $CA_j, CB_j$ with fuzzy mean respectively ar construction cost of both facilities simultaneously at a same site is also random variable $C$ with fuzzy mean . $CA_j$ is distributed according to the normal distribution with fuzzy mea $M_{1j}$ and variance $\sigma_{1j}^2$, $CB_j$ according that with fuzzy mean $M_{2j}$ and variance $\sigma_{2j}^2$, and $C$ according to fuzzy mean $M_{3j}$ and variance $\sigma_{3j}^2$. We assume that they are independent ea other. 縲 Note that if two facilities are constructed at different sites, the total constructic cost is the sum of the construction cost of A and that of B. Each mean $M_{uj}$ is a $L$ fuz number with $L(\dfrac{t - m_{uj}}{\sigma_{uj}}), u = 1, 2, 3.$

(3) The probability that total construction cost becomes below budget $f$ should be not le than the fixed probability level $\alpha$ and $f$ should be minimized where we assume that $\alpha >$ For A, B, separately constructed case at $j$, this probabilistic condition is

$$\Pr\{CA_j \leq f\} \geq \alpha \Leftrightarrow \Pr\{\frac{CA_j - m_{1j}}{\sigma_{1j}} \leq \frac{f - m_{1j}}{\sigma_{1j}}\} \geq \alpha \Leftrightarrow f \geq m_{1j} + K_\alpha \sigma_{1j}$$

where $K_\alpha$ is a $\alpha$ percentile points of the cumulative distribution function of the standa normal distribution since $\dfrac{CA_j - m_{1j}}{\sigma_{1j}}$ is a random variable according to the standard norm distribution. Similarly done, for the case of separate construction of B, we have the followi deterministic equivalent condition as $f \geq m_{2j} + K_\alpha \sigma_{2j}$and for the case that both A ar B are constructed at the same site , corresponding deterministic equivalent condition

$f \geq m_{3j} + K_\alpha \sigma_{3j}$. Summarizing we have

$f \geq m_{1j}+K_\alpha \sigma_{1j}(\mathrm{A} : site\ \ j),\ \ f \geq m_{2j}+K_\alpha \sigma_{2j}(\mathrm{B} : site\ \ j),\ \ f \geq m_{3j}+K_\alpha \sigma_{3j}(\mathrm{bothA, B} : site\ \ j)$

but if A, B are constructed at different possible sites $FP_i, FP_j$ respectively, the budget constraint is

$$f \geq (m_{1i} + m_{2j}) + K_\alpha \sqrt{\sigma_{1i}^2 + \sigma_{2j}^2 + 2\sigma_{1i2j}}$$

where $\sigma_{1i2j}$ is a covariance between $CA_i$ and $CB_j$ since $CA_i + CB_j$ is a random variable according to the normal distribution with mean $(m_{1j} + m_{2j})$ and variance $\sigma_{1i}^2 + \sigma_{2j}^2 + 2\sigma_{1i2j}$.

(4) We consider three criteria, that is, maximum distance from the construction site of A to all demand points to be minimized, minimum distance from the construction site of B to all demand points to be maximized and budget to be minimized. Let $d(i, j)$ be the distance between demand point $D_i, i = 1, 2, ..., m$ and possible construction site $FP_j, j = 1, 2, ..., n$. These are calculated using some algorithm (for example, matrix algorithm using path algebra) of the shortest path problem on the following networks $N(V, E)$ (refer to [4]):

$V = \{D_1, D_2., \cdots, D_m, (B_1^1, B_1^3), (B_1^1, B_1^4), (B_1^2, B_1^3), (B_1^2, B_1^4), \cdots (B_i^1, B_i^3), (B_i^1, B_i^4), (B_i^2, B_i^3), (B_i^2, B_i^4), \cdots (B_s^1, B_s^3), (B_s^1, B_s^4), (B_s^2, B_s^3), (B_s^2, B_s^4), FP_1, FP_2, \cdots, FP_n\}(= \{v_1, v_2, \cdots, v_m, v_{m+1}, \cdots, v_{m+4s}, v_{m+4s+1}, \cdots, v_{m+4s+n}\})$

and $E$ consists of edges corresponding to visible pairs between two vertices in $V$ where length of each edge is a rectilinear distance between corresponding pair of verticies . Two points $P^1, P^2$ are called visible each other if there exists a route connecting two points using only horizontal line segment and vertical line segment not passing through some barriers without detours. .Otherwise we call $P^1 and P^2$ as invisible. In an invisible case we cannot connect two points by horizontal line segment and vertical line segment without detour like Figure 1.)

4,,

Fig.1 An invisible point pair

For each edge, the rectilinear distance between corresponding vertices is attached as a length. Then the first criterion is $d_A(j) = \max\{d(i, j)|i = 1, 2, ..., m\}$ and $d_A(j)$ should be minimized about *j*=1,2,...,*n*. The second criterion is $d_B(j) = \min\{d(i, j)|i = 1, 2, ..., m\}$ and $d_B(j)$ should be maximized about *j*=1,2,...,*n*. The third criterion is minimum budget $F$ under the above deterministic equivalent inequality, that is,

$$F = \min\{m_{1j_A} + m_{2j_B} + K_\alpha \sqrt{\sigma_{1j_A}^2 + \sigma_{2j_B}^2 + 2\sigma_{1j_A 2j_B}}, m_{3j_C} + K_\alpha \sigma_{3j_C}\}$$

where $j_A$: the site of facility A, $j_B$: the site of facility B if separately constructed and $J_C$ is the site that both A, B are constructed at the same site $j_C$. However if we assume that

$$m_{1i} + m_{2j} + K_\alpha\sqrt{\sigma_{1i}^2 + \sigma_{2j}^2 + 2\sigma_{1i2j}} > \max\{m_{3i} + K_\alpha\sigma_{3i}, m_{3j} + K_\alpha\sigma_{3j}\}$$

for any pair of $(i,j)$, $F = \min\{m_{3i} + K_\alpha\sigma | i = 1,2,...,n\}$. Since usually there exists no site optimizing tri-criteria at a time and so we seek some non-dominated solutions for the above model (1)-(4) after definition of non-domination in the next section.

**3 Solution Procedure** First we define a solution vector $V^X = (V_1^X, V_2^X, V_3^X)$ corresponding to a solution $X$ where $X$ is denoted as $X = (j_A^X, j_B^X)$ where $J_A^X, j_B^X$ are construction sites of A and that of B respectively. Therefore

$$V_1^X = \max\{d(i, j_A^X) | i = 1,2,...,m\}, V_2^X = \min\{d(i, j_B^X) | i = 1,2,...,m\}$$

$$V_3^X = \begin{cases} m_{1j_A^X} + m_{2j_B^X} + \sqrt{\sigma^2_{1j_A^X} + \sigma^2_{2j_B^X} + \sigma^2_{12j_A^Xj_B^X}} & (j_A^X \neq j_B^X) \\ m_{3j_A^X} + K_\alpha\sigma_{j_A^X} & (j_A^X = j_B^X) \end{cases}$$

**Non-dominated Solution**

For solutions $X_1, X_2$, if
$V_1^{X_1} \leq V_1^{X_2}$, $V_2^{X_1} \geq V_2^{X_2}$, $V_3^{X_1} \leq V_3^{X_2}$ and $V^{X_1} \neq V^{X_2}$, then we call $X_1$ dominates $X_2$. If there exists no solution dominating solution $X$, then $X$ is called non-dominated solution. We seek some non-dominated solutions. Note that usually $\min\{d_A(j) | j = 1,2,...,n\} \leq \max\{d(i, j_C) | i = 1,2,...,m\}$ and $\max\{d_B(j) | j = 1,2,...,n\} \geq \min\{d(i, j_C) | i = 1,2,...,m\}$ hold where $j_C$ is the minimizer of $\min\{M_{3j} + K_\alpha\sigma_{3j} | j = 1,2,...,n\}$.

Therefore first we check whether it holds that $\min\{d_A(j) | j = 1,2,...,n\} = \max\{d(i, j_C) | i = 1,2,...,m\}$. and $\max\{d_B(j) | j = 1,2,...,n\} = \min\{d(i, j_C) | i = 1,2,...,m\}$.. If so, the optimal solution is to construct both facilities A, B at the same possible site $j_C$ as a multi-facility. Otherwise (usually this case holds), we seek some non-dominated solution as below (5)-(7).

(5) First of all, above solution constructing the multi-facility at possible site $FP_{j_c}$ is a non-dominated solution (if minimizer $j_C$is not unique, we must check the non-domination and choose non-dominated one or ones.

(6) We find the minimizer $j_A$ of $\min\{d_A(j) | j = 1,2,...,n\}$ and maximizer $j_B$ of $\max\{d_B(j) | j = 1,2,...,n\}$.Then solution that facility A is constructed at $j_A$ and B at $j_B$ is a non-dominated solution. Of course, if $j_A$ or $j_B$ is not unique, we check these solutions about non-domination and choose non-dominated one or ones.

(7)We consider the weighted convex sum of $d_A(j)$ and $d_B(j)$ , that is, $W(j) = w_1 d_A(j) + w_2 d_B(j), w_1, w_2 > 0, w_1 + w_2 = 1$ and find the minimizer $j_W$ Then a solution that both A and B are constructed at the site $j_W$ as a multi-facility is non-dominated one. Again if $j_W$ is not unique, then check the non-domination and choose non-dominated one or ones.

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**4 Conclusion** This paper considered construction of two facilities simultaneously at different site or at a same site as a multi-facility under the stochastic construction costs. Here we considered a finite possible construction sites but following are left further research problems.

(8) As for more suitable criteria, we should consider environmental load, especially for obnoxious facility.

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# CHARACTERIZATIONS OF ω-LIKE CLOSED SETS AND SEPARATION AXIOMS IN TOPOLOGICAL SPACES

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Abstract. One of the aim of the present paper is introduce the concept of $\omega^{\rho}$-closed sets in topological space $(X, \tau)$ (cf. Definition 1.4) and study topological prop-erties of their classes of sets, where $\rho : SO(X, \tau) \to P(X)$ is a function defined by $\rho(V) := V$, $\rho(V) := Int(V)$ or $\rho(V) := Int(Cl(V))$ for every semi-open set $V$ of $(X, \tau)$. Furthermore, their relation ships with other generalied closed sets are investigated (cf. Remark 2.2). Using some analogous concept of the Jankovic-Reilly decomposition of sets ([2]), the concept of $\omega^{\rho}$-closed sets is completely characterized (cf. Theorem 4.8(iii)). In Section 5 and Section 6, some new separation axioms are introduced and investigated (i.e. $(\omega^{\rho 1}, \omega^{\rho 2})$-$T^{\rho}_{1/2}$-separation axioms (cf. Definition 5.3(I)(i), Theorem 5.11, Theorem 5.13, Theorem 5.15), where $\rho 1, \rho 2, \rho \in \{id, \circ, \circ-\}$ (cf. Definition 1.3). Throughout the present paper, examples are almost stated from topics of the digital line $(\mathbb{Z}, \kappa)$ due to E. D. Khalimsky (cf. Definition 2.3).

## 1 Introduction and preliminaries

Throughout the present paper, $(X,\tau)$ represents a nonempty topological space on which no separation axioms are assumed unless otherwise mentioned and $P(X)$ denotes the power set of $X$. For a subset $A$ of $(X,\tau)$, $Cl(A)$, $Int(A)$ and $Ker(A)$ denote the closure, interior and kernel of $A$ with respect to the topological space $(X,\tau)$ respectively; i.e., $Cl(A) := \cap\{F | A \subset F \text{ and } X \setminus F \in \tau\}$, $Int(A) := \cup\{U | U \subset A \text{ and } U \in \tau\}$ and $Ker(A) := \cap\{V | A \subset V \text{ and } V \in \tau\}$. A subset $B$ of $(X,\tau)$ is said to be *semi-open* ([13, in 1963],[8]), if $B \subset Cl(Int(B))$ holds in $(X,\tau)$. And, a subset $E$ of $(X,\tau)$ is said to be *preopen* ([19, in 1982]), if $E \subset Int(Cl(E))$ holds in $(X,\tau)$. The family of all semi-open sets (resp. preopen sets) of $(X,\tau)$ is denoted by $SO(X,\tau)$ (resp. $PO(X,\tau)$). For a subset $A$ of $(X,\tau)$, $pCl(A)$ denotes the preclosure of $A$ with respect to $(X,\tau)$, i.e., $pCl(A) := \cap\{F | A \subset F \text{ and } X \setminus F \in PO(X,\tau)\}$.

We recall the following concepts of two classes of generalized closed sets of a topological space $(X,\tau)$.

**Definition 1.1** (i) ([27, in 1995], [28, in 2000;Definition 2.1],[26, in 2002]) A subset $A$ of $(X,\tau)$ is said to be $\omega$-*closed* in $(X,\tau)$, if $Cl(A) \subset U$ whenever $A \subset U$ and $U \in SO(X,\tau)$.

(ii) ([22, in 2005]) A subset $A$ of $(X,\tau)$ is said to be *weakly* $\omega$-*closed* in $(X,\tau)$, if $Cl(Int(A)) \subset U$ whenever $A \subset U$ and $U \in SO(X,\tau)$.

(iii) A subset $B$ of $(X,\tau)$ is said to be $\omega$-*open* ([27]) (resp. *weakly* $\omega$-*open* ([22, Definition 3.22])) in $(X,\tau)$, if $X \setminus B$ is $\omega$-closed (resp. weakly $\omega$-closed) in $(X,\tau)$.

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Key words and phrases — : $\omega$-closed sets, $\omega^{\circ}$-closed sets, $\omega^{\circ-}$-closed sets, g-closed sets, separation axiom $T_{1/2}$

We use the following notation and definition.

**Notation 1.2** ($\bullet 1$) $\omega C(X,\tau) := \{A|\ A \text{ is } \omega\text{-closed in } (X,\tau)\}$;
($\bullet 1'$) $\omega O(X,\tau) := \{B|\ B \text{ is } \omega\text{-open in } (X,\tau)\}$;
($\bullet 2$) ${}^{w}\omega C(X,\tau) := \{A|\ A \text{ is weakly } \omega\text{-closed in } (X,\tau)\}$;
($\bullet 2'$) ${}^{w}\omega O(X,\tau) := \{B|\ B \text{ is weakly } \omega\text{-open in } (X,\tau)\}$.

**Definition 1.3** Let $\mathcal{E}_X$ be a subfamily of $P(X)$. The following function $\rho : \mathcal{E}_X \to P(X)$ is used on the present paper: for every set $U \in \mathcal{E}_X$ and a topological space $(X,\tau)$,

(i) $\rho := \circ : \mathcal{E}_X \to P(X)$ defined by $\circ(U) := Int(U)$;
(ii) $\rho := \circ- : \mathcal{E}_X \to P(X)$ defined by $\circ - (U) := Int(Cl(U))$;
(iii) $\rho := \circ - \circ : \mathcal{E}_X \to P(X)$ defined by $\circ - \circ(U) := Int(Cl(Int(U)))$;
(iv) $\rho := -\circ : \mathcal{E}_X \to P(X)$ defined by $- \circ (U) := Cl(Int(U))$;
(v) $\rho := - \circ - : \mathcal{E}_X \to P(X)$ defined by $- \circ -(U) := Cl(Int(Cl(U)))$;
(vi) $\rho := id : \mathcal{E}_X \to P(X)$ defined by $id(U) := U$.

We define some related classes of $\omega$-like closed sets (cf. Definition 1.4, Notation 1.5).

**Definition 1.4** Let $A$ and $B$ be subsets of a topological space $(X,\tau)$. And, let $\rho : SO(X,\tau) \to P(X)$ be a function such that $\rho \in \{id, \circ, -\circ, - \circ -, \circ-, \circ - \circ\}$ (cf. Definition 1.3 above for $\mathcal{E}_X := SO(X,\tau)$). A subset $A$ is said to be *$\omega^\rho$-closed* in $(X,\tau)$, if $Cl(A) \subset \rho(U)$ holds whenever $A \subset U$ and $U \in SO(X,\tau)$. The complemet $X \setminus B$ of an $\omega^\rho$-closed set $B$ is called an *$\omega^\rho$-open set* of $(X,\tau)$.

We have the following equivalent expression: a subset $A$ is $\omega^{id}$-closed (resp. $\omega^{id}$-open) in $(X,\tau)$ if and only if $A$ is $\omega$-closed (resp. $\omega$-open) in $(X,\tau)$.

**Notation 1.5** (i) For each function $\rho : SO(X,\tau) \to P(X)$ with $\rho \in \{id, \circ, -\circ, - \circ -, \circ-, \circ - \circ\}$ (cf. Definition 1.3 above for $\mathcal{E}_X := SO(X,\tau)$), we use the following notation:
($\bullet 3^\rho$) $\omega^\rho C(X,\tau) := \{A|\ A \text{ is } \omega^\rho\text{-closed in } (X,\tau)\})$;
($\bullet 3'^\rho$) $\omega^\rho O(X,\tau) := \{U|\ U \text{ is } \omega^\rho\text{-open in } (X,\tau)\}$ (cf. Definition 1.4 above).
(ii) ($\bullet 4$) $psC(X,\tau) := \{A|\ A \text{ is } ps\text{-closed in } (X,\tau)\}$;
($\bullet 4'$) $psO(X,\tau) := \{U|\ U \text{ is } ps\text{-open in } (X,\tau)\}$.
The concept of *ps-closed sets* of (ii) above (cf. [3, Definition 2.1]) is defined as follows: a subset $A$ is called a *ps-closed set* of $(X,\tau)$ if $pCl(A) \subset U$ whenever $A \subset U$ and $U \in SO(X,\tau)$; and its complement $X \setminus A$ is called a *ps*-open set of $(X,\tau)$.
(iii) We note that $\omega^{id}C(X,\tau) = \omega C(X,\tau)$ and $\omega^{id}O(X,\tau) = \omega O(X,\tau)$ (cf. Notations 1.2, 1.5(i)).
(iv) ($\bullet 5$) $C(X,\tau) := \{F|\ F \text{ is closed in } (X,\tau)\text{,i.e., } X \setminus F \in \tau\}$;
($\bullet 6$) $PC(V,\tau) := \{F|\ F \text{ is preclosed in } (X,\tau)\text{,i.e., } X \setminus F \in PO(X,\tau)\}$.

The purposes of the present paper are to characterlize the $\omega$-like closed sets of a topological space (cf. Theorem 2.1, Theorem 3.7, Proposition 4.4, Theorem 4.8) and to investigate the $(\omega^{\rho 1}, \omega^{\rho 2})$-$T^{\rho}_{1/2}$ separation axioms where $\rho 1, \rho 2, \rho \in \{id, \circ, \circ-\}$ (cf. Theorem 5.11, Theorem 5.13, Theorem 5.15). Moreover, in Section 6, it is shown that the digital line $(\mathbb{Z}, \kappa)$ is $\omega^{\circ -}$-$T_1$ except $\mathbb{Z}_\kappa$ (cf. Definition 2.3, Theorem 6.1(iv)).

## 2 Properties on $\omega$-like closed sets

For the families in Notation 1.5($\bullet 3^\rho$), ($\bullet 4$), ($\bullet 6$) and Notation 1.2 ($\bullet 1$), ($\bullet 2$), we have the following properties.

**Theorem 2.1** (i) $\omega^{\circ}C(X,\tau) \subset \omega C(X,\tau) \subset \omega^{-\circ}C(X,\tau)$.
(ii) $\omega^{-\circ}C(X,\tau) = \omega^{-\circ-}C(X,\tau) = P(X)$.
(iii) $\omega^{\circ}C(X,\tau) \subset \omega^{\circ-}C(X,\tau) \subset \omega^{-\circ}C(X,\tau)$.
(iv) $\omega^{\circ-}C(X,\tau) = \omega^{\circ-\circ}C(X,\tau)$.
(v) ([3, Corollary 2.6 (iv), Table 1]) $psC(X,\tau) = PC(X,\tau)$.
(vi) ([26],[27],[3]) $C(X,\tau) \subset \omega C(X,\tau) \subset {}^{w}\omega C(X,\tau)$.
(vii) ${}^{w}\omega C(X,\tau) = PC(X,\tau)$.

*Proof.* **(i) - (iv)** They are proved by definitions.

**(vii)** *Proof of the equality* ${}^{w}\omega C(X,\tau) = psC(X,\tau)$: let $A \in {}^{w}\omega C(X,\tau)$. For any subset $U \in SO(X,\tau)$ such that $A \subset U$, we have that $Cl(Int(A)) \subset U$ and so $pCl(A) = A \cup Cl(Int(A)) \subset U$; and so we see $pCl(A) \subset U$. Thus, we have that ${}^{w}\omega C(X,\tau) \subset psC(X,\tau)$. Conversely, supppose that $A \in psC(X,\tau)$. Let $U \in SO(X,\tau)$ such that $A \subset U$. Then, $pCl(A) = A \cup Cl(Int(A)) \subset U$ and hence $Cl(Int(A)) \subset U$. Therefore, $A$ is ${}^{w}\omega C(X,\tau)$. We proved that $psC(X,\tau) \subset {}^{w}\omega C(X,\tau)$.

Thus we show the required equality using (v). □

**Remark 2.2** By Theorem 2.1 above, the following diagram of implications is obtained. All implications in the following diagram are not reversible (cf. Example 2.4 (i) - (v) below); and two concepts of $C(X,\tau)$ and $\omega^{\circ}C(X,\tau)$ are independent (cf. Example 2.4(vi) below).

$$\begin{tikzcd}
A \in C(X,\tau) \arrow[dr] \arrow[dd, shift left, "/" marking] & & A \in {}^{w}\omega C(X,\tau) = psC(X,\tau) = PC(X,\tau) \\
& A \in \omega C(X,\tau) \arrow[ur] \arrow[dr] & \\
A \in \omega^{\circ}C(X,\tau) \arrow[ur] \arrow[uu, shift left, "/" marking] \arrow[d] & & A \in \omega^{-\circ}C(X,\tau) = \omega^{-\circ-}C(X,\tau) = P(X) \\
A \in \omega^{\circ-}C(X,\tau) = \omega^{\circ-\circ}C(X,\tau) \arrow[urr] & &
\end{tikzcd}$$

The concept of *the digital line* $(\mathbb{Z},\kappa)$ is initiatived by E.D. Khalimsky and sometimes it is called the Khalimsky line ([10, in 1990]).

**Definition 2.3** ([10, in 1990] and references there;[11, in 1991;p.905]; e.g.,[17, in 2014;Section 3]). *The digital line* or so called *Khalimsky line* $(\mathbb{Z},\kappa)$ is the set $\mathbb{Z}$ of all integers, equipped with the topology $\kappa$ having $\{\{2m-1, 2m, 2m+1\} |\ m \in \mathbb{Z}\}$ as a subbase. *The digital plane* or *Khalimsky plane* is the Cartesian product of 2-copies of the digital line $(\mathbb{Z},\kappa)$; this topological space is denoted by $(\mathbb{Z}^2,\kappa^2)$ (cf. [12, in 1994;Definition 4])

**Example 2.4** (i) An $\omega$-closed set need not be $\omega^{\circ}$-closed (i.e., $\omega^{\circ}C(X,\tau) \not\leftarrow \omega C(X,\tau)$): we give two examples as follows.

(i-1) Let $(X,\tau) := (\mathbb{Z},\kappa)$ be the digital line (cf. Definition 2.3 above) and $A := \{2m\}$, where $m \in \mathbb{Z}$. Then, by definition of the topology $\kappa, A := \{2m\}$ is closed and so $A \in \omega C(\mathbb{Z},\kappa)$. We show $A \notin \omega^{\circ}C(\mathbb{Z},\kappa)$. Indeed, there exists a semi-open set $U := \{2m, 2m+1\}$ such that $A \subset U$; and so we have that $Int(U) = \{2m+1\}$ and $Cl(A) = \{2m\} \not\subset \{2m+1\} = Int(U)$. This shown that the set $A$ is not $\omega^{\circ}$-closed in $(\mathbb{Z},\kappa)$.

(i-2) We can give an example on the Euclidean line $(X,\tau) := (\mathbb{R},\epsilon)$. Let $A := \{x, y\}$, where $x$ and $y$ are distinct point of $(\mathbb{R},\epsilon)$. There exists a semi-open set $U := \{t \in \mathbb{R} | x \leq t < z\} \cup \{t \in \mathbb{R} | z < t \leq y\}$, where $z$ is a point with a relation $x < z < y$. Then, $A \subset U$

and $Cl(A) = \{x, y\} \not\subset Int(U)$, because $Int(U) = \{t \in \mathbb{R} | x < t < z\} \cup \{t \in \mathbb{R} | z < t < y\}$; and so $A \notin \omega^{\circ}C(\mathbb{R}, \epsilon)$. And $A$ is closed and so $A \in \omega C(\mathbb{R}, \epsilon)$.

(ii) A ${}^{w}\omega$-closed set (=preclosed set; cf. Theorem 2.1 (v)(vii)) need not be $\omega$-closed (i.e., $\omega C(X,\tau) \not\leftarrow {}^{w}\omega C(X,\tau)$ ): let $(X,\tau) := (\mathbb{Z}^2, \kappa^2)$ be the digital plane (cf. Definition 2.3 above) and $A := \{x, y\}$ a subset of $(\mathbb{Z}^2, \kappa^2)$, where $x = (2m, 2s)$ and $y = (2m + 1, 2s)$ for some integers $m$ and $s$. Then, first we show that $Cl(Int(A)) = Cl(\emptyset) = \emptyset \subset A$; and so $A \in PC(\mathbb{Z}^2, \kappa^2)$ and hence $A \in {}^{w}\omega C(\mathbb{Z}^2, \kappa^2)$ (cf. Theorem 2.1(iii)). We note that the subset $A$ of the present example (ii) is a preclosed set which is not closed in $(\mathbb{Z}^2, \kappa^2)$.

Secondly, we show that $A \notin \omega C(\mathbb{Z}^2, \kappa^2)$. Indeed, we take a subset $U := A \cup \{(2m + 1, 2s + 1)\}$; then $U$ is semi-open in $(\mathbb{Z}^2, \kappa^2)$. Indeed since $\kappa^2 := \kappa \times \kappa$, we see that $Cl(Int(U)) = Cl(\{(2m + 1, 2s + 1)\}) = \{2m, 2m + 1, 2m + 2\} \times \{2s, 2s + 1, 2s + 2\} \supset U$ hold and so $U \subset Cl(Int(U))$ (i.e., $U \in SO(\mathbb{Z}^2, \kappa^2)$). Finally, we have that $A \subset U$ and $Cl(A) \not\subset U$. Indeed, $Cl(A) = Cl(\{x\}) \cup Cl(\{y\}) = A \cup \{(2m + 2, 2s)\} \not\subset U$ hold, because $\{x\} = \{(2m, 2s)\}$ is closed and $Cl(\{y\}) = Cl(\{(2m + 1, 2s)\} = \{(2m, 2s), (2m + 1, 2s), (2m + 2, 2s)\}$ holds in $(\mathbb{Z}^2, \kappa^2)$. Therefore, $A$ is not $\omega$-closed in $(\mathbb{Z}^2, \kappa^2)$. Moreover, we have a digital geometric example in Remark 4.5(ii).

(iii) An $\omega^{\circ -}$-closed set need not be $\omega^{\circ}$-closed (i.e., $\omega^{\circ}C(X,\tau) \not\leftarrow \omega^{\circ -}C(X,\tau)$ ): let $(X,\tau)$ be a topological space defined by $X := \{a, b, c\}$ and $\tau := \{\emptyset, \{a\}, \{a, b\}, X\}$. Then, we have $SO(X,\tau) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$. Let $A := \{a, c\}$ and $U \in SO(X,\tau)$ with $A \subset U$; and so $U = \{a, c\}$ or $X$. Then, $Cl(A) = X \subset Int(Cl(U))$, because $Int(Cl(U)) = X$ for each subset $U$; hence we show $A \in \omega^{\circ -}C(X,\tau)$. Moreover, we show that the subset $A$ is not $\omega^{\circ}$-closed in $(X,\tau)$. Indeed, the subset $A$ is a semi-open set with $Cl(A) = X \not\subset Int(A) = \{a\}$. In addtion, in Remark 4.5(ii) below, we have a geometric example of the present topic.

(iv) An $\omega^{-\circ}$-closed set need not be $\omega^{\circ -}$-closed (i.e., $\omega^{\circ -}C(X,\tau) \not\leftarrow \omega^{-\circ}C(X,\tau)$ ): let $A := \{2m + 1, 2m + 2, 2m + 3, 2m + 4\}$ be a subset of $(\mathbb{Z}, \kappa)$. Since $A \in P(\mathbb{Z}) = \omega^{-\circ}C(\mathbb{Z}, \kappa)$ (cf. Theorem 2.1(ii)), we should show $A \notin \omega^{\circ -}C(\mathbb{Z}, \kappa)$. Indeed, let $U := A$; and $Cl(Int(U)) = Cl(\{2m+1, 2m+2, 2m+3\}) = \{2m, 2m+1, 2m+2, 2m+3, 2m+4\} \supset U$ and so $U \in SO(\mathbb{Z}, \kappa)$ such that $A \subset U$. For this semi-open set $U$, we have that :

- $Cl(A) = \{2m, 2m + 1, 2m + 2, 2m + 3, 2m + 4\}$ and;
- $Int(Cl(U)) = \{2m + 1, 2m + 2, 2m + 3\}$.

Thus, it is shown that $Cl(A) \not\subset Int(Cl(U))$, i.e., $A$ is not $\omega^{\circ -}$-closed set in $(\mathbb{Z}, \kappa)$.

(v) An $\omega$-closed set need not be a closed set (i.e., $C(X,\tau) \not\leftarrow \omega C(X,\tau)$ ): such example is shown by [26].

(vi) Two families $C(X,\tau)$ and $\omega^{\circ}C(X,\tau)$ are independent.

· *Proof of* $\omega^{\circ}C(X,\tau) \not\leftarrow C(X,\tau)$ : the subset $A := \{2m\}$ of $(\mathbb{Z}, \kappa)$ in (i)(i-1) is a closed singleton, where $m \in \mathbb{Z}$, and it is not $\omega^{\circ}$-closed in $(\mathbb{Z}, \kappa)$ (cf. (i)(i-1)).

· *Proof of* $C(X,\tau) \not\leftarrow \omega^{\circ}C(X,\tau)$: let $(X,\tau)$ be a topological space defined by $X := \{a, b, c\}$ and $\tau := \{\emptyset, \{a\}, \{b, c\}, X\}$. Let $A := \{b\}$ be a not closed singleton. Let $U$ be a semi-open set containing $A$; then $U = \{b, c\}$ or $X$ and so $Int(U) = U$. Then, $Cl(A) = \{b, c\} \subset Int(U) = U$ hold and so we show that $A \in \omega^{\circ}C(X,\tau)$.

## 3 More characterizations of ${}^{w}\omega$-closed sets and related Janković Reilly decompositions

In the present section, we give more characterizations of ${}^{w}\omega$-closed sets (resp. $ps$-closed sets) by Theorem 3.7 (i)(1)(2)(3) (resp. (i) (4)(5)(6)(7)) below, even if we know that ${}^{w}\omega(X,\tau) = psC(X,\tau) = PC(X,\tau)$ hold for a topological space $(X,\tau)$ (cf. Theorem 2.1 (v)(vii)). They are done by an analogy of the Janković Reilly decomposition method; and so we recall them as follows (cf. Theorem 3.1, Notation 3.2, Lemma 3.4, Lemma 3.6 below).

**Theorem 3.1** (i) ([9, Lemma 2]) *Every singleton* $\{x\}$ *of a topological space* $(X, \tau)$ *is either preopen* (i.e., $\{x\} \subset Int(Cl(\{x\}))$) *or nowhere dense* (i.e., $Int(Cl(\{x\})) = \emptyset$).

(ii) (Janković Reilly decompostion;[2, p. 40, line +10]; cf. Theorem 3.3 below) *Any topological space* $(X, \tau)$ *has the following decomposition:*

$X = X_1 \bigcup X_2$ *with* $X_1 \bigcap X_2 = \emptyset$, *where* $X_1$ *and* $X_2$ *are defined respectively by:*
(1a) $X_1 := \{x \in X |\ \{x\}$ *is nowhere dense in* $(X, \tau)\}$;
(1b) $X_2 := \{x \in X |\ \{x\}$ *is preopen in* $(X, \tau)\}$.

The decomposition $X = X_1 \cup X_2$(disjoint union) of Theorem 3.1 is usefull and it is called the *Janković Reilly decomposition* of $X$ (e.g., [2, p. 40, line +10]). Moreover, we use the following convenient notation, because we want to investigate more decompositions.

**Notation 3.2** For a subset $E$ of $(X, \tau)$, we define the following subsets of $E$:
(•2a) $E_{\mathcal{ND}} := \{x |\ x \in E$ and $\{x\}$ is nowhere dense in $(X, \tau)\}$,
(i.e., $E_{\mathcal{ND}} = X_1 \cap E$ and $X_1 = X_{\mathcal{ND}}$, cf. (1a) of Theorem 3.1(ii) above);
(•2b) $E_{\mathcal{PO}} := \{x |\ x \in E$ and $\{x\}$ is preopen in $(X, \tau)\}$,
(i.e., $E_{\mathcal{PO}} = X_2 \cap E$ and $X_2 = X_{\mathcal{PO}}$, cf. (1b) of Theorem 3.1(ii) above);
(•2c) $E_{\mathcal{SC}} := \{x |\ x \in E$ and $\{x\}$ is semi-closed in $(X, \tau)\}$;
(•2d) $E_{\omega\mathcal{O}} := \{x |\ x \in E$ and $\{x\}$ is $\omega$-open in $(X, \tau)\}$;
(•2e) $E_{\tau} := \{x |\ x \in E$ and $\{x\}$ in open in $(X, \tau)\}$;
(•2f) $E_{\mathcal{C}} := \{x |\ x \in E$ and $\{x\}$ in closed in $(X, \tau)\}$;
(•2g) $E_{\mathcal{RO}} := \{x |\ x \in E$ and $\{x\}$ in regular-open in $(X, \tau)$, i.e., $\{x\} = Int(Cl(\{x\}))\}$.

By using Notation 3.2 (•2a),(•2b) above, the *Janković Reilly decomposition* in Theorem 3.1(ii) is stated as follows.

**Theorem 3.3** (Theorem 3.1(ii) above, [9, Lemma 2]) *For any subset* $E$ *of* $(X, \tau)$, $E = E_{\mathcal{PO}} \cup E_{\mathcal{ND}}$ *and* $E_{\mathcal{PO}} \cap E_{\mathcal{ND}} = \emptyset$ *hold.*

**Lemma 3.4** (i) *For any subset* $E$ *of* $(X, \tau)$, $E = E_{\mathcal{SC}} \cup E_{\omega\mathcal{O}}$ *holds.*

(ii) *For a topological space* $(X, \tau)$ *and a subset* $E$ *of* $(X, \tau)$,
(1) $X_{\mathcal{PO}} \cap X_{\mathcal{SC}} = (X_{\mathcal{PO}})_{\mathcal{SC}} = X_{\mathcal{RO}}$ *and* $X_{\mathcal{ND}} \cap X_{\omega\mathcal{O}} = (X_{\mathcal{ND}})_{\omega\mathcal{O}} \subset X_{\tau}$ *hold, and*
(2) $E_{\mathcal{PO}} \cap E_{\mathcal{SC}} = (E_{\mathcal{PO}})_{\mathcal{SC}} = E_{\mathcal{RO}}$ *and* $E_{\mathcal{ND}} \cap E_{\omega\mathcal{O}} = (E_{\mathcal{ND}})_{\omega\mathcal{O}} \subset E_{\tau}$ *hold.*

(iii) *Suppose one of the following properties:*

(a) $E_{\mathcal{ND}} = \emptyset$ *and* $E_{\mathcal{RO}} = \emptyset$; (b) $E_{\tau} = \emptyset$ *and* $(E_{\mathcal{PO}})_{\omega\mathcal{O}} = \emptyset$.

*Then,* $E_{\mathcal{SC}} \cap E_{\omega\mathcal{O}} = \emptyset$ *holds; and so the union* $E_{\mathcal{SC}} \cup E_{\omega\mathcal{O}}$ *of* (i) *is a disjoint union under the assumptions* (a) *or* (b) *above.*

*Proof.* **(i)** Let $x \in E$. We consider the following two cases.

Case 1. $\{x\}$ is not semi-closed in $(X, \tau)$: for this case, we show that $x \in E_{\omega\mathcal{O}}$. Indeed, $X$ is a unique semi-open set containing $X \setminus \{x\}$. Thus, $X \setminus \{x\}$ is $\omega$-closed in $(X, \tau)$ and so $\{x\}$ is an $\omega$-open set (i.e., $x \in E_{\omega\mathcal{O}}$).

Case 2. $\{x\}$ is semi-closed: for this case, it is shown that $x \in E_{\mathcal{SC}}$, by definition.

Therefore, using two cases, we have $E \subset E_{\mathcal{SC}} \cup E_{\omega\mathcal{O}}$; the converse inequality is trivial, by the definition of (•2c) and (•2d) in Notation 3.2. Thus we show the equality: $E = E_{\mathcal{SC}} \cup E_{\omega\mathcal{O}}$.

**(ii)** They are shown by using definitions.

**(iii)** In general, by using Theorem 3.1 (i.e., Theorem 3.3), it is shown that: $E_{\mathcal{SC}} \cap E_{\omega\mathcal{O}} = \{(E_{\mathcal{PO}} \cup E_{\mathcal{ND}})_{\mathcal{SC}}\} \cap \{(E_{\mathcal{PO}} \cup E_{\mathcal{ND}})_{\omega\mathcal{O}}\} = \{(E_{\mathcal{PO}})_{\mathcal{SC}} \cup (E_{\mathcal{ND}})_{\mathcal{SC}}\} \cap \{(E_{\mathcal{PO}})_{\omega\mathcal{O}} \cup (E_{\mathcal{ND}})_{\omega\mathcal{O}}\}$. We prove that $E_{\mathcal{SC}} \cap E_{\omega\mathcal{O}} = \emptyset$ holds under one of our assumptions (a), (b).
Case 1. we assume (a): for this case, we show that $E_{\mathcal{SC}} \cap E_{\omega\mathcal{O}} \subset (E_{\mathcal{PO}})_{\mathcal{SC}} \cup (E_{\mathcal{ND}})_{\mathcal{SC}} \subset$

$(E_{\mathcal{PO}})_{\mathcal{SC}} \cup E_{\mathcal{ND}} = E_{\mathcal{RO}} \cup E_{\mathcal{ND}} = \emptyset$ (cf. (ii)(2) above and the assumption (a)).
Case 2. we assume (b): for this case, we show that $E_{\mathcal{SC}} \cap E_{\omega\mathcal{O}} \subset (E_{\mathcal{PO}})_{\omega\mathcal{O}} \cup (E_{\mathcal{ND}})_{\omega\mathcal{O}} \subset (E_{\mathcal{PO}})_{\omega\mathcal{O}} \cup E_{\tau} = \emptyset$(cf. (ii)(2) above and the assumption (b)). □

**Remark 3.5** (i) The property ($X = X_{\mathcal{SC}} \cup X_{\omega\mathcal{O}}$) of Lemma 3.4 (i) above does not imply a disjoint union in general. For example, let $(X, \tau)$ be a topological space defined by $X := \{a, b, c\}$ and $\tau := \{\emptyset, \{a\}, \{b, c\}, X\}$. Then, a sigleton $\{a\}$ is semi-closed and $\omega$-open; and so $a \in X_{\mathcal{SC}} \cap X_{\omega\mathcal{O}}$.

(ii) For the digital line $(\mathbb{Z}, \kappa)$, we have the following datum on the subsets defined Lemma 3.4: $\mathbb{Z}_{\mathcal{PO}} = \{2m+1 | m \in \mathbb{Z}\}$=$\mathbb{Z}_{\kappa}$ (e.g. [6, Theorem 2.1(i)(a)]), $\mathbb{Z}_{\mathcal{ND}} = \{2m | m \in \mathbb{Z}\}$; and so we have the decomposition $\mathbb{Z} = \mathbb{Z}_{\mathcal{PO}} \cup \mathbb{Z}_{\mathcal{ND}}$. On the other hands, we have that $\mathbb{Z}_{\mathcal{SC}} = \mathbb{Z}$, $\mathbb{Z}_{\omega\mathcal{O}} = \{2m+1 | m \in \mathbb{Z}\}$; for a nonempty set $E$, $E_{\mathcal{ND}} = \{2m \in E | m \in \mathbb{Z}\}$ and $E_{\mathcal{RO}} = \{2m+1 \in E | m \in \mathbb{Z}\}$=$E_{\kappa}$ and $(E_{\mathcal{PO}})_{\omega\mathcal{O}} = E_{\mathcal{PO}}$.

We need the following lemma in order to prove Theorem 3.7 below; Lemma 3.6 (iii) and (iv) are applied; we recall the definitions of $sKer(\bullet)$ and $pKer(\bullet)$: for a subset $A$ of $(X, \tau)$, $sKer(A) := \bigcap\{U |\ U \in SO(X, \tau) \text{ and } A \subset U\}$ and $pKer(A) := \bigcap\{V |\ V \in PO(X, \tau) \text{ and } A \subset V\}$.

**Lemma 3.6** (cf. [4, Proposition 2.1]) *Let $B$ be a subset of $(X, \tau)$. Then, we have following properties.*

(i) [4, Proposition 2.1] $(sCl(B))_{\mathcal{PO}} \subset sKer(B)$.
(ii) [26, Proposition 2.2.18] $(Cl(B))_{\mathcal{PO}} \subset sKer(B)$.
(iii) $(Cl(Int(B)))_{\mathcal{PO}} \subset sKer(B)$.
(iv) $(pCl(B))_{\mathcal{PO}} \subset sKer(B)$.
(v) $(sKer(B))_{\mathcal{SC}} \subset B \subset sKer(B)$.
(vi) $((Cl(B))_{\mathcal{PO}})_{\mathcal{C}} \subset pKer(B)$.
(vi)' $((sCl(B))_{\mathcal{PO}})_{\mathcal{C}} \subset pKer(B)$.
(vi)" $((pCl(B))_{\mathcal{PO}})_{\mathcal{C}} \subset pKer(B)$.

*Proof.* **(iii)** Since $Cl(Int(B)) \subset Cl(B)$ holds and $E_{\mathcal{PO}} \subset F_{\mathcal{PO}}$ holds if $E \subset F$ in general, we have that $(Cl(Int(B)))_{\mathcal{PO}} \subset (Cl(B))_{\mathcal{PO}}$; and so, by (ii), it is shown that $(Cl(Int(B)))_{\mathcal{PO}} \subset sKer(B)$ holds.

**(iv)** This is proved by using (ii), because $pCl(E) \subset Cl(E)$ holds for any subset $E$ of $(X, \tau)$.

**(v)** We prove only the implication $(sKer(B))_{\mathcal{SC}} \subset B$. Let $x \in (sKer(B))_{\mathcal{SC}}$ and assume that $x \notin B$. Since $X \setminus \{x\} \in SO(X, \tau)$ and $B \subset X \setminus \{x\}$, it is shown that $sKer(B) \subset X \setminus \{x\}$. Then we have that $\{x\} \subset sKer(B) \subset X \setminus \{x\}$; and hence this is a contradiction.

**(vi)** Let $x \in ((Cl(B))_{\mathcal{PO}})_{\mathcal{C}}$. Suppose that $x \notin pKer(B)$. There exists a set $V \in PO(X, \tau)$ such that $B \subset V$ and $x \notin V$. Taking the set $X \setminus V$, then $X \setminus V$ is preclosed in $(X, \tau)$ and $x \in X \setminus V$. Then, we have that:
$\{x\} \cup Cl(Int(\{x\})) = pCl(\{x\}) \subset pCl(X \setminus V)$=$X \setminus V$; and so
$(\cdot\ 1)$ $Cl(Int(\{x\})) \subset X \setminus V$. Since $x \in Cl(B)$ and $B \subset V$, $(\cdot 2)$ $x \in Cl(\{x\}) \subset Cl(V)$. Since $x \in X_{\mathcal{PO}}$, we have that $(\cdot 3)$ $\{x\} \subset Int(Cl(\{x\}))$; and so we have that: $(\cdot 4)$ the set $Int(Cl(\{x\}))$ is an open set containing $x$ such that $x \in Cl(V)$.

By $(\cdot 2)$ and $(\cdot 4)$, it is shown that: $(\cdot 5)$ $Int(Cl(\{x\})) \cap V \neq \emptyset$. By using $(\cdot 1)$ and an assumption that $x \in X_{\mathcal{C}}$, it is shown that $Int(Cl(\{x\})) \cap V \subset Cl(Int(Cl(\{x\}))) \cap V = Cl(Int(\{x\})) \cap V \subset (X \setminus V) \cap V = \emptyset$. Therefore, we have that $Int(Cl(\{x\})) \cap V$=$\emptyset$; this contradicts the property $(\cdot 5)$ above.

**(vi)'** (resp. **(vi)"**) Since $sCl(B) \subset Cl(B)$ (resp. $pCl(B) \subset Cl(B)$) holds for every set $B$ of $(X, \tau)$, (vi)' (resp.(vi)") is obtaned by (vi). □

Finally, we have the following characterizations of weakly $\omega$-closed sets (i.e., ${}^{w}\omega$-closed sets) and $ps$-closed sets as follows.

**Theorem 3.7** (i) (cf. Theorem 2.1(v)(vi)) *For a subset $B$ of $(X,\tau)$, the following properties are equivalent:*
(1) *$B$ is ${}^{w}\omega$-closed in $(X,\tau)$;*
(2) $(Cl(Int(B)))_{\mathcal{ND}} \subset B$;
(3) $Cl(Int(B)) \subset sKer(B)$;
(4) *$B$ is $ps$-closed in $(X,\tau)$* (i.e., *$B$ is $(SO(X,\tau),PO(X,\tau))^{id}$-closed*);
(5) $(pCl(B))_{\mathcal{ND}} \subset B$;
(6) $pCl(B) \subset sKer(B)$;
(7) *$B$ is preclosed in $(X,\tau)$.*
(ii) *For a topological space $(X,\tau)$, ${}^{w}\omega O(X,\tau)$ forms a generalized topology of $X$ in the sense of Lugojan* ([15]) *such that $\tau \subset \omega O(X,\tau) \subset {}^{w}\omega O(X,\tau) = PO(X,\tau)$.*

*Proof.* **(i) (1)⇒(2)** Let $x \in (Cl(Int(B)))_{\mathcal{ND}}$. Suppose that $x \notin B$. The singleton $\{x\}$ is semi-closed, because $\{x\}$ is nowhere dense (i.e., $Int(Cl(\{x\})) = \emptyset$) and so $X \setminus \{x\}$ is a semi-open set containing $B$. By (1), $Cl(Int(B)) \subset X \setminus \{x\}$. We have a contradiction that $x \in X \setminus \{x\}$.
**(2)⇒(3)** Using Theorem 3.3, Lemma 3.6(iii) and (2), we have $Cl(Int(B))$-$=(Cl(Int(B)))_{\mathcal{PO}} \cup (Cl(Int(B)))_{\mathcal{ND}} \subset sKer(B) \cup B = sKer(B)$.
**(3)⇒(1)** Let $U \in SO(X,\tau)$ such that $B \subset U$. By definition of the concept of $sKer(\cdot)$ and (3), it is shown that $sKer(B) \subset U$ and so $Cl(Int(B)) \subset U$. Therefore, the set $B$ is ${}^{w}\omega$-closed in $(X,\tau)$.
**(4)⇒(5)** Let $x \in (pCl(B))_{\mathcal{ND}}$. Suppose that $x \notin B$. The singleton $\{x\}$ is semi-closed and so $X \setminus \{x\}$ is a semi-open set containing $B$. By (4), $pCl(B) \subset X \setminus \{x\}$. We have a contradiction that $x \in X \setminus \{x\}$.
**(5)⇒(6)** Using Theorem 3.3, Lemma 3.6(iv) and the assumption (5), we have that: $pCl(B)=(pCl(B))_{\mathcal{PO}} \cup (pCl(B))_{\mathcal{ND}} \subset sKer(B) \cup B = sKer(B)$.
**(6)⇒(4)** Let $U \in SO(X,\tau)$ such that $B \subset U$. By definition of the concept of $sKer(\cdot)$ and (6), it is shown that $sKer(B) \subset U$ and so $pCl(B) \subset U$. Therefore, the set $B$ is $ps$-closed in $(X,\tau)$.
**(6)⇒(7)** It follow from definition and (6) that the set $B$ is a $ps$-closed set. Indeed, let $U \in SO(X,\tau)$ such that $B \subset U$; and so $pCl(B) \subset sKer(B) \subset U$; thus $B \in psC(X,\tau)$. Using Theorem 2.1 (v), $B$ is preclosed.
**(7)⇒(1) and (1)⇒ (4)** They are obtained by using Theorem 2.1 (v),(vii).
**(ii)** These properties are obviously obtained by properties on $PC(X,\tau)$, because ${}^{w}\omega C(X,\tau) = PC(X,\tau)$ holds (cf.(i)). However, we attempt to prove them from the Janković Reilly decompositions method point of view. Let $\{B_i |\ i \in \Gamma\}$ be a family of ${}^{w}\omega$-closed sets in $(X,\tau)$ and let $B := \bigcap\{B_i |\ i \in \Gamma\}$. We have $Cl(Int(B)) \subset Cl(Int(B_i))$ for each $i \in \Gamma$ and so $(Cl(Int(B)))_{\mathcal{ND}} \subset \bigcap\{(Cl(Int(B_i)))_{\mathcal{ND}} |\ i \in \Gamma\} \subset \bigcap\{B_i |\ i \in \Gamma\} = B$ (cf. (i) (1)⇒(2)). Namely, by the equivalente property (2)⇔(1) in (i), the set $B$ is ${}^{w}\omega$-closed in $(X,\tau)$. It is obvious that $\emptyset \in {}^{w}\omega O(X,\tau)$ and $X$ *in* ${}^{w}\omega O(X,\tau)$. Thus, it is shown that ${}^{w}\omega O(X,\tau)$ is a generalized topology of $X$ in the sense of Lugojan ([15]). □

**Remark 3.8** Using Janković Reilly decomposition method (cf. Theorem 3.3), we show an alternative proof of Theorem 2.1(v), i.e., $psC(X,\tau) = PC(X,\tau)$ hold (cf. [3, Corollary 2.6 (iv), Table 1]). First we show that $psC(X,\tau) \subset PC(X,\tau)$. Let $A \in psC(X,\tau)$ and $x \in pCl(A)$. We claim that $x \in A$. We recall that $pCl(A) = (pCl(A))_{\mathcal{PO}} \cup (pCl(A))_{\mathcal{ND}}$. When $x \in (pCl(A))_{\mathcal{PO}}, \{x\}$ is preopen and so $\{x\} \cap A \neq \emptyset$ (i.e., $x \in A$). When $x \in (pCl(A))_{\mathcal{ND}}$, it is obtained that $x \in A$, by Theorem 3.7 (i)(4)⇒(5). Therefore, for both cases, we have $x \in A$ whenever $x \in pCl(A)$, i.e., $A \in PC(X,\tau)$ and so $psC(X,\tau) \subset PC(X,\tau)$. The converse implication is obvious.

In the end of the present Section 3, we apply Lemma 3.4 (i) to an alternative characterization of the $\omega$-closed sets; the equivalent property (3) $\Leftrightarrow$ (4) in Theorem 3.9 below is shown by using Lemma 3.4(i).

**Theorem 3.9** (Sheik John [26] for (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3)) *For a subset $B$ of $(X,\tau)$, the following properties are equivalent:*

(1) *$B$ is $\omega$-closed in $(X,\tau)$;*

(2) *$(Cl(B))_{\mathcal{ND}} \subset B$;*

(3) *$Cl(B) \subset sKer(B)$;*

(4) (a) *$(Cl(B))_{\mathcal{SC}} \subset B$ and* (b) *$(Cl(B))_{\omega\mathcal{O}} \subset sKer(B)$ hold.*

*Proof.* (3)$\Rightarrow$(4) First we claim that $(sKer(B))_{\mathcal{SC}} \subset B$. Indeed, let $x \in (sKer(B))_{\mathcal{SC}}$ and assume that $x \notin B$. Since the set $X \setminus \{x\} \in SO(X,\tau)$ and $B \subset X \setminus \{x\}$, $sKer(B) \subset X \setminus \{x\}$. Then, we have that $\{x\} \subset X \setminus \{x\}$ and so this is a contradiction. Thus, we show that $(sKer(B))_{\mathcal{SC}} \subset B$. By using (3), it is shown that $(Cl(B))_{\mathcal{SC}} \subset (sKer(B))_{\mathcal{SC}} \subset B$; and so (a) is proved. The property (b) is obtained by (3), because $(Cl(B))_{\omega\mathcal{O}} \subset Cl(B) \subset sKer(B)$ hold.

(4)$\Rightarrow$(3): Using Lemma 3.4 (i) and (4), we have that $Cl(B)=(Cl(B))_{\mathcal{SC}} \cup (Cl(B))_{\omega\mathcal{O}} \subset B \cup sKer(B)=sKer(B)$. That is, $Cl(B) \subset sKer(B)$ holds. □

## 4 Some properties of $\omega^\rho$-closed sets, where $\rho \in \{\circ, \circ-\}$

After some characteriations of $\omega^\rho$-closedness (cf. Proposition 4.4), we add a complete characterization of the $\omega^\rho$-closedness, where $\rho : SO(X,\tau) \to P(X)$ is a function such that $\rho \in \{\circ, \circ-\}$ (cf. Theorem 4.8(iii)).

**Theorem 4.1** (i) *The union of two $\omega^\circ$-closed* (resp. *$\omega^{\circ-}$-closed*) *sets is $\omega^\circ$-closed* (resp. *$\omega^{\circ-}$-closed*).

(ii) *If $A$ is $\omega^\circ$-closed* (resp. *$\omega^{\circ-}$-closed*) *and $A \subset B \subset Cl(A)$, then $B$ is $\omega^\circ$-closed* (resp. *$\omega^{\circ-}$-closed*).

(iii) *If $A$ is $\omega^\circ$-closed* (resp. *$\omega^{\circ-}$-closed*), *then $Cl(A) \backslash A$ does not contain any nonempty semi-closed* (resp. *semi-closed and semi-open set*).

*Proof.* **(i)** Let $A, B \in \omega^\circ C(X,\tau)$ (resp. $A, B \in \omega^{\circ-}C(X,\tau)$) and $U \in SO(X,\tau)$ such that $A \cup B \subset U$. Then, it follows from assumptions that $Cl(A \cup B) = Cl(A) \cup Cl(B) \subset Int(U)$ (resp. $Cl(A \cup B) \subset Int(Cl(U))$), because $Cl(A) \subset Int(U)$ and $Cl(B) \subset Int(U)$ hold (resp. $Cl(A) \subset Int(Cl(U))$ and $Cl(B) \subset Int(Cl(U))$ hold). Thus, we show that $A \cup B \in \omega^\circ C(X,\tau)$ (resp. $A \cup B \in \omega^{\circ-}C(X,\tau)$).

**(ii)** Let $U \in SO(X,\tau)$ such that $B \subset U$. Then, by assumptions, it is shown that $Cl(B) = Cl(A), A \subset U$ and so $Cl(B) \subset Int(U)$ (resp. $Cl(B) \subset Int(Cl(U))$), i.e., $B \in \omega^\circ C(X,\tau)$ (resp. $B \in \omega^{\circ-}C(X,\tau)$).

**(iii)** Case 1. $A \in \omega^\circ C(X,\tau)$: suppose that $Cl(A) \setminus A$ contains a semi-closed set $F$. Since $A \subset X \setminus F$ and $X \setminus F \in SO(X,\tau), Cl(A) \subset Int(X \setminus F)$ holds. Thus, we have that $Cl(F) = X \setminus (Int(X \setminus F)) \subset X \setminus Cl(A)$ and so $Cl(A) \subset X \setminus Cl(F)$. We have that $F \subset Cl(F) \cap Cl(A) \subset (X \setminus Cl(A)) \cap Cl(A)$, because $F \subset Cl(A)$ holds; and hence $F = \emptyset$.

Case 2. $A \in \omega^{\circ-}C(X,\tau)$: suppose that $Cl(A) \setminus A$ contains a semi-closed and semi-open set $F$. Since $A \subset X \setminus F$ and $X \setminus F \in SO(X,\tau), Cl(A) \subset Int(Cl(X \setminus F))$ holds. Thus, we have that $Cl(Int(F)) = X \setminus (Int(Cl(X \setminus F))) \subset X \setminus Cl(A)$ and so $Cl(A) \subset X \setminus Cl(Int(F))$. Then, we have that $F \subset Cl(Int(F)) \cap Cl(A) \subset (X \setminus Cl(A)) \cap Cl(A)$, because $F \subset Cl(A)$ and $F$ is semi-open; and hence $F = \emptyset$. $\square$

Moreover, as continuation of Notation 3.2, we prepare the following notation.

**Notation 4.2** For a subset $E$ of $(X, \tau)$, we define the following families: (cf. Defintion 1.4)
($\bullet$3*a*) $E_{\omega^\circ\mathcal{O}} := \{x|\ x \in E$ and $\{x\}$ is $\omega^\circ$-open set of $(X, \tau)\}$;
($\bullet$3*b*) $E_{\omega^{\circ-}\mathcal{O}} := \{x|\ x \in E$ and $\{x\}$ is $\omega^{\circ-}$-open set of $(X, \tau)\}$;
($\bullet$3c) $E_{\mathcal{PC}} := \{x|\ x \in E$ and $\{x\}$ is preclosed in $(X, \tau)\}$.

**Lemma 4.3** *For a topological space $(X, \tau)$ and a subset $E$ of $(X, \tau)$, we have the following properties* (cf. Notation 3.2, Notation 4.2).
(i) $X = X_{\mathcal{SC}} \cup X_{\omega^\circ\mathcal{O}}$ *and* $E = E_{\mathcal{SC}} \cup E_{\omega^\circ\mathcal{O}}$ *hold.*
(ii) $X = (X_{\mathcal{SC}} \cap X_\tau) \cup X_{\omega^{\circ-}\mathcal{O}}$ *and* $E = (E_{\mathcal{SC}} \cap E_\tau) \cup E_{\omega^{\circ-}\mathcal{O}}$ *hold.*
(iii) $X = (X_{\mathcal{SC}} \cap X_{\mathcal{PO}}) \cup X_{\omega^{\circ-}\mathcal{O}}$ *and* $E = (E_{\mathcal{SC}} \cap E_{\mathcal{PO}}) \cup E_{\omega^{\circ-}\mathcal{O}}$ *hold.*

*Proof.* **(i)** First, let $x \in X$. Suppose that $x \notin X_{\mathcal{SC}}$. We claim that $x \in X_{\omega^\circ\mathcal{O}}$. Indeed, let $U$ be any semi-open set containing $X \setminus \{x\}$. Then, $U = X$, because $X \setminus \{x\}$ is not semi-open and so $X$ is a unique semi-open set containing $X \setminus \{x\}$. Thus, $Cl(X \setminus \{x\}) \subset U = X = Int(U)$, i.e., $X \setminus \{x\}$ is $\omega^\circ$-closed, i.e. $x \in X_{\omega^\circ\mathcal{O}}$. Therefore, we have that $X = X_{\mathcal{SC}} \cup X_{\omega^\circ\mathcal{O}}$ holds. And, for the final property that $E = E_{\mathcal{SC}} \cup E_{\omega^\circ\mathcal{O}}$, the proof is obvious, because of the facts that $E_{\mathcal{SC}}{=}E \cap X_{\mathcal{SC}}$ and $E_{\omega^\circ\mathcal{O}}{=}E \cap X_{\omega^\circ\mathcal{O}}$ for any subset $E$ of $(X, \tau)$.

**(ii)** First, let $x \in X$ and suppose that $x \in X \setminus (X_{\mathcal{SC}} \cap X_\tau)$. We claim that $x \in X_{\omega^{\circ-}\mathcal{O}}$. Let $U \in SO(X, \tau)$ such that $X \setminus \{x\} \subset U$. Then, $U = X$ or $U = X \setminus \{x\}$.

Case 1. $x \notin X_{\mathcal{SC}}$: by similar argument of the proof of (i), it is shown that $X \setminus \{x\} \notin SO(X, \tau)$ and so $U = X$ and $Cl(X \setminus \{x\}) \subset X = Int(Cl(U))$.

Case 2. $x \notin X_\tau$: for this case, if $U = X$, then $Cl(X \setminus \{x\}) \subset X = Int(Cl(X)) = Int(Cl(U))$; if $U = X \setminus \{x\}$, then $X \setminus \{x\} \neq Cl(X \setminus \{x\}){=}X$-
$={Int}(X){=}Int(Cl(X \setminus \{x\})) = Int(Cl(U))$.

By both cases, $X \setminus \{x\}$ is $\omega^{\circ-}$-closed in $(X, \tau)$, i.e., $x \in \omega^{\circ-}O(X, \tau)$ under the assumption that the point $x$ satiesfies Case 1 or Case 2 above. Therefore, we show that, for a point $x \in X, x \in X_{\mathcal{SC}} \cap X_\tau$ or $x \in X_{\omega^{\circ-}\mathcal{O}}$, i.e., $X \subset (X_{\mathcal{SC}} \cap X_\tau) \cup X_{\omega^{\circ-}\mathcal{O}}$ holds; and hence we have the required first equality. Since $E_\mathcal{E} = E \cap X_\mathcal{E}$ holds where the symbol $\mathcal{E} \in \{\mathcal{SC}, \tau, \omega^{\circ-}\mathcal{O}\}$, we have the final equality using the firsr property above.

**(iii)** By using (ii) above and the following fact that $E_\tau \subset E_{\mathcal{PO}}$ holds, it is shown that $E = (E_{\mathcal{SC}} \cap E_\tau) \cup E_{\omega^{\circ-}\mathcal{O}} \subset (E_{\mathcal{SC}} \cap E_{\mathcal{PO}}) \cup E_{\omega^{\circ-}\mathcal{O}}$ hold. Hence, we have the required equalities. $\square$

We have the following property: ($\bullet$) For a subset $A$ of $(X, \tau)$, $(Cl(A))_\tau \subset A$ holds. Indeed, let $x \in (Cl(A))_\tau$. Suppose that $x \notin A$. Since $A \subset X \setminus \{x\}$ and $\{x\}$ is open, i.e., $X \setminus \{x\}$ is closed, we have that $Cl(A) \subset Cl(X \setminus \{x\}) = X \setminus \{x\}$; and so we have that $x \in Cl(A) \subset X \setminus \{x\}$; this contradicts $x \notin X \setminus \{x\}$. ($\square$)

For an $\omega^\rho$-closed set $A$, where $\rho \in \{\circ, \circ-\}$, we have an analogouse form of the property ($\bullet$) above and Theorem 3.7 (cf. Proposition 4.4 and Remark 4.5 below).

**Proposition 4.4** (i) *If $A$ is an $\omega^{\circ-}$-closed set of $(X, \tau)$, then* $((Cl(A))_{\mathcal{PC}})_{\mathcal{SO}} \subset A$ (cf. Notations 3.2($\bullet$2e),4.2($\bullet$3d); Remark 4.5 (i), (ii)).
(ii) *If $A$ is an $\omega^\circ$-closed set of $(X, \tau)$, then* $(Cl(A))_{\mathcal{SC}} \subset A$ (cf. Remark 4.5 (iii),(iv)).
(iii) *If $A$ is an $\omega^{\circ-}$-closed set of $(X, \tau)$, then* $((Cl(A))_{\mathcal{SC}})_{\mathcal{SO}} \subset A$ (cf. Remark 4.5 (vii),(viii)).

(iv) *If $A$ is an $\omega^{\circ -}$-closed set of $(X, \tau)$, then* $((Cl(A))_{ND})_{\mathcal{SO}} \subset A$ (cf. Remark 4.5 (v),(vi)).

*Proof.* **(i)** First, we recall that $(E_{\mathcal{PC}})_{\mathcal{SO}} = E_{\mathcal{PC}} \cap E_{\mathcal{SO}}$ holds for any set $E \subset X$. Let $x \in ((Cl(A))_{\mathcal{PC}})_{\mathcal{SO}}$. Suppose that $x \notin A$. Since $A \subset X \setminus \{x\}$ and $X \setminus \{x\}$ is preopen (i.e., $X \setminus \{x\} \subset Int(Cl(X \setminus \{x\})))$, the set $Int(Cl(X \setminus \{x\}))$ is a semi-open set containing $A$. Since $A$ is $\omega^{\circ -}$-closed, we have that $Cl(A) \subset Int(Cl(\{Int(Cl(X \setminus \{x\}))\})) = Int(Cl(X \setminus \{x\})) = X \setminus Cl(Int(\{x\}))$ ; and so $x \in X \setminus Cl(Int(\{x\}))$, i.e., $(*)$ $x \notin Cl(Int(\{x\}))$. On the other hand, it follows from the assumption ($x \in ((Cl(A))_{\mathcal{PC}})_{\mathcal{SO}} \subset X_{\mathcal{SO}}$) for the point $x$ that $\{x\} \subset Cl(Int(\{x\}))$ holds; this contradicts the property $(*)$ above.

**(ii)** Let $x \in (Cl(A))_{\mathcal{SC}}$. And suppose that $x \notin A$. Then, $A \subset X \setminus \{x\}$ and $X \setminus \{x\} \in SO(X, \tau)$, we have $Cl(A) \subset Int(X \setminus \{x\})$; and so $x \in Int(X \setminus \{x\}) = X \setminus Cl(\{x\})$, i.e., $x \notin Cl(\{x\})$; this contradicts the property: $E \subset Cl(E)$ for any subset $E$.

**(iii)** Let $x \in ((Cl(A))_{\mathcal{SC}})_{\mathcal{SO}}$ such that $x \notin A$. Since $A \subset X \setminus \{x\}$ and $X \setminus \{x\} \in SO(X, \tau)$ and $A$ is $\omega^{\circ -}$-closed, we have that $Cl(A) \subset Int(Cl(X \setminus \{x\}))=X \setminus Cl(Int(\{x\}))$. Since $X \setminus x \in X_{\mathcal{SC}}$ , $Int(Cl(X \setminus \{x\})) \subset X \setminus \{x\}$ and so $x \in X \setminus \{x\}$; this is a contradiction.

**(iv)** It is known that $E_{ND} \subset E_{\mathcal{SC}}$ holds for any set $E$ of a topological space $(X, \tau)$. Then, for the given $\omega^{\circ -}$-closed set $A$, by (iii) above, it is obtained that $((Cl(A))_{ND})_{\mathcal{SO}} \subset ((Cl(A))_{\mathcal{SC}})_{\mathcal{SO}} \subset A$. □

**Remark 4.5** (i) The converse of Proposition 4.4 (i) is not true from the following example. Let $A := \{2m+1\}$ be a subset of the digital line $(\mathbb{Z}, \kappa)$. First, we claim that $A$ is not $\omega^{\circ -}$-closed in $(\mathbb{Z}, \kappa)$. Indeed, the set $A$ is semi-open; and, take $U := A \in SO(\mathbb{Z}, \kappa)$; then, we have that $Cl(A) = \{2m, 2m+1, 2m+2\} \not\subset Int(Cl(U))=\{2m+1\}$; and so $A$ is not $\omega^{\circ -}$-closed in $(\mathbb{Z}, \kappa)$. Finally, we show that $((Cl(A))_{\mathcal{PC}})_{\mathcal{SO}}=(\{2m, 2m+2\})_{\mathcal{SO}}=\emptyset \subset A$ hold.

(ii) Let $A := \{0\} \cup \{2s+1 \in \mathbb{Z} | \ s \in \mathbb{Z}\}$ be an open set of $(\mathbb{Z}, \kappa)$. Then, $A$ is an example of the $\omega^{\circ -}$-closed set which satisfies Proposition 4.4(i)). Indeed, let $U \in SO(\mathbb{Z}, \kappa)$ such that $A \subset U$. Since $A \in \kappa \subset SO(\mathbb{Z}, \kappa)$, we have that $Cl(A) = \mathbb{Z}=Int(\mathbb{Z})=Int(Cl(A)) \subset Int(Cl(U))$; and so $A$ is $\omega^{\circ -}$-closed in $(\mathbb{Z}, \kappa)$. Moreover, $(((Cl(A))_{\mathcal{PC}})_{\mathcal{SO}}=(\mathbb{Z}_{\mathcal{PC}})_{\mathcal{SO}}=(\{2s| \ s \in \mathbb{Z}\})_{\mathcal{SO}}=\emptyset \subset A$ hold in $(\mathbb{Z}, \kappa)$. On the other hand, the present set $A$ is an example which is not $\omega^{\circ}$-closed in $(\mathbb{Z}, \kappa)$. Indeed, take $U := A \in SO(\mathbb{Z}, \kappa)$; and so $Cl(A)=\mathbb{Z} \not\subset Int(U)=A$; by Definition 1.4, $A$ is not $\omega^{\circ}$-closed. Moreover, since $(Cl(A))_{\mathcal{SC}}=\mathbb{Z} \not\subset A$ holds, the set $A$ is not $\omega^{\circ}$-closed in $(\mathbb{Z}, \kappa)$ (cf. Proposition 4.4(ii)).

(iii) The converse of Proposition 4.4 (ii) is not true from the following example. Let $A := \{2m, 2m+1, 2m+2\}$ be a subset of $(\mathbb{Z}, \kappa)$ and the semi-open set $U := A$. It is shown that $Cl(A) = A \not\subset Int(U)=\{2m+1\}$; and so $A$ is not $\omega^{\circ}$-closed. On the other hands, $(Cl(A))_{\mathcal{SC}}=\mathbb{Z}_{\mathcal{SC}} \cap Cl(A)=\mathbb{Z} \cap A=A$ hold in $(\mathbb{Z}, \kappa)$.

(iv) Using contraposition of Proposition 4.4(ii), we can find any examples of non-$\omega^{\circ}$-closed sets. For example, the subset $A := \{2m+1\}$ given by (i) above is not $\omega^{\circ}$-closed in $(\mathbb{Z}, \kappa)$. Indeed, $(Cl(A))_{\mathcal{SC}}=\mathbb{Z}_{\mathcal{SC}} \cap Cl(A)=\mathbb{Z} \cap Cl(A)=\{2m, 2m+1, 2m+2\} \not\subset A$; and so $A$ is not $\omega^{\circ}$-closed in $(\mathbb{Z}, \kappa)$.

(v) We have an example of an $\omega^{\circ -}$-closed set $A$ which satisfies Proposition 4.4 (iii). We consider the $\omega^{\circ -}$-closed set $A$ of (ii) above, say $A := \{0\} \cup \{2s+1 \in \mathbb{Z} | \ s \in \mathbb{Z}\}$. Indeed, since $(Cl(A))_{\mathcal{SC}}=\mathbb{Z}_{\mathcal{SC}}=\mathbb{Z}$, we have that $(((Cl(A))_{\mathcal{SC}})_{\mathcal{SO}}=\mathbb{Z}_{\mathcal{SO}}=\{2s+1 \in \mathbb{Z} | \ s \in \mathbb{Z}\} \subset A$.

(vi) The converse of Proposition 4.4 (iii) is not true. Let $A := \{2s+1 \in \mathbb{Z} | \ s \in \mathbb{Z}\} \setminus \{1\}$ be an open set of $(\mathbb{Z}, \kappa)$. Then, we have that $Cl(A)=\mathbb{Z} \setminus \{1\}$ and so $((Cl(A))_{\mathcal{SC}})_{\mathcal{SO}}=$ - $(Cl(A))_{\mathcal{SO}}=(\mathbb{Z} \setminus \{1\})_{\mathcal{SO}}=A$, because any singleton $\{x\}$ is semi-closed, any odd singleton $\{2s+1\}$ is semi-open and any even singleton $\{2s\}$ is not semi-open in $(\mathbb{Z}, \kappa)$, where

$s \in \mathbb{Z}$. And, the set $A$ is not $\omega^{\circ-}$-closed in $(\mathbb{Z}, \kappa)$. Indeed, there exists a semi-open set $U := A$ such that $A \subset U$; and so $Cl(A)=\mathbb{Z} \setminus \{1\} \not\subset \{z \in \mathbb{Z}|z \leq -1\} \cup \{z \in \mathbb{Z}|3 \leq z\}=Int(\mathbb{Z} \setminus \{1\})=Int(Cl(A))$; and hence the set $A$ is not $\omega^{\circ-}$-closed in $(\mathbb{Z}, \kappa)$.

(vii) The converse of Propositon 4.4(iv) is not true. The following subset $A := \{2m-2, 2m-1, 2m+1, 2m+2\}$ of $(\mathbb{Z}, \kappa)$ is an example of non-$\omega^{\circ-}$-closed sets. Indeed, we know that $A \in SO(\mathbb{Z}, \kappa)$ such that $A \subset A$; and so $Cl(A)=A \cup \{2m\} \not\subset \{2m-1, 2m, 2m+1\}=Int(Cl(A))$; thus $A$ is not $\omega^{\circ-}$-closed. Moreover, $((Cl(A))_{\mathcal{ND}})_{\mathcal{SO}}=(\{2m-2, 2m, 2m+2\})_{\mathcal{SO}}=\emptyset \subset A$ hold.

(viii) (cf. Proposition 4.4(iv)) For the $\omega^{\circ-}$-closed set $A :=\{0\} \cup \{2s+1|\ s \in \mathbb{Z}\}$ of (ii) above, we check the following property: $((Cl(A))_{ND})_{\mathcal{SO}} \subset A$. Indeed, $((Cl(A))_{\mathcal{ND}})_{\mathcal{SO}}$ - $=(\mathbb{Z}_{\mathcal{ND}})_{\mathcal{SO}}=(\{2s|\ s \in \mathbb{Z}\})_{\mathcal{SO}}=\emptyset \subset A$ hold.

We define some analogouse concepts of the sets $Ker(\bullet)$ and $sKer(\bullet)$ (cf. Definition 4.6) and we characterize the $\omega^{\rho}$-closedness of a subset, where $\rho : SO(X, \tau) \to P(X)$ is a function such that $\rho \in \{\circ, \circ-\}$ (cf. Theorem 4.8(iii) below).

**Definition 4.6** For a subset $A$ of $(X, \tau)$ and a function $\rho : SO(X, \tau) \to P(X)$ with $\rho \in \{id, \circ-, \circ\}$, we define the following subsets:

(·) $s^{\rho}Ker(A) := \bigcap\{W|\ W \in SO(X, \tau)$ and $A \subset \rho(W)\}$;

(·)' $s^{\rho}Ker^{'}(A) := \bigcap\{\rho(W)|\ W \in SO(X, \tau)$ and $A \subset \rho(W)\}$;

(·)" $s^{\rho}Ker_1(A) := \bigcap\{\rho(W)|\ W \in SO(X, \tau)$ and $A \subset W\}$.

We note that $s^{id}Ker(A)=s^{id}Ker^{'}(A)=s^{id}Ker_1(A)=sKer(A)$ hold.

**Proposition 4.7** (i) *For any subset $A$ of a topological space $(X, \tau)$, we have the following properties:*
(i-1) *$s^{\circ}Ker_1(A) \subset s^{\circ}Ker'(A) \subset s^{\circ}Ker(A)$;*
(i-2) *$s^{\circ-}Ker(A) \subset s^{\circ}Ker(A)$;*
(i-3) *$A \subset s^{\circ}Ker(A)$.*

(ii) (ii-1) *There exists a subset $A$ of $(\mathbb{Z}, \kappa)$ such that $s^{\circ-}Ker(A) \subsetneqq A$.*
(ii-2) *There exists a subset $A$ of $(\mathbb{Z}, \kappa)$ such that $s^{\circ}Ker_1(A) \subsetneqq A$ and $s^{\circ-}Ker_1(A) \subsetneqq A$.*

*Proof* **(i) (i-1)** Let $\rho := \circ$ throughout the present proof of (i-1).
*Proof of $s^{\circ}Ker_1(A) \subset s^{\circ}Ker'(A)$*: let $x$ be any point such that $x \notin s^{\circ}Ker'(A)$. Then, by Definition 4.6(·)', there exists a subset $W \in SO(X, \tau)$ such that $x \notin \rho(W) = Int(W)$ and $A \subset \rho(W) = Int(W)$; and so $x \notin s^{\circ}Ker_1(A)$ (cf. Definition 4.6(·)"), because $\rho(W) \subset W$ holds for $\rho = \circ$.

*Proof of $s^{\circ}Ker'(A) \subset s^{\circ}Ker(A)$*: let $x$ be any point such that $x \notin s^{\circ}Ker(A)$. Then, by Definition 4.6(·), there exists a subset $W \in SO(X, \tau)$ such that $x \notin W$ and $A \subset \rho(W)$; and so $x \notin s^{\circ}Ker'(A)$, because $\rho(W) \subset W$ and so $x \notin \rho(W)$ holds for $\rho = \circ$.

**(i-2)** Let $x$ be any pont such that $x \notin s^{\circ}Ker(A)$. Then, by Definition 4.6(·), there exists a subset $W \in SO(X, \tau)$ such that $x \notin W$ and $A \subset Int(W)$; and so $x \notin s^{\circ-}Ker(A)$ (cf. Definition 4.6(·)), because $A \subset Int(W) \subset Int(Cl(W))$ holds.

**(i-3)** Let $x$ be any point such that $x \notin s^{\circ}Ker(A)$. Then, by Definition 4.6(·), there exists a subset $W \in SO(X, \tau)$ such that $x \notin W$ and $A \subset Int(W)$; and so $x \notin A$, because $Int(W) \subset W$.

**(ii) (ii-1)** We prepare the following notation: $K^{\rho}_{A}(X, \tau) := \{S|\ S \in SO(X.\tau)$ and $A \subset \rho(S)\}$, where $\rho : SO(X, \tau) \to P(X)$ be a function and a subset $A$ of a topological space $(X, \tau)$. Then, $(*)$ $s^{\rho}Ker(A)=\bigcap\{W|\ W \in K^{\rho}_{A}(X, \tau)\}$ holds.

Let $(X, \tau)$ be the digital line $(\mathbb{Z}, \kappa)$ and $\rho := \circ-$. Let $A := \{0\} \cup \{2s+1|\ s \in \mathbb{Z}\}$ and $W_0 := A \setminus \{0\}$. Then, since $A \subset \rho(W_0) = Int(Cl(W_0)) = \mathbb{Z}$, $A \subset \rho(A) = \mathbb{Z}$ and

$W_0, A \in SO(\mathbb{Z},\kappa)$, we have that $W_0 \in K^{\rho}_{A}(\mathbb{Z},\kappa)$ and $A \in K^{\rho}_{A}(\mathbb{Z},\kappa)$. Therefore, we have that $s^{\circ-}Ker(A) \subset W_0 \subsetneqq A$ holds for the set $A$.

**(ii-2)** Let $\rho \in \{\circ, \circ-\}$ and let $A := \{-5, 0, 1, 5\}$ be a subset of $(\mathbb{Z},\kappa)$. Then, $A \in SO(\mathbb{Z},\kappa)$ and $\rho(A) = \{-5, 1, 5\}$ for the function $\rho \in \{\circ, \circ-\}$. We are able to take the set $W := A$ as a semi-open set $W$ in the set $s^{\rho}Ker_1(A) := \bigcap\{\rho(W) |\ W \in SO(\mathbb{Z},\kappa)$ and $A \subset W\}$, then it is obtained that $s^{\rho}Ker_1(A) \subset \rho(A) = \{-5, 1, 5\} \subsetneqq \{-5, 0, 1, 5\} = A$; and hence $s^{\rho}Ker_1(A) \subsetneqq A$ for the present set $A$ and $\rho \in \{\circ, \circ-\}$. □

**Theorem 4.8** *Let $A$ be a subset of $(X,\tau)$.*

(i) *If $A$ is $\omega^{\circ}$-closed in $(X,\tau)$, then $Cl(A) \subset s^{\circ}Ker(A)$* (cf. Remark 4.9 (i) below).

(ii) *If $A$ is $\omega^{\circ-}$-closed in $(X,\tau)$, then $(Cl(A))_{\mathcal{PO}} \subset s^{\circ-}Ker(A)$* (cf. Remark 4.9 (ii) below).

(iii) *$A$ is an $\omega^{\rho}$-closed set of $(X,\tau)$ if and only if $Cl(A) \subset s^{\rho}Ker_1(A)$ holds, where $\rho : SO(X,\tau) \to P(X)$ is a function such that $\rho \in \{\circ, \circ-\}$.*

*Proof* **(i)** Throughout the present proof, let $\rho := \circ : SO(X,\tau) \to P(X)$ be the function defined by $\rho(U) := Int(U)$ for every set $U \in SO(X,\tau)$. Let $x \in Cl(A)$. Suppose that $x \notin s^{\rho}Ker(A)$. There exists a subset $V \in SO(X,\tau)$ such that $x \notin V$ and $A \subset \rho(V)$ (cf. Definition 4.6 (i)). Since $A$ is $\omega^{\circ}$-closed and $\rho(V) = Int(V) \in \tau \subset SO(X,\tau)$, we have that $Cl(A) \subset Int(\rho(V)) = Int(Int(V)) \subset V$ and so $x \in V$; and hence this is a contradiction.

**(ii)** Throughout the present proof, let $\rho := \circ- : SO(X,\tau) \to P(X)$ be the function defined by $\rho(U) := Int(Cl(U))$ for every set $U \in SO(X,\tau)$. Let $x \in (Cl(A))_{\mathcal{PO}}$. Suppose that $x \notin s^{\circ-}Ker(A)$. There exists a subset $V \in SO(X,\tau)$ such that $x \notin V$ and $A \subset \rho(V)$ (cf. Definition 4.6 (i)). Since $A$ is $\omega^{\circ-}$-closed and $\rho(V)) \in \tau \subset SO(X,\tau)$, we have that $Cl(A) \subset Int(Cl(\rho(V)))) = Int(Cl(Int(Cl(V)))) \subset Cl(V)$ and so $x \in Cl(V)$. Thus, it is proved that $(*1)$: $Int(Cl(\{x\})) \cap V \neq \emptyset$, because $x \in Cl(V)$, $x \in Int(Cl(\{x\}))$ and $Int(Cl(\{x\})) \in \tau$. On the other hands, since $x \in X \setminus V$ and $X \setminus V \in SC(X,\tau)$ hold, we have that $\{x\} \cup Int(Cl(\{x\})) = sCl(\{x\}) \subset sCl(X \setminus V) = X \setminus V$; and so $Int(Cl(\{x\})) \subset X \setminus V$; and hence we have that $Int(Cl(\{x\})) \cap V \subset (X \setminus V) \cap V = \emptyset$; this contradicts $(*1)$ above.

**(iii)** (*Necessity*) Let $x \in Cl(A)$. Suppose that $x \notin s^{\rho}Ker_1(A)$ (cf. Definition 4.6(·)”). There exists a subset $V \in SO(X,\tau)$ such that $x \notin \rho(V)$ and $A \subset V$. Since $A$ is $\omega^{\rho}$-closed, we have that $Cl(A) \subset \rho(V)$; and so $x \in \rho(V)$; and hence this is a contradiction.

(*Sufficiency*) Assume that $Cl(A) \subset s^{\rho}Ker_1(A)$. Let $V \in SO(X,\tau)$ such that $A \subset V$. Then, by definition, it is shown that $s^{\rho}Ker_1(A) \subset \rho(V)$ holds, where $s^{\rho}Ker_1(A) := \bigcap\{\rho(W) |\ W \in SO(X,\tau)$ and $A \subset W\}$. Therefore, $Cl(A) \subset \rho(V)$ hold, whenever $V \in SO(X,\tau)$ and $A \subset V$; thus $A$ is $\omega^{\rho}$-closed in $(X,\tau)$ (cf. Definition 4.6(·)”). □

**Remark 4.9** (i) The converse of Theorem 4.8(i) is not true from the same example given by Remark 4.5(iii). Namely, let $A := \{2m, 2m+1, 2m+2\}$ be a subset of the digital line $(\mathbb{Z},\kappa)$, where $m \in \mathbb{Z}$; then $A$ is not $\omega^{\circ}$-closed in $(\mathbb{Z},\kappa)$ (cf. Remark 4.5 (iii)). And, it is obtained that $Cl(A) \subset s^{\circ}Ker(A)$ holds, because $Cl(A) = A$ for the present set $A$ and $B \subset s^{\circ}Ker(B)$ holds, in general, for every set $B$ of a topological space $(X,\tau)$.

(ii) The converse of Theorem 4.8(ii) is not true from the same example given by Remark 4.5(i). Indeed, let $A := \{2m+1\}$ be a subset of the digital line $(\mathbb{Z},\kappa)$, where $m \in \mathbb{Z}$; then $A$ is not $\omega^{\circ-}$-closed in $(\mathbb{Z},\kappa)$. And, we note that $(Cl(A))_{\mathcal{PO}} = (\{2m, 2m+1, 2m+2\})_{\mathcal{PO}} = A$. If $W \in SO(\mathbb{Z},\kappa)$ and $A \subset Int(Cl(W))$, then $A \subset W$; and so we show that $A \subset s^{\circ-}Ker(A)$. Therefore, we have that $(Cl(A))_{\mathcal{PO}} \subset s^{\circ-}Ker(A)$ holds in $(\mathbb{Z},\kappa)$.

**Remark 4.10** Using the concepts of $(Cl(\bullet))_{\mathcal{PO}}$, it is possible to define the following *$\omega^{\rho}$-like closed sets*, where $\rho : SO(X,\tau) \to P(X)$ is a function such that $\rho \in \{\circ, \circ-\}$:
$(\cdot 1)$ a subset $A$ of $(X,\tau)$ is said to be $\omega^{\rho}_{(\mathcal{PO})}$*-closed, if* $(Cl(A))_{\mathcal{PO}} \subset \rho(V)$ *holds whenever* $A \subset V$ *and* $V \in SO(X,\tau)$.
$(\cdot 2)$ $\omega^{\rho}_{(\mathcal{PO})}C(X,\tau) := \{A|\ A$ is $\omega^{\rho}_{(\mathcal{PO})}$-closed in $(X,\tau)\}$, where $\rho \in \{\circ, \circ-\}$. Then, we prove the following properties:
$(\cdot 3)$ $\omega^{\rho}_{(\mathcal{PO})}C(X,\tau)$=$P(X)$ holds (i.e. every set is $\omega^{\rho}_{(\mathcal{PO})}$-closed in $(X,\tau)$). Namely, let $A$ be a set of $(X,\tau)$. Then $(Cl(A))_{\mathcal{PO}} \subset \rho(W)$ holds whenever $A \subset W$ and $W \in SO(X,\tau)$, where $\rho \in \{\circ, \circ-\}$.
$(\cdot 4)$ $(Cl(A))_{\mathcal{PO}} \subset s^{\circ}Ker_1(A) \subset s^{\circ-}Ker_1(A)$ hold (cf. Definition 4.6 (i)").

*Proof of* $(\cdot 3)$. Let $A$ be a subset of $(X,\tau)$. By Lemma 3.6 (ii), it is well known that, $(*1)$ $(Cl(A))_{\mathcal{PO}} \subset sKer(A)$ holds. Let $W \in SO(X,\tau)$ such that $A \subset W$. Take a point $x \in (Cl(A))_{\mathcal{PO}}$ (i.e., $x \in Cl(A)$ and $\{x\} \subset Int(Cl(\{x\}))$).

Case 1. $\rho = \circ$: we suppose that $x \notin \rho(W) = Int(W)$. Since $x \in X \setminus Int(W) = Cl(X \setminus W)$, $Cl(X \setminus W)$ is semi-closed and $x \in Int(Cl(\{x\}))$, we have that $Int(Cl(\{x\})) = \{x\} \cup Int(Cl(\{x\})) = sCl(\{x\}) \subset sCl(Cl(X \setminus W)) = Cl(X \setminus W)$; and so $Int(Cl(\{x\})) \subset X \setminus Int(W)$. Thus, we show that $(*2)$ $Int(Cl(\{x\})) \cap Int(W) = \emptyset$. On the other hands, we use the property that $(*)$ $(Cl(A))_{\mathcal{PO}} \subset sKer(A)$; and so $x \in sKer(A)$. Then, for the given set $W \in SO(X,\tau)$ such that $A \subset W$, we show that $x \in sKer(A) \subset W$; and so $x \in W$. Since $x \in W \subset Cl(Int(W))$ and $x \in Int(Cl(\{x\})) \in \tau$, it is obtained that $(*3)$ $Int(Cl(\{x\})) \cap Int(W) \neq \emptyset$; and hence $(*3)$ contradicts $(*2)$ above. Therefore, we proved that the property that $x \in \rho(W) = Int(W)$ holds for any point $x \in (Cl(A))_{\mathcal{PO}}$. Namely, $(Cl(A))_{\mathcal{PO}} \subset \rho(W) = Int(W)$ holds for any set $W \in SO(X,\tau)$ such that $A \subset W$.

Case 2. $\rho = \circ-$: by the result for Case 1 above, it is obtained that $(Cl(A))_{\mathcal{PO}} \subset Int(W) \subset Int(Cl(W)) = \rho(W)$ holds for any set $W \in SO(X,\tau)$ such that $A \subset W$. $(\diamond)$.

*Proof of* $(\cdot 4)$. Let $A \in P(X)$. First, we recall that (cf. Definiton 4.6) $s^{\circ}Ker_1(A) = \bigcap\{Int(S)|\ S \in \mathcal{K}_{1,A}\}$, where $\mathcal{K}_{1,A} := \{S'|S' \in SO(X,\tau)$ and $A \subset S'\}$. Then, by $(\cdot\ 3)$ for $\rho = \circ$, it is obtained that $(Cl(A))_{\mathcal{PO}} \subset Int(W)$ holds for any set $W \in \mathcal{K}_{1,A}$; and hence $(Cl(A))_{\mathcal{PO}} \subset \bigcap\{Int(W)|\ W \in \mathcal{K}_{1,A}\} = s^{\circ}Ker_1(A)$ holds. And, we prove the last implication:$(*)$ $s^{\circ}Ker_1(A) \subset s^{\circ-}Ker_1(A)$ for any subset $A$ of $(X,\tau)$. Indeed, let $x \notin s^{\circ-}Ker_1(A)$. There exists a set $W \in SO(X,\tau)$ such that $x \notin Int(Cl(W))$ and $A \subset W$. Since $x \notin Int(W), A \subset W$ and $W \in SO(X,\tau)$, we have that $x \notin \bigcap\{Int(W')|W' \in SO(X,\tau)$ and $A \subset W'\} = s^{\circ}Ker_1(A)$. $(\diamond)$

## 5 $(\omega,\omega)$-$T^{\rho}_{1/2}$ spaces and related separation axioms, where $\rho \in \{id, \circ, \circ-\}$

We recall that, by definition due to Levine [14], a topological space $(X,\tau)$ is said to be $T_{1/2}$ if every generalized closed set (shortly, g.closed set) is closed in $(X,\tau)$. And, by Dunham [5], it is shown that $(X,\tau)$ is $T_{1/2}$ if and only if every singleton $\{x\}$ is closed or open in $(X,\tau)$, where $x \in X$ (cf. [5], e.g., [7]). Moreover, it is well known that the separation axiom $T_{1/2}$ is placed between the axioms $T_0$ and $T_1$ ([14]).

In order to introduce the concept of $(\omega,\omega)$-$T^{\rho}_{1/2}$ spaces (cf. Definition 5.3) and related separation axioms, we prepare the concept of a general form of "g.closed sets" (cf. Definition 5.2). The purpose of the present section is to prove Theorem 5.11, Theorem 5.13 and Theorem 5.15.

Throughout the present paper, let $(\mathcal{E}_X, \mathcal{E}'_X)$ be an ordered pair of two families $\mathcal{E}_X$ and $\mathcal{E}'_X$ of subsets in a topological space $(X,\tau)$ such that
$(\bullet 1)$ $\{\emptyset, X\} \subset \mathcal{E}_X$ and $\{\emptyset, X\} \subset \mathcal{E}'_X$.

**Notation 5.1** (i) (e.g., [18, in 1996; (2.1)], [16, in 1999;Definition 2.1], [20, in 2003;Definiton 3.2]) Let $A$ be a subset of $(X,\tau)$ and $(\mathcal{E}_X, \mathcal{E}'_X)$ be an ordered pair satisfying $(\bullet 1)$ above.

(•2) $\mathcal{E}_X$-$Cl(A) := \bigcap\{F|\ A \subset F$ and $X \setminus F \in \mathcal{E}_X\}$;
(•2)' $\mathcal{E}'_X$-$Cl(A) := \bigcap\{F|\ A \subset F$ and $X \setminus F \in \mathcal{E}'_X\}$.

(ii) ([26, in 2002]) (•3) $\omega Cl(A) := \omega O(X,\tau)$-$Cl(A)$ (cf. (i)(•2) above for the case where $\mathcal{E}_X = \omega O(X,\tau)$) ([27, in 1995], [28, in 2000;Defintion 3.1]));
(•4) $\omega^\mu Cl(A) :=\omega^\mu O(X,\tau)$-$Cl(A)$, where $\mu : SO(X,\tau) \to P(X)$ is a function such that $\mu \in \{id, \circ, \circ-\}$ and $A \subset X$ (cf. (i)(•2) above for the case where $\mathcal{E}_X = \omega^\mu O(X,\tau)$, Notation 1.5(•$3^\mu$)$'$).

**Definition 5.2** (I) Let $\rho 1 : SO(X,\tau) \to P(X)$ and $\rho 2 : SO(X,\tau) \to P(X)$ be two functions such that $\rho 1 \in \{id, \circ, \circ-\}$ and $\rho 2 \in \{id, \circ, \circ-\}$; and $\rho : \omega^{\rho 1}O(X,\tau) \to P(X)$ be a function such that $\rho \in \{id, \circ, \circ-\}$.

A subset $A$ of a topological space $(X,\tau)$ is said to be:
$(\omega^{\rho 1}, \omega^{\rho 2})$-$g^\rho$*.closed* in $(X,\tau)$, if $\omega^{\rho 2}Cl(A) \subset \rho(V)$ holds whenever $V \in \omega^{\rho 1}O(X,\tau)$ with $A \subset V$ (cf. Notation 1.5(•3'$^\rho$)); this may be called as $(\omega^{\rho 1}, \omega^{\rho 2})$-*generalized closed set with degree* $\rho$. Sometimes, an "$(\omega^{id}$, $\omega^{id})$-$g^{id}$.closed" set is said simply to be "$(\omega$, $\omega)$-$g$.closed".

(II) (cf. [18, Definition 2.10] for $\rho = id$) Let $\rho : \mathcal{E}_X \to P(X)$ be a function with $\rho \in \{id, \circ, \circ-\}$. A subset $A$ of $(X,\tau)$ is said to be:
$(\mathcal{E}_X, \mathcal{E}'_X)$-$g^\rho$*.closed* in $(X,\tau)$, if $\mathcal{E}'_X$-$Cl(A) \subset \rho(V)$ holds whenever $A \subset V$ and $V \in \mathcal{E}_X$; this may be called as $(\mathcal{E}_X, \mathcal{E}'_X)$-*generalized closed with degree* $\rho$ .

We note that: a subset $A$ is $(\omega^{\rho 1}$, $\omega^{\rho 2})$-$g^\rho$.closed in $(X,\tau)$ if and only if $A$ is - $(\omega^{\rho 1}O(X,\tau)$,$\omega^{\rho 2}O(X,\tau))$-$g^\rho$.closed in $(X,\tau)$ in the sense of Definition 5.2 (II) for $\mathcal{E}_X$:= - $\omega^{\rho 1}O(X,\tau)$, $\mathcal{E}'_X$:=$\omega^{\rho 2}O(X,\tau)$). The above pairs $(\omega^{\rho 1}$, $\omega^{\rho 2})$ and $(\omega^{\rho 1}O(X,\tau)$,$\omega^{\rho 2}O(X,\tau))$ imply the ordered pairs.

First, using Definition 5.2 above, we define the concept on $(\omega^{\rho 1}$,$\omega^{\rho 2})$-$T^\rho_{1/2}$ spaces and also it's general forms $(\mathcal{E}_X$,$\mathcal{E}'_X)$-$T^\rho_{1/2}$ spaces. Especially, the concept of $(\mathcal{E}_X$,$\mathcal{E}'_X)$-$T^{id}_{1/2}$ spaces is defined in [18, in 1996;Definition 2.19].

**Definition 5.3** (I) Let $\rho 1 : SO(X,\tau) \to P(X)$ and $\rho 2 : SO(X,\tau) \to P(X)$ be two functions such that $\rho 1 \in \{id, \circ, \circ-\}$ and $\rho 2 \in \{id, \circ, \circ-\}$; and let $\rho : \omega^{\rho 1}O(X,\tau) \to P(X)$ be a function such that $\rho \in \{id, \circ, \circ-\}$.

For the fixed functions $\rho 1$, $\rho 2$ and $\rho$, a topological space $(X,\tau)$ is said to be:

(i) $(\omega^{\rho 1}$,$\omega^{\rho 2})$-$T^\rho_{1/2}$, if $A$ is $\omega^{\rho 2}$-closed (cf. Definition 1.4;i.e., $X \setminus A \in \omega^{\rho 2}O(X,\tau)$) for every $(\omega^{\rho 1}$,$\omega^{\rho 2})$-$g^\rho$.closed set $A$, (cf. Definition 5.2(I));

(ii) *weak* $(\omega^{\rho 1}$,$\omega^{\rho 2})$-$T^\rho_{1/2}$, where $\rho 2 \neq id$, if $\omega^{\rho 2}Cl(A) = A$ holds for every $(\omega^{\rho 1}$,$\omega^{\rho 2})$-$g^\rho$.closed set $A$, where $\omega^{\rho 2}Cl(A)$:=$\omega^{\rho 2}O(X,\tau)$-$Cl(A)$ (cf. Definition 5.2(I), Notation 5.1).

(II) Let $(\mathcal{E}_X, \mathcal{E}'_X)$ be an ordered pair and let $\rho : \mathcal{E}_X \to P(X)$ be a fixed function such that $\rho \in \{id, \circ, \circ-\}$. A topological space $(X,\tau)$ is said to be:

(i) an $(\mathcal{E}_X$,$\mathcal{E}'_X)$-$T^\rho_{1/2}$ *space*, if $X \setminus A \in \mathcal{E}'_X$ holds for every $(\mathcal{E}_X$,$\mathcal{E}'_X)$-$g^\rho$.closed set $A$ (cf. [18, Definition 2.19] for $\rho = id$).

(ii) a *weak* $(\mathcal{E}_X$,$\mathcal{E}'_X)$-$T^\rho_{1/2}$ *space*, if $\mathcal{E}'_X$-$Cl(A) = A$ holds for every $(\mathcal{E}_X$,$\mathcal{E}'_X)$-$g^\rho$.closed set $A$ (cf. Definition 5.2(II), Notation 5.1).

We investigate some relations between "*weak* $(\mathcal{E}_X$,$\mathcal{E}'_X)$-$T^\rho_{1/2}$ *spaces*" and "$(\mathcal{E}_X$,$\mathcal{E}'_X)$-$T^\rho_{1/2}$ *spaces*" (cf. Lemma 5.5), applying the following Lemma 5.4 due to Noiri and Popa ([20, in 2003;Lemma 3.3], [21, in 2000]).

**Lemma 5.4** ([20, in 2003;Lemma 3.3], [21, in 2000]) *For a minmal structure* $m_X$ *on a nonempty set* $X$ (i.e., $\emptyset \in m_X, X \in m_X$ *and* $m_X \subset P(X)$), *the following are equivalent:*
(1) $m_X$ *has property, say* $(\mathcal{B})_{m_X}$*: if the union of any family of subsets belonging to* $m_X$ *belongs to* $m_X$*;*
(2) *if* $m_X$*-*$Int(V) = V$*, then* $V \in m_X$*;*
(3) *if* $m_X$*-*$Cl(F) = F$*, then* $X \setminus F \in m_X$.

**Lemma 5.5** *Let* $(X, \tau)$ *be a topological space and* $(\mathcal{E}_X, \mathcal{E}'_X)$ *an ordered pair of given familes* $\mathcal{E}_X$ *and* $\mathcal{E}'_X$ *such that* $\{\emptyset, X\} \subset \mathcal{E}_X \cap \mathcal{E}'_X$.

*For each function* $\rho : \mathcal{E}_X \to P(X)$ *with* $\rho \in \{id, \circ, \circ-\}$*, we have the following properties.*

(i) *Every* $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\rho}_{1/2}$ *space* $(X, \tau)$ *is weak* $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\rho}_{1/2}$.

(ii) *Suppose that* $\mathcal{E}'_X$ *has property* $(\mathcal{B})_{\mathcal{E}'_X}$*: the union of any family of subsets belonging to* $\mathcal{E}'_X$ *belongs to* $\mathcal{E}'_X$ (cf. Lemma 5.4). *Then, every weak* $(\mathcal{E}_X,\mathcal{E}'_X)$-$T^{\rho}_{1/2}$ *space* $(X, \tau)$ *is* $(\mathcal{E}_X,\mathcal{E}'_X)$-$T^{\rho}_{1/2}$.

*Proof.* **(i)** Let $A$ be an $(\mathcal{E}_X, \mathcal{E}'_X)$-$g^{\rho}$ closed set in $(X, \tau)$. Then, by assumption, it is obtained that $X \setminus A \in \mathcal{E}'_X$; and so $\mathcal{E}'_X$-$Cl(A) := \bigcap\{F |\ A \subset F$ and $X \setminus F \in \mathcal{E}'_X\} = A$ hold. Therefore, $(X, \tau)$ is weak $(\mathcal{E}_X,\mathcal{E}'_X)$-$T^{\rho}_{1/2}$.

**(ii)** Let $A$ be an $(\mathcal{E}_X,\mathcal{E}'_X)$-$g^{\rho}$ closed set in $(X, \tau)$. Since $(X, \tau)$ is weak $(\mathcal{E}_X,\mathcal{E}'_X)$-$T^{\rho}_{1/2}$, we have $\mathcal{E}'_X$-$Cl(A) = A$. Since $(\mathcal{B})_{\mathcal{E}'_X}$ is supposed, by Lemma 5.4, it is obtained that $X \setminus A \in \mathcal{E}'_X$. Therefore, $(X, \tau)$ is $(\mathcal{E}_X,\mathcal{E}'_X)$-$T^{\rho}_{1/2}$. □

**Remark 5.6** (i) The following properties on a topological space $(X, \tau)$ are equivalent for a fixed function $\rho : \mathcal{E}_X \to P(X)$ with $\rho \in \{id, \circ, \circ-\}$ and a fixed function $\rho 1 : SO(X, \tau) \to P(X)$ with $\rho 1 \in \{id, \circ, \circ-\}$:

(1) $(X, \tau)$ is $(\omega^{\rho 1},\omega^{id})$-$T^{\rho}_{1/2}$ (cf. Definition 5.3(I)(i);

(2) $(X, \tau)$ is *weak* $(\omega^{\rho 1}O(X, \tau),\omega O(X, \tau))$-$T^{\rho}_{1/2}$ (cf. Definition 5.3(II)(ii));

(3) $(X, \tau)$ is $(\omega^{\rho 1}O(X, \tau),\omega O(X, \tau))$-$T^{\rho}_{1/2}$ (cf. Definition 5.3(II)(ii)).

Indeed, they are obtained by definitions and the well known fact that, for a subset $A$ of $(X, \tau)$, $X \setminus A \in \omega O(X, \tau)$ if and only if $\omega Cl(A) = A$ holds, where $\omega Cl(A)$:=$\omega O(X, \tau)$-$Cl(A)$. By [26], $\omega O(X, \tau)$ has property $(\mathcal{B})_{\omega O(X,\tau)}$; and so the equivalences are obtained by Lemma 5.5.

(ii) The concept of an $(\omega^{id},\omega^{id})$-$T^{id}_{1/2}$ space is called an $(\omega,\omega)$-$T_{1/2}$ space or an $\omega$-$T_{1/2}$ space.

**Lemma 5.7** (i) *The following properties on a topological space* $(X, \tau)$ *are equivalent: let* $\mathcal{E}_X$ *and* $\mathcal{E}'_X$ *be two families satisfying the condition that* $\{\emptyset, X\} \subset \mathcal{E}_X \cap \mathcal{E}'_X$.

(1) $(X, \tau)$ *is weak* $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\circ}_{1/2}$*;*

(2) $(*1)$*: if* $x \in X$*, then* $X \setminus \{x\} \in \mathcal{E}_X$ *or* $\mathcal{E}'_X$-$Cl(X \setminus \{x\}) = X \setminus \{x\}$ *hold;*

(3) $(X, \tau)$ *is weak* $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{id}_{1/2}$.

(ii) *Every weak* $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\circ -}_{1/2}$ *topological space* $(X, \tau)$ *is weak* $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\rho}_{1/2}$*, where* $\rho \in \{id, \circ\}$.

(iii) *Suppose that* $(*2)$*: if* $x \in X$*, then* $X \setminus \{x\} \in \mathcal{E}_X \cap SC(X, \tau)$ *or* $\mathcal{E}'_X$-$Cl(X \setminus \{x\}) = X \setminus \{x\}$ *hold. Then,* $(X, \tau)$ *is weak* $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\circ -}_{1/2}$.

*Proof.* **(i)** (1)⇒(2) We suppose that $X \setminus \{x\} \notin \mathcal{E}_X$. Let $U \in \mathcal{E}_X$ be any set such that $X \setminus \{x\} \subset U$. Then we have that $U = X$ only; and so $\mathcal{E}'_X$-$Cl(X \setminus \{x\}) \subset \mathcal{E}'_X$-$Cl(X) = X = Int(U)$. Thus, we have that $X \setminus \{x\}$ is $(\mathcal{E}_X, \mathcal{E}'_X)$-$g^{\circ}$.closed (cf. Definition 5.2(II)).

By assumption (cf. Defintion 5.3(II)(ii)), it is shown that $\mathcal{E}'_X\text{-}Cl(X \setminus \{x\}) = X \setminus \{x\}$ holds. Therefore, we have $(*1)$.

(2)$\Rightarrow$ (3) Let $A$ be an $(\mathcal{E}_X, \mathcal{E}'_X)$-$g^{id}$.closed set. We claim that $\mathcal{E}'_X\text{-}Cl(A) = A$. Let $x \in \mathcal{E}'_X\text{-}Cl(A)$; and we suppose that $x \notin A$; and so $A \subset X \setminus \{x\}$.

Case 1. $\mathcal{E}'_X\text{-}Cl(X \setminus \{x\}) = X \setminus \{x\}$: for this case, we have that $x \in \mathcal{E}'_X\text{-}Cl(A) \subset \mathcal{E}'_X\text{-}Cl(X \setminus \{x\}) = X \setminus \{x\}$; and so $x \in X \setminus \{x\}$; this is a contradiction.

Case 2. $X \setminus \{x\} \in \mathcal{E}_X$: for this case, since $A \subset X \setminus \{x\}$, where $X \setminus \{x\} \in \mathcal{E}_X$, and $A$ is $(\mathcal{E}_X, \mathcal{E}'_X)$-$g^{id}$.closed, we have that $x \in \mathcal{E}'_X\text{-}Cl(A) \subset X \setminus \{x\}$ (cf. Definition 5.2(II)); and so $x \in X \setminus \{x\}$; this is also a contradiction.

By all cases, we have contradictions; and so we prove that $\mathcal{E}'_X\text{-}Cl(A) \subset A$ holds. Since $A \subset \mathcal{E}'_X\text{-}Cl(A)$, we have the required equality $\mathcal{E}'_X\text{-}Cl(A) = A$; and hence $(X, \tau)$ is weak $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{id}_{1/2}$ (cf. Defintion 5.3(II)).

(3)$\Rightarrow$(1) Let $A$ be an $(\mathcal{E}_X, \mathcal{E}'_X)$-$g^{\circ}$.closed set of $(X, \tau)$. Then, by Definition 5.2(II), it is shown that the set $A$ is $(\mathcal{E}_X, \mathcal{E}'_X)$-$g^{id}$.closed. Using the assumption (3), we have that $\mathcal{E}'_X\text{-}Cl(A) = A$; and so $(X, \tau)$ is $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\circ}_{1/2}$ (cf. Definition 5.3(II)).

**(ii)** We prove the property $(*1)$ of (i) above. Indeed, we suppose that $X \setminus \{x\} \notin \mathcal{E}_X$. Let $U \in \mathcal{E}_X$ be any set such that $X \setminus \{x\} \subset U$. Then we have that $U = X$ only; and so $\mathcal{E}'_X\text{-}Cl(X \setminus \{x\}) \subset \mathcal{E}'_X\text{-}Cl(U) = \mathcal{E}'_X\text{-}Cl(X) = X = Int(Cl(U))$. Thus, we have that $X \setminus \{x\}$ is $(\mathcal{E}_X, \mathcal{E}'_X)$-$g^{\circ-}$.closed (cf. Definition 5.2(II)). It is shown that $\mathcal{E}'_X\text{-}Cl(X \setminus \{x\}) = X \setminus \{x\}$ holds, because $(X, \tau)$ is weak $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\circ-}_{1/2}$. Therefore, we have $(*1)$; and so $(X, \tau)$ is $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\rho}_{1/2}$, where $\rho \in \{id, \circ\}$ (cf. (i) above).

**(iii)** Let $A$ be an $(\mathcal{E}_X, \mathcal{E}'_X)$-$g^{\circ-}$.closed set. We claim that $\mathcal{E}'_X\text{-}Cl(A) = A$. Indeed, let $x \in \mathcal{E}'_X\text{-}Cl(A)$. And we suppose that $x \notin A$; and so $A \subset X \setminus \{x\}$.

Case 1. $\mathcal{E}'_X\text{-}Cl(X \setminus \{x\}) = X \setminus \{x\}$: for this case, we have that $x \in \mathcal{E}'_X\text{-}Cl(A) \subset \mathcal{E}'_X\text{-}Cl(X \setminus \{x\}) = X \setminus \{x\}$; and so $x \in X \setminus \{x\}$; this is a contradiction.

Case 2. $X \setminus \{x\} \in \mathcal{E}_X \cap SC(X, \tau)$: for this case, since $A \subset X \setminus \{x\}$, $X \setminus \{x\} \in \mathcal{E}_X$ and $A$ is $(\mathcal{E}_X, \mathcal{E}'_X)$-$g^{\circ-}$.closed, we have that $x \in \mathcal{E}'_X\text{-}Cl(A) \subset Int(Cl(X \setminus \{x\})) = X \setminus Cl(Int(\{x\}))$. We have that $x \in X \setminus Cl(Int(\{x\}))$. Namely, we have that $\{x\} \not\subset Cl(Int(\{x\}))$,i.e., $\{x\}$ is not semi-open in $(X, \tau)$. This contradicts one of the assumptions: $X \setminus \{x\} \in SC(X, \tau)$ (i.e., $\{x\}$ is semi-open in $(X, \tau)$).

Thus, for both cases, we have contradictions; and so we show that $\mathcal{E}'_X\text{-}Cl(A) \subset A$; and so $A = \mathcal{E}'_X\text{-}Cl(A)$; and hence $(X, \tau)$ is weak $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\circ-}_{1/2}$. □

**Remark 5.8** The following diagram shows the implications in Lemma 5.7 above: under the assumption that $\{\emptyset, X\} \subset \mathcal{E}_X \cap \mathcal{E}'_X$.

$$\begin{tikzcd}
 & weak\ (\mathcal{E}_X, \mathcal{E}'_X)\text{-}T^{id}_{1/2} \arrow[r, shift left] \arrow[d, shift right] & (*1) \text{ of Lemma 5.7(i)(2)} \arrow[l, shift left] \\
weak\ (\mathcal{E}_X, \mathcal{E}'_X)\text{-}T^{\circ-}_{1/2} \arrow[ur] \arrow[dr] & & \\
(*2) \text{ of Lemma 5.7(iii)} \arrow[u] & weak\ (\mathcal{E}_X, \mathcal{E}'_X)\text{-}T^{\circ}_{1/2} \arrow[uu, shift right] &
\end{tikzcd}$$

We investigate the following properties on "$(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\rho}_{1/2}$", corresponding to Lemma 5.7 above.

**Lemma 5.9** (i) *Let* $\rho : \mathcal{E}_X \to P(X)$ *be a fixed function such that* $\rho \in \{id, \circ, \circ-\}$. *Suppose that* $(X, \tau)$ *is an* $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\rho}_{1/2}$ *topological space. Then,*
$(*1')$: *if* $x \in X$ *then* $X \setminus \{x\} \in \mathcal{E}_X$ *or* $\{x\} \in \mathcal{E}'_X$.

(ii) *Suppose that* $\mathcal{E}'_X$ *has property* $(\mathcal{B})_{\mathcal{E}'_X}$ (cf. Lemma 5.5(ii)). *If* $(*1')$ *of* (i) *above holds, then* $(X, \tau)$ *is* $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\rho}_{1/2}$, *where* $\rho \in \{id, \circ\}$. *And, every* $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\circ-}_{1/2}$ *topological space is* $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{id}_{1/2}$ *and* $(\mathcal{E}_X, \mathcal{E}'_X)$-$T^{\circ}_{1/2}$.

(iii) *Suppose that* $\mathcal{E}'_X$ *has property* $(\mathcal{B})_{\mathcal{E}'_X}$ *and that* (∗2'): *if* $x \in X$ *then* $X \setminus \{x\} \in \mathcal{E}_X \cap SC(X,\tau)$ *or* $\{x\} \in \mathcal{E}'_X$. *Then,* $(X,\tau)$ *is* $(\mathcal{E}_X,\mathcal{E}'_X)$-$T^{\circ -}_{1/2}$.

*Proof.* **(i)** Let $\{x\}$ be a singleton in $(X,\tau)$. We suppose that $X \setminus \{x\} \notin \mathcal{E}_X$. Let $U \in \mathcal{E}_X$ be any set such that $X \setminus \{x\} \subset U$. Then, $U = X$ holds only; and so $\mathcal{E}'_X$-$Cl(X \setminus \{x\}) \subset \mathcal{E}_X$-$Cl(U) = \mathcal{E}_X$-$Cl(X) = X = \rho(U)$, where $\rho \in \{id, \circ, \circ-\}$. Thus, we have that $X \setminus \{x\}$ is $(\mathcal{E}_X,\mathcal{E}'_X)$-$g^{\rho}$.closed (cf. Definition 5.3(II)). By assumption, it is shown that $X \setminus (X \setminus \{x\}) \in \mathcal{E}'_X$ and so $\{x\} \in \mathcal{E}'_X$.

**(ii)** First, suppose that (∗1') holds. For a singleton $\{x\}$ such that $\{x\} \in \mathcal{E}'_X$, it is shown that $\mathcal{E}'_X$-$Cl(X \setminus \{x\}) = X \setminus \{x\}$ holds (cf. Notation 5.1(I)(i)). Then, the given assumption (∗1') implies the assumption (∗1) of Lemma 5.7(i)(2), i.e., $X \setminus \{x\} \in \mathcal{E}_X$ or $\mathcal{E}'_X$-$Cl(X \setminus \{x\}) = X \setminus \{x\}$ hold. Thus, by Lemma 5.7(i), $(X,\tau)$ is weak $(\mathcal{E}_X,\mathcal{E}'_X)$-$T^{\rho}_{1/2}$, where $\rho \in \{id, \circ\}$; and, by Lemma 5.5(ii), $(X,\tau)$ is $(\mathcal{E}_X,\mathcal{E}'_X)$-$T^{\rho}_{1/2}$, where $\rho \in \{id, \circ\}$. Finally, suppose that $(X,\tau)$ is $(\mathcal{E}_X,\mathcal{E}'_X)$-$T^{\circ -}_{1/2}$. Then, by (i) above, it is shown that the property (∗1') holds; and so, by the first property of the present (ii), the space $(X,\tau)$ is $(\mathcal{E}_X,\mathcal{E}'_X)$-$T^{\rho}_{1/2}$, where $\rho \in \{id, \circ\}$.

**(iii)** Let $\{x\} \in \mathcal{E}'_X$. It is shown that $\mathcal{E}'_X$-$Cl(X \setminus \{x\}) = X \setminus \{x\}$ holds; and so the assumption (∗2') implies the assumption (∗2) of Lemma 5.7(iii), i.e., $X \setminus \{x\} \in \mathcal{E}_X \cap SC(X,\tau))$ or $\mathcal{E}'_X$-$Cl(X \setminus \{x\}) = X \setminus \{x\})$ hold. Thus, by Lemma 5.7(iii) and Lemma 5.5(ii), it is shown that $(X,\tau)$ is $(\mathcal{E}_X,\mathcal{E}'_X)$-$T^{\circ -}_{1/2}$. □

**Remark 5.10** The following diagram is shown by the above implications in Lemma 5.9: under the assumption $(\mathcal{B})_{\mathcal{E}'_X}$;

$$\begin{tikzcd}
 & (\mathcal{E}_X,\mathcal{E}'_X)\text{-}T^{id}_{1/2} \arrow[dd, shift right] \\
(\mathcal{E}_X,\mathcal{E}'_X)\text{-}T^{\circ -}_{1/2} \arrow[ur] \arrow[dr] & \\
 & (\mathcal{E}_X,\mathcal{E}'_X)\text{-}T^{\circ}_{1/2} \arrow[uu, shift right]
\end{tikzcd}$$

Using Lemma 5.7 for $\mathcal{E}_X := \omega^{\rho 1}O(X,\tau)$ and $\mathcal{E}'_X := \omega^{\rho 2}O(X,\tau)$, the concept of "weak $(\omega^{\rho 1},\omega^{\rho 2})$-$T^{\rho}_{1/2}$ spaces" is characterized by the following Theorem 5.11, where $\rho 1 : SO(X,\tau) \to P(X)$ and $\rho 2 : SO(X,\tau) \to P(X)$ are functions such that $\rho 1 \in \{id, \circ, \circ-\}$ and $\rho 2 \in \{id, \circ, \circ-\}$ and $\rho : \omega^{\rho 1}O(X,\tau) \to P(X)$ is a function such that $\rho \in \{id, \circ, \circ-\}$; (cf. Definition 5.3 (I)(ii)).

**Theorem 5.11** *Let* $\rho 1 : SO(X,\tau) \to P(X)$ *and* $\rho 2 : SO(X,\tau) \to P(X)$ *be two functions such that* $\rho 1 \in \{id, \circ, \circ-\}$ *and* $\rho 2 \in \{id, \circ, \circ-\}$.

(i) *The following properties are equivalent*:

(1) *a topological space* $(X,\tau)$ *is weak* $(\omega^{\rho 1},\omega^{\rho 2})$-$T^{\circ}_{1/2}$;

(2) (∗1): *if* $x \in X$ *then* $\{x\}$ *is* $\omega^{\rho 1}$*-closed* (cf. Definition d75) (i.e., $X \setminus \{x\} \in \omega^{\rho 1}O(X,\tau)$) *or* $\omega^{\rho 2}Cl(X \setminus \{x\}) = X \setminus \{x\}$;

(3) $(X,\tau)$ *is weak* $(\omega^{\rho 1},\omega^{\rho 2})$-$T^{id}_{1/2}$.

(ii) *Every weak* $(\omega^{\rho 1},\omega^{\rho 2})$-$T^{\circ -}_{1/2}$ *topological space is weak* $(\omega^{\rho 1},\omega^{\rho 2})$-$T^{\rho}_{1/2}$, *where* $\rho \in \{id, \circ\}$.

(iii) *Suppose that* (∗2): *if* $x \in X$ *then* $X \setminus \{x\} \in \omega^{\rho 1}O(X,\tau) \cap SC(X,\tau)$ *or* $\omega^{\rho 2}Cl(X \setminus \{x\}) = X \setminus \{x\}$. *Then,* $(X,\tau)$ *is weak* $(\omega^{\rho 1},\omega^{\rho 2})$-$T^{\circ -}_{1/2}$. □

**Remark 5.12** The following diagrams are obtained by Theorem 5.11(i) and (ii) above: for fixed functions $\rho 1 \in \{id, \circ, \circ-\}$ and $\rho 2 \in \{id, \circ, \circ-\}$ (cf. Remark 5.8),

$$\begin{tikzcd}
weak\ (\omega^{\rho 1},\omega^{\rho 2})\text{-}T^{id}_{1/2} \arrow[d, shift right] \arrow[rr, phantom, "\rightleftharpoons"] & & \{x\}\in\omega^{\rho 1}C(X,\tau) \text{ or } \omega^{\rho 2}Cl(X\setminus\{x\}) = X\setminus\{x\}(\forall x\in X) \\
 & weak\ (\omega^{\rho 1},\omega^{\rho 2})\text{-}T^{\circ -}_{1/2} \arrow[ul] \arrow[dl] & \\
weak\ (\omega^{\rho 1},\omega^{\rho 2})\text{-}T^{\circ}_{1/2} \arrow[u, shift right] & &
\end{tikzcd}$$

In Definition 5.3 (II)(i), especially we consider the case where $\mathcal{E}_X := \omega^{\rho 1}O(X,\tau)$ ($\rho 1 : SO(X,\tau)\rightarrow P(X)$ is a function such that $\rho 1\in\{id,\circ,\circ -\}$) and $\mathcal{E}'_X{:=}\omega O(X,\tau)$; and so we have the following propeties on "$(\omega^{\rho 1},\omega)$-$T^{\rho}_{1/2}$" spaces using Lemma 5.9 above and Definition 5.3 (II)(i), where $\rho : \omega^{\rho 1}O(X,\tau)\rightarrow P(X)$ is a function such that $\rho\in\{id,\circ,\circ -\}$. We note that the family $\mathcal{E}'_X := \omega O(X,\tau)$ has property $(\mathcal{B})_{\mathcal{E}'_X}$ (cf. Remark 5.6 above;[26]).

**Theorem 5.13** *For a fixed function* $\rho 1 : SO(X,\tau)\rightarrow P(X)$ *with* $\rho 1\in\{id,\circ,\circ -\}$*, we have the following properties.*

(i) *The following properties are equivalent:*

(1) *a topological space* $(X,\tau)$ *is* $(\omega^{\rho 1},\omega)$-$T^{id}_{1/2}$*;*

(2) *if* $x\in X$ *then* $\{x\}$ *is* $\omega^{\rho 1}$*-closed or* $\{x\}$ *is* $\omega$*-open;*

(3) $(X,\tau)$ *is* $(\omega^{\rho 1},\omega)$-$T^{\circ}_{1/2}$.

(ii) *Every* $(\omega^{\rho 1},\omega)$-$T^{\circ -}_{1/2}$ *topological space is* $(\omega^{\rho 1},\omega)$-$T^{id}_{1/2}$ *and* $(\omega^{\rho 1},\omega)$-$T^{\circ}_{1/2}$.

(iii) *Suppose that if* $x\in X$ *then* $\{x\}$ *is* $\omega^{\rho 1}$*-closed and semi-open* (i.e. $X\setminus\{x\}\in\omega^{\rho 1}O(X,\tau)$ *and* $\{x\}\in SO(X,\tau)$)*, or* $\{x\}$ *is* $\omega$*-open, then* $(X,\tau)$ *is* $(\omega^{\rho 1},\omega)$-$T^{\circ -}_{1/2}$. □

**Remark 5.14** The following diagram is obtained by Theorem 5.13(i) and (ii) above:

$$\begin{tikzcd}
 & (\omega^{\rho 1},\omega)\text{-}T^{id}_{1/2} \arrow[d, shift right] \arrow[r, phantom, "\rightleftharpoons"] & \{x\}\in\omega^{\rho 1}C(X,\tau)\cup\omega O(X,\tau)\ (\forall x\in X) \\
(\omega^{\rho 1},\omega)\text{-}T^{\circ -}_{1/2} \arrow[ur] \arrow[dr] & & \\
 & (\omega^{\rho 1},\omega)\text{-}T^{\circ}_{1/2} \arrow[u, shift right] &
\end{tikzcd}$$

In Defintion 5.3(II)(i), especially we consider the case where $\mathcal{E}_X := \omega^{\rho 1}O(X,\tau)$ ($\rho 1 : SO(X,\tau)\rightarrow P(X)$ is function such that $\rho 1\in\{id,\circ,\circ -\}$), $\mathcal{E}'_X := \omega^{\circ}O(X,\tau)$ (resp. $\mathcal{E}'_X := \omega^{\circ -}O(X,\tau)$) and a function $\rho : SO(X,\tau)\rightarrow P(X)$ with $\rho\in\{id,\circ,\circ -\}$; and so we have the following properties on "$(\omega^{\rho 1},\omega^{\circ})$-$T^{\rho}_{1/2}$" (resp. "$(\omega^{\rho 1},\omega^{\circ -})$-$T^{\rho}_{1/2}$") spaces, using Lemma 5.9 and Definition 5.3 (I) above.

**Theorem 5.15** *For fixed functions* $\rho 1 : SO(X,\tau)\rightarrow P(X)$ *with* $\rho 1\in\{id,\circ,\circ -\}$ *and* $\mu : SO(X,\tau)\rightarrow P(X)$ *with* $\mu\in\{\circ,\circ -\}$*, we have the following properties.*

(i) *For a fixed function* $\rho : \omega^{\rho 1}O(X,\tau)\rightarrow P(X)$ *with* $\rho\in\{id,\circ,\circ -\}$*, if* $(X,\tau)$ *is* $(\omega^{\rho 1},\omega^{\mu})$-$T^{\rho}_{1/2}$*, then* $\{x\}\in\omega^{\rho 1}C(X,\tau)\cup\omega^{\mu}O(X,\tau)$ *for each singleton* $\{x\}$ *of* $(X,\tau)$.

(ii) *Suppose that* $\omega^{\mu}O(X,\tau)$ *has property* $(\mathcal{B})_{\omega^{\mu}O(X,\tau)}$ *for* $\mu\in\{\circ,\circ -\}$*. Then, the following properties are equivalent:*

(1) $(X,\tau)$ *is* $(\omega^{\rho 1},\omega^{\mu})$-$T^{id}_{1/2}$*;*

(2) *if* $x\in X$ *then* $\{x\}\in\omega^{\rho 1}C(X,\tau)\cup\omega^{\mu}O(X,\tau)$*;*

(3) $(X,\tau)$ *is* $(\omega^{\rho 1},\omega^{\mu})$-$T^{\circ}_{1/2}$.

(iii) *Suppose that* $\omega^{\mu}O(X,\tau)$ *has property* $(\mathcal{B})_{\omega^{\mu}O(X,\tau)}$ *for* $\mu\in\{\circ,\circ -\}$*. Then, every* $(\omega^{\rho 1},\omega^{\mu})$-$T^{\circ -}_{1/2}$ *topological space is* $(\omega^{\rho 1},\omega^{\mu})$-$T^{id}_{1/2}$ *and* $(\omega^{\rho 1},\omega^{\mu})$-$T^{\circ}_{1/2}$.

(iv) *Suppose that* $\omega^{\mu}O(X,\tau)$ *has property* $(\mathcal{B})_{\omega^{\mu}O(X,\tau)}$ *for* $\mu \in \{\circ, \circ-\}$. *Then, if* $\{x\} \in (\omega^{\rho 1}C(X,\tau) \cap SO(X,\tau)) \cup \omega^{\mu}O(X,\tau)$ *for each* $x \in X$, *then* $(X,\tau)$ *is* $(\omega^{\rho 1}, \omega^{\mu})$-$T_{1/2}^{\circ-}$. □

(•) In the end of the present section, we define the concepts of $\omega^{\circ}$-$T_i$ spaces, $\omega^{\circ-}$-$T_i$ spaces and $\omega$-$T_i$ spaces for each integer $i \in \{1, 0\}$ (cf. Definition 5.16 (II) below). The following Definition 5.16 (I) (i.e., $\mathcal{E}_X$-$T_i$ separation axioms, where $i \in \{0,1\}$) are well known by many authors; for examples, they are defined on a generalized topology, say $\lambda$, due to [1, in 2002] and they are investigated on $(X,\lambda)$ by [24, in 2011; for i=1],[25, in 2016;Definition 1.7 (for i=1), Definition 1.8 (for i=1/2), Defintion 3.1 (for i=3/4)]. We give Definition 5.16 (I) in order to explain the concepts of $\omega^{\rho}$-$T_i (i \in \{0,1\})$ accurately (cf. Definiton 5.16 (II)).

Let $X \times X$ be the direct product of $X$ and $\triangle(X) := \{(x,x) |\ x \in X\}$ the diagonal set of $X$; and $(X \times X) \setminus \triangle(X) := \{(x,y) \in X \times X | x \neq y\}$.

**Definition 5.16** (I)([1], [24],[25]) A topological space $(X,\tau)$ is said to be:

(i) $\mathcal{E}_X$-$T_1$, if for each $(x,y) \in (X \times X) \setminus \triangle(X)$ there exist subsets $U$ and $V$ belonging to $\mathcal{E}_X$ such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$;

(ii) $\mathcal{E}_X$-$T_0$, if for each $(x,y) \in (X \times X) \setminus \triangle(X)$ there exists a subset $U$ belonging to $\mathcal{E}_X$ such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$ (i.e., $U \in \mathcal{E}_X$ contains exactly one of two points).

(II) For each integer $i \in \{0,1\}$ and a function $\rho : SO(X,\tau) \to P(X)$ with $\rho \in \{id, \circ, \circ-\}$, a topological space $(X,\tau)$ is said to be $\omega^{\rho}$-$T_i$ , if $(X,\tau)$ is $\omega^{\rho}O(X,\tau)$-$T_i$ (in the sense of (I) for $\mathcal{E}_X = \omega^{\rho}O(X,\tau)$) (cf. Notation 1.5 (i)). Sometimes, the separation axiom $\omega^{id}$-$T_i$ is denoted by $\omega$-$T_i$, where $i \in \{0,1\}$.

The followng properties are well known; (ii) is obtained by using (i) below and Lemma 5.4.

**Theorem 5.17** (i) *The following properties* (1) *and* (2) *are equivalent:*

(1) *a topological space* $(X,\tau)$ *is* $\mathcal{E}_X$-$T_1$;

(2) *for each singleton* $\{x\}, \mathcal{E}_X$-$Cl(\{x\}) = \{x\}$ *holds.*

(ii) *Suppose that* $\mathcal{E}_X$ *has property* $(\mathcal{B})_{\mathcal{E}_X}$. *Then,* (1), (2) *above and the following property* (3) *are equivalent.*

(3) *For each singleton* $\{x\}$, $X \setminus \{x\} \in \mathcal{E}_X$ *holds.* □

We investigate some relations among $\omega^{\rho 1}$-$T_i$ spaces for a function $\rho 1 : SO(X,\tau) \to P(X)$ with $\rho 1 \in \{id, \circ, \circ-\}$ and a fixed number $i$ with $i \in \{0, 1/2, 1\}$.

**Theorem 5.18** (i) *Every* $T_i$ *space is* $\omega$-$T_i$ *for each* $i \in \{0, 1/2, 1\}$, *where a symbol* $\omega$-$T_{1/2}$ *means the separation axiom:* $(\omega,\omega)$-$T_{1/2}^{id}$ (cf. Definition 5.3 (I)($*$1)).

(ii) *Every* $\omega^{\circ}$-$T_i$ *space is* $\omega$-$T_i$ *and* $\omega^{\circ-}$-$T_i$ *for each* $i \in \{0,1\}$ (cf. Theorem 5.13(ii) for the case where $i = 1/2$).

*Proof* **(i)** Since $\tau \subset \omega O(X,\tau)$, the case where of $i \in \{0,1\}$ is proved by Definition 5.16 for $\mathcal{E}_X := \omega O(X,\tau)$. By [5, Theorem 2.5], it is shown that if $(X,\tau)$ is $T_{1/2}$ then every singleton $\{x\}$ of $(X,\tau)$ is open or closed; and so it is $\omega$-open or $\omega$-closed. Then, the proof of the case where of $i = 1/2$ is obtained by Theorem 5.13(i) for $\rho 1 = id$.

**(ii)** Since $\omega^{\circ}O(X,\tau) \subset \omega O(X,\tau)$ and $\omega^{\circ}O(X,\tau) \subset \omega^{\circ-}O(X,\tau)$ holds (cf. Theorem 2.1), the proof of (ii) is obtained by Definition 5.16. □

We investigate some relations among $\omega^{\rho 1}$-$T_0$ spaces, $\omega^{\rho 1}$-$T_1$ spaces and $(\omega^{\rho 1}, \omega^{\rho 1})$-$T^{\rho}_{1/2}$ spaces, where $\rho 1 : SO(X,\tau) \to P(X)$ is a function such that $\rho 1 \in \{id, \circ, \circ-\}$ and $\rho = id : \omega^{\rho 1}O(X,\tau) \to P(X)$ (cf. Definition 5.3(I) and Definition 5.16 (II)).

**Theorem 5.19** *We have the following diagram of implications.*

(i) $\omega$-$T_1$ $\Rightarrow$ $(\omega,\omega)$-$T^{id}_{1/2}(= \omega$-$T_{1/2})$ $\Rightarrow$ $\omega$-$T_0$.

(ii) *Let* $\mu : SO(X,\tau) \to P(X)$ *be a function such that* $\mu \in \{\circ, \circ-\}$. *Supose that* $\omega^{\mu}O(X,\tau)$ *has property property* $(\mathcal{B})_{\omega^{\mu}O(X,\tau)}$. *Then,*

$\omega^{\mu}$-$T_1$ $\Rightarrow$ $(\omega^{\mu},\omega^{\mu})$-$T^{id}_{1/2}$.

(iii) *Let* $b : SO(X,\tau) \to P(X)$ *be a fixed function such that* $b \in \{\circ, \circ-\}$. *Then,*

$(\omega^{b},\omega^{b})$-$T^{id}_{1/2}$ $\Rightarrow$ $\omega^{b}$-$T_0$.

*Proof* **(i)** $\cdot(\omega$-$T_1 \Rightarrow (\omega,\omega)$-$T^{id}_{1/2})$: Suppose that $(X,\tau)$ is $\omega$-$T_1$, i.e., $\omega O(X,\tau)$-$T_1$. By Theorem 5.17(i) for $\mathcal{E}_X := \omega O(X,\tau)$, it is shown that $\omega O(X,\tau)$-$\mathrm{Cl}(\{x\}) = \{x\}$ for each singleton $\{x\}$ of $(X,\tau)$; and so, by Theorem 5.17(ii) for $\mathcal{E}_X := \omega O(X,\tau)$, it is shown that every singleton $\{x\}$ is $\omega$-closed (i.e., $X \setminus \{x\} \in \omega O(X,\tau)$), because $\omega O(X,\tau)$ has property $(\mathcal{B})_{\omega O(X,\tau)}$ (cf. Remark 5.6(i)). Using Theorem 5.13(i) for $\rho 1 = id$, we have that the space $(X,\tau)$ is $(\omega,\omega)$-$T^{id}_{1/2}$ (cf. Remark 5.6(ii)).

$\cdot((\omega,\omega)$-$T^{id}_{1/2} \Rightarrow \omega$-$T_0)$: Suppose that $(X,\tau)$ is $(\omega,\omega)$-$T^{id}_{1/2}$. By Theorem 5.13(i) for $\rho 1 = id$, every singleton $\{x\}$ is $\omega$-closed or $\omega$-open. For a pair of distinct points $x$ and $y$, we consider the following cases:

Case 1. $\{x\} \in \omega O(X,\tau)$ and $\{y\} \in \omega O(X,\tau)$: for this case, $\{x\}$ is the required set belonging to $\mathcal{E}_X := \omega O(X,\tau)$ such that $x \in \{x\}$ and $y \notin \{x\}$.

Case 2. $\{x\} \in \omega O(X,\tau)$ and $\{y\} \in \omega C(X,\tau)$: for this case, $\{x\} \in \mathcal{E}_X := \omega O(X,\tau)$ such that $x \in \{x\}$ and $y \notin \{x\}$.

Case 2'. $\{x\} \in \omega C(X,\tau)$ and $\{y\} \in \omega O(X,\tau)$: for this case, $\{y\} \in \mathcal{E}_X := \omega O(X,\tau)$ such that $y \in \{y\}$ and $x \notin \{y\}$.

Case 3. $\{x\} \in \omega C(X,\tau)$ and $\{y\} \in \omega C(X,\tau)$: for this case, $X \setminus \{y\} \in \mathcal{E}_X := \omega O(X,\tau)$ such that $x \in X \setminus \{y\}$ and $y \notin X \setminus \{y\}$.

Therefore $(X,\tau)$ is $\omega$-$T_0$ (cf. Definition 5.16(II) for $\rho = id$).

**(ii)** Let $x \in X$. By Theorem 5.17(ii) for $\mathcal{E}_X := \omega^{\mu}O(X,\tau)$, it is shown that the singleton $\{x\}$ is $\omega^{\mu}$-closed (i.e., $X \setminus \{x\} \in \omega^{\mu}O(X,\tau)$); and so, by Theorem 5.15(ii) for the case where $\rho 1 = \mu$, $(X,\tau)$ is $(\omega^{\mu},\omega^{\mu})$-$T^{id}_{1/2}$.

**(iii)** Let $(X,\tau)$ be an $(\omega^{b},\omega^{b})$-$T^{id}_{1/2}$ space, where $b \in \{\circ, \circ-\}$. Let $x \neq y$ be two points of $X$. Then, by Theorem 5.15(i) for $\mathcal{E}_X := \omega^{b}O(X,\tau)$ and $\rho 1 = \mu = b$, it is shown that, the singleton $\{x\}$ is $\omega^{b}$-closed or $\{x\}$ is $\omega^{b}$-open. Then, $(X,\tau)$ is $\omega^{b}$-$T_0$. □

## 6 An example satisfying a separation axiom: "$\omega^{\circ-}$-$T_1$ except a subset $A$" of $(\mathbb{Z},\kappa)$

In the last section, we prove the following properties: Theorem 6.1 on some separation axioms of the digital line $(\mathbb{Z},\kappa)$.

**Theorem 6.1** *Let* $(\mathbb{Z},\kappa)$ *be the digital line and* $\mathbb{Z}_{\kappa} := \{2s+1 | s \in \mathbb{Z}\}$. *We have the following properties of* $(\mathbb{Z},\kappa)$.

(i) $(\mathbb{Z},\kappa)$ *is* $(\omega,\omega)$-$T^{id}_{1/2}$.

(ii) $(\mathbb{Z},\kappa)$ *is not* $\omega^{\circ}$-$T_0$.

(iii) $(\mathbb{Z},\kappa)$ *is not* $\omega^{\circ-}$-$T_0$.

(iv) $(\mathbb{Z},\kappa)$ *is* $\omega^{\circ-}$-$T_1$ *except* $\mathbb{Z}_{\kappa}$.

In the end of the present section, we prove the Theorem 6.1 above, after recalling of definitions (i.e., Definitions 6.2, 6.3) and preparing some propositions (i.e., Propositions 6.4,6.5).

**Definition 6.2** Suppose that $|X| > 1$. Let $A$ be a proper subset of $X$. A topological space $(X, \tau)$ is said to be:

$\mathcal{E}_X$-$T_1$ *except* $A$, if the following properties (1) and (2) are satisfied:

(1) for every ordered pair $(x, y) \in (X \setminus A) \times (X \setminus A)$ such that $x \neq y$, there exists a set $V \in \mathcal{E}_X$ such that $x \in V$ and $y \notin V$ and there exists a set $V_1 \in \mathcal{E}_X$ such that $x \notin V_1$ and $y \in V_1$;

(2) for every ordered pair $(a, b) \in A \times A$ such that $a \neq b$, there does not exist any subsets $V \in \mathcal{E}_X$ and $V_1 \in \mathcal{E}_X$ such that $a \in V$ and $b \notin V$, and $b \in V_1$ and $a \notin V_1$.

Put $\mathcal{E}_X := \omega^{\circ -}O(X, \tau)$ in Defintion 6.2; then we have the following definition.

**Definition 6.3** Suppose that $|X| > 1$ and $A$ is a proper subset of $X$. A topologcal space $(X, \tau)$ is said to be $\omega^{\circ -}$-$T_1$ *except* $A$, if the space $(X, \tau)$ is $\omega^{\circ -}O(X, \tau)$-$T_1$ except $A$ in the sense of Definition 6.2.

**Proposition 6.4** *Let* $(\mathbb{Z}, \kappa)$ *be the digital line and* $\{2m\}$ *and* $\{2s+1\}$ *be two singletons of* $(\mathbb{Z}, \kappa)$, *where* $m, s \in \mathbb{Z}$.

(i) $\{2m\} \in \omega C(\mathbb{Z}, \kappa)$, $\{2m\} \notin \omega O(\mathbb{Z}, \kappa)$; $\{2s+1\} \notin \omega C(\mathbb{Z}, \kappa)$, $\{2s+1\} \in \omega O(\mathbb{Z}, \kappa)$.

(ii) $\{2m\} \notin \omega^{\circ -}C(\mathbb{Z}, \kappa)$, $\{2m\} \in \omega^{\circ -}O(\mathbb{Z}, \kappa)$; $\{2s+1\} \notin \omega^{\circ -}C(\mathbb{Z}, \kappa)$, $\{2s+1\} \notin \omega^{\circ -}O(\mathbb{Z}, \kappa)$.

(iii) *For every singleton* $\{x\}$ *of* $(\mathbb{Z}, \kappa)$, $\{x\} \notin \omega^{\circ}C(\mathbb{Z}, \kappa)$ *and* $\{x\} \notin \omega^{\circ}O(\mathbb{Z}, \kappa)$.

*Proof.* **(i)** It is well known that $\{2m\}$ is not open and it is closed in $(\mathbb{Z}, \kappa)$ and $\{2s+1\}$ is open and it is not closed in $(\mathbb{Z}, \kappa)$. Since $\omega O(\mathbb{Z}, \kappa) = \kappa$ holds by [17, Theorem 4.6], and hence we have that $\{2m\} \in \omega C(\mathbb{Z}, \kappa) \setminus \omega O(\mathbb{Z}, \kappa)$ and $\{2s+1\} \in \omega O(\mathbb{Z}, \kappa) \setminus \omega C(\mathbb{Z}, \kappa)$ hold.

**(ii)** · *Proof of* $\{2m\} \notin \omega^{\circ -}C(\mathbb{Z}, \kappa)$: there exists a semi-open set $V := \{2m, 2m+1\}$ such that $\{2m\} \subset V$ and $Cl(\{2m\}) = \{2m\} \not\subset Int(Cl(V))$, because of $Int(Cl(V)) = Int(\{2m, 2m+1, 2m+2\}) = \{2m+1\}$; and so $\{2m\}$ is not $\omega^{\circ -}$-closed in $(\mathbb{Z}, \kappa)$ (i.e., $\{2m\} \notin \omega^{\circ -}C(\mathbb{Z}, \kappa)$).

· *Proof of* $\{2m\} \in \omega^{\circ -}O(\mathbb{Z}, \kappa)$: let $E := \mathbb{Z} \setminus \{2m\}$. Let $V$ be a semi-open set containing $E$; then $V = E$ or $V = \mathbb{Z}$. Since $Cl(E) = \mathbb{Z}$ and $Int(Cl(E)) = \mathbb{Z}$ hold, we have that $Cl(E) \subset Int(Cl(V))$; and so $E := \mathbb{Z} \setminus \{2m\}$ is $\omega^{\circ -}$-closed in $(\mathbb{Z}, \kappa)$. Hence $\{2m\}$ is $\omega^{\circ -}$-open (i.e., $\{2m\} \in \omega^{\circ -}O(\mathbb{Z}, \kappa)$).

· *Proof of* $\{2s+1\} \notin \omega^{\circ -}C(\mathbb{Z}, \kappa)$: there exists a semi-open set $V := \{2s+1\}$ such that $\{2s+1\} \subset V$ and $\mathrm{Cl}(\{2s+1\}) = \{2s, 2s+1, 2s+2\} \not\subset Int(Cl(V))$, because of $Int(Cl(V)) = Int(\{2s, 2s+1, 2s+2\}) = \{2s+1\}$; and so $\{2s+1\}$ is not $\omega^{\circ -}$-closed in $(\mathbb{Z}, \kappa)$ (i.e., $\{2s+1\} \notin \omega^{\circ -}C(\mathbb{Z}, \kappa)$).

· *Proof of* $\{2s+1\} \notin \omega^{\circ -}O(\mathbb{Z}, \kappa)$: let $E := \mathbb{Z} \setminus \{2s+1\}$. Let $V := E$; and so $V$ is a semi-open set containing $E$. Since $Cl(E) = E$ and $Int(Cl(V)) = Int(E) = \mathbb{Z} \setminus \{2s, 2s+1, 2s+2\}$ hold, we have that $Cl(E) = E \not\subset Int(Cl(V))$; and so $E := \mathbb{Z} \setminus \{2s+1\}$ is not $\omega^{\circ -}$-closed in $(\mathbb{Z}, \kappa)$. Hence $\{2s+1\} \notin \omega^{\circ -}O(\mathbb{Z}, \kappa)$.

**(iii)** Let $x = 2m$ or $x = 2s+1$, where $m \in \mathbb{Z}$ and $s \in \mathbb{Z}$.

· *Proof of* $\{2m\} \notin \omega^{\circ}C(\mathbb{Z}, \kappa)$: by using the properties for $(\mathbb{Z}, \kappa)$ of Theorem 2.1(iii) (i.e., $\omega^{\circ}C(\mathbb{Z}, \kappa) \subset \omega^{\circ -}C(\mathbb{Z}, \kappa)$) and the corresponding property of the present (ii) (i.e., $\{2m\} \notin \omega^{\circ -}C(\mathbb{Z}, \kappa)$), it is shown that $\{2m\} \notin \omega^{\circ}C(\mathbb{Z}, \kappa)$.

· *Proof of* $\{2m\} \notin \omega^{\circ}O(\mathbb{Z}, \kappa)$: by using the property for $(X, \tau)$ of Theorem 2.1(i) (i.e., $\omega^{\circ}C(X, \tau) \subset \omega C(X, \tau)$) and definitions, it is shown that $\omega^{\circ}O(X, \tau) \subset \omega O(X, \tau)$ holds in general. By using the corresponding property of the proof of (i), it is obtained that $\{2m\} \notin \omega^\circ O(\mathbb{Z}, \kappa)$.

$\cdot$ *Proof of* $\{2s+1\} \notin \omega^\circ C(\mathbb{Z}, \kappa)$: by using the property for $(\mathbb{Z}, \kappa)$ of Theorem 2.1(iii) (i.e., $\omega^\circ C(\mathbb{Z}, \kappa) \subset \omega^{\circ -} C(\mathbb{Z}, \kappa)$) and the corresponding property of the present (ii) (i.e., $\{2s+1\} \notin \omega^{\circ -} C(\mathbb{Z}, \kappa)$), it is shown that $\{2s+1\} \notin \omega^\circ C(\mathbb{Z}, \kappa)$.

$\cdot$ *Proof of* $\{2s+1\} \notin \omega^\circ O(\mathbb{Z}, \kappa)$: by using the same property for $(\mathbb{Z}, \kappa)$ of Theorem 2.1(iii) (cf. Proof of $\{2m\} \notin \omega^\circ O(\mathbb{Z}, \kappa)$) and the corresponding property of the present (ii) (i.e., $\{2s+1\} \notin \omega^{\circ -} O(\mathbb{Z}, \kappa)$), it is shown that $\{2s+1\} \notin \omega^\circ O(\mathbb{Z}, \kappa)$. □

**Proposition 6.5** (i) (i-1) *If* $U \in \omega^\circ O(\mathbb{Z}, \kappa)$ *and* $2m \in U$ *for some integer* $m$, *then* $\{2m-1, 2m, 2m+1\} \subset U$.

(i-2) *If* $U \in \omega^\circ O(\mathbb{Z}, \kappa)$ *and* $2s+1 \in U$ *for some integer* $s$, *then* $\{2s-1, 2s, 2s+1, 2s+2, 2s+3\} \subset U$.

(i-3) $\omega^\circ O(\mathbb{Z}, \kappa) = \{\emptyset, \mathbb{Z}\}$ *holds.*

(ii) (ii-1) *If* $V \in \omega^{\circ -} O(\mathbb{Z}, \kappa)$ *and* $2s+1 \in V$ *for some integer* $s$, *then* $\{2s-1, 2s, 2s+1, 2s+2, 2s+3\} \subset V$.

(ii-2) *The following properties on a nonempty subset* $V$ *are equivalent:*
(1) $V \in \omega^{\circ -} O(\mathbb{Z}, \kappa)$ *and* $2s+1 \in V$ *for some integer* $s$;
(2) $V = \mathbb{Z}$ *holds.*

(ii-3) $\omega^{\circ -} O(\mathbb{Z}, \kappa) = \{E \mid E \subset \mathbb{Z}_F\} \cup \{\emptyset, \mathbb{Z}\}$ *holds, where* $\mathbb{Z}_F := \{2m \mid m \in \mathbb{Z}\}$. *Especially,* $\mathbb{Z}_F \in \omega^{\circ -} O(\mathbb{Z}, \kappa)$ *holds.*

(ii-4) *Every nonempty subset of* $\mathbb{Z}_\kappa$ *is not* $\omega^{\circ -}$*-open in* $(\mathbb{Z}, \kappa)$(i.e., $\{2m+1 \mid m \in E\} \notin \omega^{\circ -} O(\mathbb{Z}, \kappa)$, *where* $E \subset \mathbb{Z}$ *with* $E \neq \emptyset$). *Especially,* $\mathbb{Z}_\kappa \notin \omega^{\circ -} O(\mathbb{Z}, \kappa)$ *holds.*

*Proof.* **(i) (i-1)** Since $\{2m\} \in SC(\mathbb{Z}, \kappa)$ and so $\mathbb{Z} \setminus \{2m\}$ is a semi-open set. And, it follows from assumptions that $\mathbb{Z} \setminus \{2m\}$ contains the set $\mathbb{Z} \setminus U$ which is $\omega^\circ$-closed. Then, $Cl(\mathbb{Z} \setminus U) \subset Int(\mathbb{Z} \setminus \{2m\}) = \mathbb{Z} \setminus \{2m\}$; and so we have that $\mathbb{Z} \setminus Int(U) \subset \mathbb{Z} \setminus \{2m\}$, i.e., $2m \in Int(U)$. There exists the smallest open set $\{2m-1, 2m, 2m+1\}$ containing $2m$ such that $\{2m-1, 2m, 2m+1\} \subset Int(U) \subset U$ (e.g., [17, Definition 3.3 and its near part]).

**(i-2)** Since $\mathbb{Z} = \mathbb{Z}_{\mathcal{SC}} \cup \mathbb{Z}_{\omega^\circ \mathcal{O}}$ (cf. Lemma 4.3(i)), we consider the following cases: $\{2s+1\} \in SC(\mathbb{Z}, \kappa)$ or $\{2s+1\} \in \omega^\circ O(\mathbb{Z}, \kappa)$. By Proposition 6.4(iii), $\{2s+1\} \notin \omega^\circ O(\mathbb{Z}, \kappa)$; and so we consider the case where $\{2s+1\} \in SC(\mathbb{Z}, \kappa)$. Since $\mathbb{Z} \setminus \{2s+1\}$ is a semi-open set containing $\mathbb{Z} \setminus U$ and the set $\mathbb{Z} \setminus U$ is an $\omega^\circ$-closed set, we have that $Cl(\mathbb{Z} \setminus U) \subset Int(\mathbb{Z} \setminus \{2s+1\}) = \mathbb{Z} \setminus Cl(\{2s+1\}) = \mathbb{Z} \setminus \{2s, 2s+1, 2s+2\}$. Thus, we have that $\{2s, 2s+1, 2s+2\} \in Int(U)$. Since $2s \in Int(U)$ (resp. $2s+2 \in Int(U)$), the minimal open set containing $2s$ (resp. $2s+2$) is included in $Int(U)$,i.e., $\{2s-1, 2s, 2s+1\} \subset Int(U)$ (resp. $\{2s+1, 2s+2, 2s+3\} \subset Int(U)$.

**(i-3)** Let $U \in \omega^\circ O(\mathbb{Z}, \kappa)$ such that $U \neq \emptyset$. Then, by (i-1) and (i-2) above, it is shown that there exists an odd point, say $2u+1 \in U$, where $u \in \mathbb{Z}$. We claim that $\mathbb{Z} \subset U$. Indeed, let $z \in \mathbb{Z}$ be a point.

Case 1. $z = 2s$, where $s \in \mathbb{Z}$: for the present case, if $2s < 2u+1$, then we can take the following sequence of points, say $\{z_i\}_{i=1}^{k}$, where $k := 2(u-s+1)$ and $z_i := 2u+2-i$ $(1 \leq i \leq k)$, where ; then, $z_1 = 2u+1 \in U$ and $z_k = 2s = z$; and by using (i-1) and (i-2) above, we show inductively, that $z_i \in U$ $(2 \leq i \leq k)$ and hence $z \in U$. If $2s > 2u+1$, then we can take the following sequence of points, say $\{z'_i\}_{i=1}^{k'}$, $k' := 2(s-u)$ and $z'_i := 2u+i$ $(1 \leq i \leq k')$; then, $z'_1 = 2u+1 \in U$ and $z'_{k'} = z$; and by a similar arguments of the above case, it is shown that $z'_i \in U$ $(2 \leq i \leq k')$; and so $z \in U$. Thus, we proved that $z = 2s \in U$ holds for any cases.

Case 2. $z = 2t+1$, where $t \in \mathbb{Z}$: for the present case, let $z \neq 2u+1$. If $z < 2u+1$, then we can constract the following sequence of points, say $\{x_i\}_{i=1}^{k}$, where $k := u-t+1$, and $x_i := 2u + 1 - 2(i - 1)(1 \leq i \leq k)$; then $x_1 = 2u + 1 \in U$ and $x_k = z$; and by using (i-2) above, we show inductively, that $x_i \in U(2 \leq i \leq k)$; and so $z \in U$. If $2u + 1 < z$, then we can constract the following sequence of points, say $\{x'_i\}_{i=1}^{k'}$, where $k' := t - u + 1$, and $x'_i := 2u + 1 + 2(i - 1)(1 \leq i \leq k')$; then $x'_1 = 2u + 1 \in U$ and $x'_{k'} = z$; and by using (i-2) above, we show inductively, that $x'_i \in U(2 \leq i \leq k')$; and so $z \in U$.

Therefore, we prove that $\mathbb{Z} \subset U$ and so $U = \mathbb{Z}$.

**(ii) (ii-1)** · *Proof of* $\{2s, 2s+2\} \subset V$. Since $\mathbb{Z} \setminus \{2s+1\}$ is a semi-open set containing $\mathbb{Z} \setminus V$ and $\mathbb{Z} \setminus V$ is $\omega^{\circ -}$-closed, we have that $Cl(\mathbb{Z} \setminus V) \subset Int(Cl(\mathbb{Z} \setminus \{2s + 1\}))=\mathbb{Z} \setminus \{2s, 2s + 1, 2s + 2\}$. Thus, we have that $\{2s, 2s + 1, 2s + 2\} \subset Int(V)$.
Since $2s \in Int(V)$ (resp. $2s + 2 \in Int(V)$) and the set $\{2s - 1, 2s, 2s + 1\}$ (resp. $\{2s + 1, 2s + 2, 2s + 3\}$) is the ninimal open set containing the point $2s$ (resp. $2s + 2$), we have that $\{2s - 1, 2s, 2s + 1\} \subset V$ (resp. $\{2s + 1, 2s + 2, 2s + 3\} \subset V$). Therefore, we show the required property that $\{2s - 2 + j | 1 \leq j \leq 5\} \subset V$.

**(ii-2) (1)⇒(2)** In order to prove that $\mathbb{Z} \subset V$, let $z \in \mathbb{Z}$ be a point. First, it is claimed that:
$(*1)$ if $z = 2m + 1$ for some integer $m$, then $z \in V$.

*Proof of* $(*1)$: **(Case 1)** $z := 2m + 1$ and $z < 2s + 1$; for the present case, we apply (ii-1) for the point $2s + 1 \in V$ and $V \in \omega^{\circ -}O(X, \tau)$. And, it is shown inductively that there exists a finite sequence of points $\{y_i\}_{i=1}^{k}$ such that:
$(*2)_i$ $y_i \in V$ $(1 \leq i \leq k)$, where $y_i := 2s + 1 - 2i$ and $k := s - m$.
Indeed, by (ii-1) above for the odd point $2s + 1 \in V$, it is shown that $\{2s - 1, 2s, 2s + 1, 2s + 2, 2s + 3\} \subset V$. Thus, $2s - 1 \in V$; and so $y_1 = 2s + 1 - 2 \in V$. Then, we show that $(*2)_i$ holds for $i = 1$. In order to prove $(*2)_i$ by finite induction on $i$ $(1 \leq i \leq k)$, suppose that $y_r \in V$, where $1 < r < k$ and $y_r := 2s + 1 - 2r$. Since $y_r$ is odd and $y_r \in V$, by (ii-1) above, it is shown that $\{y_r - 2, y_r - 1,\ y_r,\ y_r + 1, y_r + 2\} \subset V$. Thus, we have that $y_{r+1} = 2s + 1 - 2(r + 1) = 2s + 1 - 2r - 2 = y_r - 2 \in V$, i.e., we have that $(*2)_i$ holds for $i = r + 1$. Then, by finte induction on $i$ $(1 \leq i \leq k)$, it is shown that $y_k \in V$; and hence $z = 2m + 1 = 2s + 1 - 2(s - m) = y_{s-m} = y_k \in V$. Thus, we show that $z \in V$ for the present Case 1.
**(Case 1')**. $z = 2m + 1 \in \mathbb{Z}$ and $2s + 1 < z$: for the present case, we apply (ii-1) above for the point $2s + 1 \in V$ and $V \in \omega^{\circ -}O(X, \tau)$. By an argument similar to that in the proof of Case 1 above, it is shown inductively that there exists a sequence of points $\{y'_i\}_{i=1}^{k'}$ such that :
$(*2)'_i$ $y'_i \in V$ holds for each integer $i$ with $1 \leq i \leq k'$, where $y'_i := (2s + 1) + 2i$ and $k' := m - s$. Thus, we show that $z \in V$ for the present Case 1'.
Finally, it is claimed that:
$(*3)$ if $z = 2m$ for some integer $m$, then $z \in V$.

*Proof of* $(*3)$: by $(*1)$ above, it is shown that $2u + 1 \in V$ for any odd point $2u + 1 \in \mathbb{Z}$. Then, take the odd point $z + 1 = 2m + 1$; and so $2m + 1 \in V$. Here, by using (ii-1) above for the point $2m + 1 \in V$ and $V \in \omega^{\circ -}O(\mathbb{Z}, \kappa)$, it is shown that $\{2m - 2, 2m, 2m + 1, 2m + 2, 2m + 3\} \subset V$; and so $z := 2m \in V$.

Therefore, we conclude that $z \in V$ for any point $z \in \mathbb{Z}$ (i.e., $\mathbb{Z} = V$).

**(2)⇒(1)** Suppose $V = \mathbb{Z}$. By definitions, it is obvious that $\mathbb{Z} \in \omega^{\circ -}O(\mathbb{Z}, \kappa)$ and there exists an odd point $2s + 1 \in V = \mathbb{Z}$, where $s \in \mathbb{Z}$.

**(ii-3)** First, we prove that:
$(*4)$ $\omega^{\circ -}O(\mathbb{Z}, \kappa) \subset \{E | \ E \subset \mathbb{Z}_F\} \cup \{\emptyset, \mathbb{Z}\}$. Indeed, let $V \in \omega^{\circ -}O(\mathbb{Z}, \kappa)$ such that $V \notin \{\emptyset, \mathbb{Z}\}$. Then, by (ii-2) above, it is shown that $2s + 1 \notin V$ holds for every integer $s \in \mathbb{Z}$, i.e., $V \subset \mathbb{Z}_F :=\{2m | \ m \in \mathbb{Z}\}$. Thus, we proved $(*4)$. Secondly, we prove that:
$(*5)$ $\{E | \ E \subset \mathbb{Z}_F\} \cup \{\emptyset, \mathbb{Z}\} \subset \omega^{\circ -}O(\mathbb{Z}, \kappa)$ holds. Let $V \subset \mathbb{Z}_F$ with $V \notin \{\emptyset, \mathbb{Z}\}$. Then,

$V = \{2m|\ m \in A\}$, where $A \subset \mathbb{Z}$. In order to prove hat $\mathbb{Z} \setminus V \in \omega^{\circ-}C(\mathbb{Z}, \kappa)$, let $U$ be a semi-open set such that $\mathbb{Z} \setminus V \subset U$. Since $\mathbb{Z}_\kappa$=$\{2s+1|s \in \mathbb{Z}\} \subset \mathbb{Z} \setminus V$, it is shown that $\mathbb{Z}$=$Cl(\mathbb{Z}_\kappa) \subset Cl(\mathbb{Z} \setminus V) \subset Cl(U)$; and so $\mathbb{Z} = Cl(U)$ and $Cl(\mathbb{Z} \setminus V) = \mathbb{Z} = Int(Cl(U))$ hold. Thus, we prove that $\mathbb{Z} \setminus V$ is $\omega^{\circ-}$-closed, i.e., $V \in \omega^{\circ-}O(\mathbb{Z}, \kappa)$.
Finally, by $(*5)$ above, it is especially shown that $\mathbb{Z}_F \in \omega^{\circ-}O(\mathbb{Z}, \kappa)$.

**(ii-4)** Let denote $V := \{2m+1|\ m \in A\}$, where $A \subset \mathbb{Z}$ with $A \neq \emptyset$. Then, $V \notin \{E|\ E \subset \mathbb{Z}_F\} \cup \{\emptyset, \mathbb{Z}\}$ ;and so, by (ii-3) above, $V \notin \omega^{\circ-}O(\mathbb{Z}, \kappa)$. Especially, $\mathbb{Z}_\kappa \notin \omega^{\circ-}O(\mathbb{Z}, \kappa)$. □

**Remark 6.6** The converse of Proposition 6.5(ii)(ii-1) is not true. Indeed, Let $V := \{2s-1, 2s, 2s+1, 2s+2, 2s+3\}$ be a subset of $(\mathbb{Z}, \kappa)$, where $s \in \mathbb{Z}$. Then, there exists a semi-open set $W := \mathbb{Z} \setminus V$ such that $\mathbb{Z} \setminus V \subset W$ and $Cl(\mathbb{Z} \setminus V)$=$Cl(W) = W \not\subset Int(Cl(W))$. Then, $\mathbb{Z} \setminus V \notin \omega^{\circ-}C(\mathbb{Z}, \kappa)$,i.e., $V \notin \omega^{\circ-}O(\mathbb{Z}, \kappa)$ holds, even if $2s+1 \in V$ and $\{2s-1, 2s, 2s+1, 2s+2, 2s+3\} \subset V$.

**Proof of Theorem 6.1:**

**Proof of (i)** It is well known that $(\mathbb{Z}, \kappa)$ is $T_{1/2}$ and so it is $(\omega, \omega)$-$T^{id}_{1/2}$ (cf. [5, Theorem 2.5], Theorem 5.18 (i)).

**Proof of (ii)** Let $x := 2m \in \mathbb{Z}$ and $U$ be any $\omega^\circ$-open set such that $x \in U$. By Proposition 6.5(i)(i-3), it is shown that $U = \mathbb{Z}$ and so $2m+1 \in U$. Thus, there exists a pair of distinct points $2m$ and $2m+1$ of $(\mathbb{Z}, \kappa)$ which does not satisfy the condition of the $\omega^\circ$-$T_0$ (cf. Definition 5.16 for $\mathcal{E}_\mathbb{Z} := \omega^\circ O(\mathbb{Z}, \kappa)$).

**Proof of (iii)** Let $x := 2s+1$ and $y := 2s+3$ be two points of $(\mathbb{Z}, \kappa)$, where $s \in \mathbb{Z}$. And, let $V$ (resp. $V_1$) be any $\omega^{\circ-}$-open set containing the point $x$ (resp. $y$). By Proposition 6.5(ii)(ii-2)(1)⇒(2), it is shown that $V = \mathbb{Z}$ (resp. $V_1 = \mathbb{Z}$), and so $y \in V$ (resp. $x \in V$). Thus, $(\mathbb{Z}, \kappa)$ is not $\omega^{\circ-}$-$T_0$ (cf. Definition 5.16).

**Proof of (iv)** First, we recall that $\mathbb{Z}_\kappa := \{2u+1|\ u \in \mathbb{Z}\}$. Let $(x, y) \in (\mathbb{Z} \setminus \mathbb{Z}_\kappa) \times (\mathbb{Z} \setminus \mathbb{Z}_\kappa)$ be an ordered pair of points such that $x \neq y$. Since $x = 2m$ for some integer $m$, there exists a set $V \in \omega^{\circ-}O(\mathbb{Z}, \kappa)$ (cf. Proposition 6.4(ii)), where $V := \{2m\}$, such that $x \in V$ and $y \notin V$. And, since $y = 2s$ for some integer $s$ with $s \neq m$, there exists a set $V_1 \in \omega^{\circ-}O(\mathbb{Z}, \kappa)$, where $V_1 := \{2s\}$, such that $x \notin V_1$ and $y \in V_1$. Thus, one of the properties of $\omega^{\circ-}$-$T_1$-ness except $\mathbb{Z}_\kappa$ is satiesfied (cf. (1) of Definition 6.2 and Definition 6.3).

Finally, let $(a, b) \in \mathbb{Z}_\kappa \times \mathbb{Z}_\kappa$ be any ordered pair of points $a$ and $b$ such that $a \neq b$. Let $V_a$ (resp. $W_b$) be any $\omega^{\circ-}$-open set such that $a \in V_a$ (resp. $b \in W_b$). Then, by Proposition 6.5(ii)(ii-2) (1)⇒(2), it is shown that $V_a = \mathbb{Z}$; and so $b \in V$ (resp. $W_b = \mathbb{Z}$ and so $a \in W_b$). Thus, the property (2) for $A := \mathbb{Z}_\kappa$ in Definition 6.2 of $\omega^{\circ-}$-$T_1$-ness except $\mathbb{Z}_\kappa$ is satisfied.

Therefore, the digital line $(\mathbb{Z}, \kappa)$ is $\omega^{\circ-}$-$T_1$ except $\mathbb{Z}_\kappa$. □

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# COMMON INVARIANT SUBSPACES OF A FAMILY OF TOEPLITZ OPERATORS

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Abstract. Let $\Phi$ be a subset of $L^\infty$ containing $H^\infty$ and $T_\Phi$ the family of Toeplitz operators $\{T_\varphi\}_{\varphi\in\Phi}$. In this paper, we study invariant subspaces of $T_\Phi$ and their properties. Moreover, we provide a concrete description of nontrivial invariant subspaces of $T_\Phi$ for some $\Phi$.

**1 Introduction** Let $\Gamma$ be the unit circle centered at the origin in the complex plane, and $H^2(\Gamma^n)$ be the Hardy space on $\Gamma^n$. In [5], the second author showed that $H^2(\Gamma)$ has a certain rigidity (see Theorem 2.1 stated below), and pointed out that $H^2(\Gamma^2)$ does not have this property. The purpose of this paper is to study this phenomenon with examples.

We introduce notions in this paper. Let $L^2(\Gamma^n)$ be the usual $L^2$ space with respect to the normalized Lebesgue measure on $\Gamma^n$. Let $P$ be the orthogonal projection from $L^2(\Gamma^n)$ onto $H^2(\Gamma^n)$. For $\varphi \in L^\infty(\Gamma^n)$, we define

$$T_\varphi f = P(\varphi f) \quad (f \in H^2).$$

Then $T_\varphi$ is called the Toeplitz operator with symbol $\varphi$. For a subset $\Phi$ in $L^\infty(\Gamma^n)$, $T_\Phi$ denotes the set of Toeplitz operators whose symbols are in $\Phi$, that is, we set

$$T_\Phi = \{T_\varphi : \varphi \in \Phi\}.$$

The collection of all closed subspaces of $H^2(\Gamma^n)$ invariant under every $T_\varphi \in T_\Phi$ is denoted by $\operatorname{Lat} T_\Phi$. Throughout this paper, we assume that $H^\infty \subseteq \Phi \subseteq L^\infty$.

This paper consists of five sections. In Section 2, we consider one variable Hardy space and recall results in [5]. In Section 3, we introduce some classes of functions in order to study $\operatorname{Lat} T_\Phi$. In Section 4, we study $\operatorname{Lat} T_\Phi$ for some $\Phi$'s. In Section 5, we show that $\operatorname{Lat} T_\Phi$ is nontrivial for some $\Phi$, and present examples of invariant subspaces of $T_z$ and $T_w$.

**2 A certain rigidity of $H^2(\Gamma)$** The following theorem was given in [5], which shows that $H^2(\Gamma)$ has a certain rigidity.

**Theorem 2.1** ([5])**.** *If* $\Phi = H^\infty(\Gamma) \cup \{\varphi\}$ *for* $\varphi \in L^\infty(\Gamma) \setminus H^\infty(\Gamma)$, *then* $\operatorname{Lat} T_\Phi = \{\langle 0\rangle, H^2(\Gamma)\}$.

The original proof is based on the theory of uniform algebras. We shall give another proof to this theorem.

*Proof.* In this proof, we will write $H^2 = H^2(\Gamma)$, $H^\infty = H^\infty(\Gamma)$ and so on. Suppose that $\mathcal{M} \in \operatorname{Lat} T_\Phi$ and $\mathcal{M}$ is nontrivial. Then, $\mathcal{M}$ is an invariant subspace of $H^2$. Hence, there exists a non-constant inner function $q$ such that $\mathcal{M} = qH^2$ by Beurling's theorem. We note that $T_\varphi \mathcal{M} \subset \mathcal{M}$ is equivalent to that

$$P_{H^2}\varphi q H^2 \subset qH^2.$$

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Hence, for any function $h \in H^2$, there exists a function $g_h \in H^2$ such that $P_{H^2}(\varphi q h) = q g_h$. Then we have that $P_{H^2}(\varphi q h - q g_h) = 0$, and which is equivalent to that $\varphi q h - q g_h \in \overline{H_0^2}$, where $\overline{H_0^2} = L^2 \ominus H^2$. Therefore we have that

$$\varphi q h \in \mathcal{M} \oplus \overline{H_0^2} \quad (h \in H^2). \tag{2.1.1}$$

In particular, for $h = 1$, there exist $g_1 \in H^2$ and $k \in H_0^2$ such that

$$\varphi q = q g_1 + \overline{k}. \tag{2.1.2}$$

Put $\mathcal{N} = H^2 \ominus \mathcal{M}$. Multiplying both sides of (2.1.2) by $h \in H^\infty$, we obtain

$$\begin{aligned}\varphi q h &= \{P_{\mathcal{M}} + P_{\mathcal{N}} + (I_{L^2} - P_{H^2})\}(q g_1 h + \overline{k} h)\\ &= (q g_1 h + P_{\mathcal{M}} \overline{k} h) \oplus P_{\mathcal{N}} \overline{k} h \oplus (I_{L^2} - P_{H^2}) \overline{k} h.\end{aligned}$$

Then, by (2.1.1), we note that

$$P_{\mathcal{N}} \overline{k} h = P_{\mathcal{N}} \varphi q h = 0.$$

Let $\mathbb{D}$ be the open unit disc in the complex plane. Now, setting

$$k = \sum_{j=1}^{\infty} c_j z^j, \quad k_n = \sum_{j=1}^{n} c_j z^j \quad \text{and} \quad s_\lambda = \frac{1}{1 - \overline{\lambda} z} \quad (\lambda \in \mathbb{D}),$$

we have that

$$\begin{aligned}\|P_{\mathcal{N}} \overline{k_n} s_\lambda\| &= \|P_{\mathcal{N}} \overline{k_n} s_\lambda - P_{\mathcal{N}} \overline{k} s_\lambda\|\\ &\leq \|\overline{k_n} s_\lambda - \overline{k} s_\lambda\|\\ &\leq \|s_\lambda\|_\infty \|k_n - k\|\\ &\to 0\end{aligned}$$

as $n \to \infty$. On the other hand,

$$\begin{aligned}P_{\mathcal{N}} \overline{k_n} s_\lambda &= P_{\mathcal{N}} T_{k_n}^* s_\lambda\\ &= P_{\mathcal{N}} \overline{k_n(\lambda)} s_\lambda\\ &\to P_{\mathcal{N}} \overline{k(\lambda)} s_\lambda\end{aligned}$$

as $n \to \infty$. Therefore $P_{\mathcal{N}} \overline{k(\lambda)} s_\lambda = 0$ for any $\lambda \in \mathbb{D}$. If $k(\lambda) \neq 0$ for some $\lambda$, then $P_{\mathcal{N}} s_\lambda = 0$. However,

$$P_{\mathcal{N}} s_\lambda = \frac{1 - \overline{q(\lambda)} q}{1 - \overline{\lambda} z} \neq 0.$$

Hence $k(\lambda) = 0$ for all $\lambda \in \mathbb{D}$. Then we see that $\varphi q = q g_1$ in (2.1.2), and which implies $\varphi = g_1 \in H^2$. This contradicts that $\varphi \in L^\infty \setminus H^\infty$. □

From Theorem 2.1, in $H^2(\Gamma)$, $\mathrm{Lat}\, T_\Phi$ has only trivial invariant subspaces if $\Phi$ contains $H^\infty(\Gamma)$ properly. On the other hand, in the case of $H^2(\Gamma^2)$, $\mathrm{Lat}\, T_\Phi$ may not be $\{\langle 0 \rangle, H^2(\Gamma^2)\}$ even if $\Phi$ properly contains $H^\infty(\Gamma^2)$. The following is an example.

*Example* 2.2. We set $\mathcal{M} = zH^2(\Gamma^2) + wH^2(\Gamma^2)$. Then $\mathcal{M} \in \mathrm{Lat}\, T_\Phi$ for $\Phi = H^\infty(\Gamma^2) \cup \{\overline{z} w\}$.

We will see more examples in Section 5.

**3 $\mathcal{M}_\Phi$, $\mathcal{M}^\Phi$ and $K^\Phi_{\mathcal{M}}$** We focus on the structure of $H^2(\Gamma^2)$, so that we will write $L^2 = L^2(\Gamma^2)$, $H^2 = H^2(\Gamma^2)$ and so on, if no confusion occurs. In this section, some classes of functions which play important roles in this paper are introduced.

**Definition 3.1.** *Let $\varphi$ be a function in $L^\infty$. For $\mathcal{M} \in \operatorname{Lat} T_\varphi$, we put*

$$\mathcal{M}_\varphi = \{f \in \mathcal{M} : \varphi f \in \mathcal{M}\} \quad \textit{and} \quad \mathcal{M}^\varphi = \mathcal{M} \ominus \mathcal{M}_\varphi.$$

*Moreover, let $\Phi$ be a subset of $L^\infty$. For $\mathcal{M} \in \operatorname{Lat} T_\Phi$, we put*

$$\mathcal{M}_\Phi = \bigcap_{\varphi \in \Phi} \mathcal{M}_\varphi \quad \textit{and} \quad \mathcal{M}^\Phi = \mathcal{M} \ominus \mathcal{M}_\Phi.$$

*Example* 3.1. $\mathcal{M}_{\overline{z}} = z\mathcal{M}$ and $\mathcal{M}^{\overline{z}} = \mathcal{M} \ominus z\mathcal{M}$. Further, if $\Phi = H^\infty \cup \{\overline{z}, \overline{w}\}$, then $\mathcal{M}_\Phi = zw\mathcal{M}$ and $\mathcal{M}^\Phi = \mathcal{M} \ominus zw\mathcal{M}$.

We are mainly interested in the case where $\Phi$ is a subset of $L^\infty$ which contains $H^\infty$ properly. We shall give some general facts on $\mathcal{M}_\Phi$ and $\mathcal{M}^\Phi$.

**Proposition 3.2.** *Let $\Phi$ be a subset of $L^\infty$ which contains $H^\infty$ properly. Then $\mathcal{M}_\Phi$ is an invariant subspace in $H^2$.*

*Proof.* It suffices to show that $\mathcal{M}_\varphi$ is an invariant subspace for any $\varphi \in \Phi$. If $f \in \mathcal{M}_\varphi$ then $\varphi f \in \mathcal{M}$. It follows from this that $z\varphi f \in \mathcal{M}$, that is, $zf \in \mathcal{M}_\varphi$. Hence $\mathcal{M}_\varphi$ is invariant under multiplication by $z$. Moreover, if $f_n \in \mathcal{M}_\varphi$ and $f_n \to f$ $(n \to \infty)$, then $f \in \mathcal{M}$ and $\varphi f_n \to \varphi f$ $(n \to \infty)$ in $\mathcal{M}$. Hence we have that $f \in \mathcal{M}_\varphi$, that is, $\mathcal{M}_\varphi$ is closed. These conclude that $\mathcal{M}$ is an invariant subspace in $H^2$. □

In order to give the next theorem on $\mathcal{M}^\Phi$, we need a lemma.

**Lemma 3.3.** *Let $\Phi$ be a subset of $L^\infty$ which contains $H^\infty$ properly. Suppose that $\mathcal{M} \in \operatorname{Lat} T_\Phi$. For any $f \in H^\infty$, we define $Q_f = P_{\mathcal{M}^\Phi} T_f|_{\mathcal{M}^\Phi}$. Then*

$$Q_{fg} = Q_f Q_g \quad (f \textit{ and } g \in H^\infty).$$

*Proof.* It follows from Proposition 3.2 that

$$\begin{aligned} Q_{fg} - Q_f Q_g &= P_{\mathcal{M}^\Phi} T_{fg} P_{\mathcal{M}^\Phi} - P_{\mathcal{M}^\Phi} T_f P_{\mathcal{M}^\Phi} T_g P_{\mathcal{M}^\Phi} \\ &= P_{\mathcal{M}^\Phi} T_f (P_{\mathcal{M}} - P_{\mathcal{M}^\Phi}) T_g P_{\mathcal{M}^\Phi} \\ &= P_{\mathcal{M}^\Phi} T_f P_{\mathcal{M}_\Phi} T_g P_{\mathcal{M}^\Phi} \\ &= 0. \end{aligned}$$

□

**Theorem 3.4.** *Let $\Phi$ be a subset of $L^\infty$ which contains $H^\infty$ properly. If $\mathcal{M} \in \operatorname{Lat} T_\Phi$ then $\dim \mathcal{M}^\Phi = \infty$.*

*Proof.* Suppose $\dim \mathcal{M}^\Phi = n < \infty$. Then, by Lemma 3.3, there exists a finite Blaschke product $b_1(z)$ such that $Q_{b_1(z)} = 0$. Hence we have $b_1(z)\mathcal{M}^\Phi \subset \mathcal{M}_\Phi$. Further, it follows from Proposition 3.2 that $b_1(z)\mathcal{M}_\Phi \subset \mathcal{M}_\Phi$, that is,

$$b_1(z)\varphi\mathcal{M} \subset \mathcal{M} \quad (\varphi \in \Phi).$$

Similarly, there exists a finite Blaschke product $b_2(w)$ such that

$$b_2(w)\varphi\mathcal{M} \subset \mathcal{M} \quad (\varphi \in \Phi).$$

Hence $b_1(z)\varphi$ and $b_2(w)\varphi$ belong to $H^2$ for all $\varphi \in \Phi$. Therefore we have

$$\varphi \in \overline{b_1(z)}H^2 \cap \overline{b_2(w)}H^2 \subset H^2.$$

However, this is a contradiction. □

Next, we introduce a kind of complement of $\mathcal{M}$ in our problem.

**Definition 3.2.** *For $\mathcal{M} \in \operatorname{Lat} T_\Phi$ and $\varphi \in \Phi$, put*

$$K = \{\overline{f} : f \in L^2 \ominus H^2\}$$

*and*

$$K_{\mathcal{M}}^{\varphi} = \{k \in K : \overline{k} = \varphi f - g \text{ for some } f \text{ and } g \in \mathcal{M}\},$$

*where $\overline{f}$ denotes the complex conjugate of $f$. Moreover, we set*

$$K_{\mathcal{M}}^{\Phi} = \bigcup_{\varphi \in \Phi} K_{\mathcal{M}}^{\varphi}.$$

If $\varphi \in H^\infty$ and $k \in K_{\mathcal{M}}^{\varphi}$, then there exist $f$ and $g \in \mathcal{M}$ such that $\overline{k} = \varphi f - g$. However, it follows from $\overline{K} \cap \mathcal{M} = \langle 0 \rangle$ that $k = 0$, that is, $K_{\mathcal{M}}^{\varphi} = \langle 0 \rangle$ for $\varphi \in H^\infty$, so that we may define

$$K_{\mathcal{M}}^{\Phi} = \bigcup_{\varphi \in \Phi \setminus H^\infty} K_{\mathcal{M}}^{\varphi}.$$

*Remark* 3.5. In $H^2(\Gamma)$,

$$K = \{\overline{f} : f \in L^2(\Gamma) \ominus H^2(\Gamma)\} = H_0^2(\Gamma)$$

and we have already dealt with $K_{\mathcal{M}}^{\varphi}$ in the proof of Theorem 2.1 (see (2.1.1)), implicitly.

Next, we study the properties of $K_{\mathcal{M}}^{\Phi}$ used in the rest of this paper.

**Lemma 3.6.** *Let $\mathcal{M}$ be a closed subspace in $H^2$, and $\Phi$ be a subset of $L^\infty$ which contains $H^\infty$.*

(1) *$\mathcal{M} \in \operatorname{Lat} T_\Phi$ if and only if $\varphi\mathcal{M} \subset \mathcal{M} + \overline{K_{\mathcal{M}}^{\varphi}}$ for all $\varphi \in \Phi$.*

(2) *If $\mathcal{M} \in \operatorname{Lat} T_\Phi$, then $(I_{L^2} - P_{\mathcal{M}})\varphi\mathcal{M}^{\varphi} = \overline{K_{\mathcal{M}}^{\varphi}}$ for all $\varphi \in \Phi$.*

*Proof.* (1) First we show the 'if' part. For any $\varphi \in \Phi$ and $f \in \mathcal{M}$, there exist $g \in \mathcal{M}$ and $k \in K_{\mathcal{M}}^{\varphi}$ such that $\varphi f = g + \overline{k}$. From this equality, we have $T_\varphi f = g \in \mathcal{M}$. Hence we see that $\mathcal{M} \in \operatorname{Lat} T_\Phi$. Next, we show the 'only if' part. Suppose that $\mathcal{M}$ is in $\operatorname{Lat} T_\Phi$. For any $\varphi \in \Phi$ and $f \in \mathcal{M}$, there exist $g \in \mathcal{M}$, $h \in H^2 \ominus \mathcal{M}$ and $k \in K$ such that

$$\varphi f = g + h + \overline{k}.$$

From this equality, we have $P(\varphi f) = g + h$. Since $P(\varphi f)$ and $g$ are in $\mathcal{M}$, $h$ must be 0. Therefore we see that $\varphi f = g + \overline{k}$ and that $k \in K_{\mathcal{M}}^{\varphi}$ by the definition of $K_{\mathcal{M}}^{\varphi}$.
(2) Since $\mathcal{M}$ contains $\mathcal{M}^{\varphi}$, for any $f \in \mathcal{M}^{\varphi}$ there exist $g \in \mathcal{M}$ and $k \in K_{\mathcal{M}}^{\varphi}$ such that $\varphi f = g + \overline{k}$ by (1). Then we see

$$(I_{L^2} - P_{\mathcal{M}})\varphi f = (I_{L^2} - P_{\mathcal{M}})(g + \bar{k}) = \bar{k}.$$

Therefore we have $(I_{L^2} - P_{\mathcal{M}})\varphi\mathcal{M}^{\varphi} \subset \overline{K^{\varphi}_{\mathcal{M}}}$. On the other hand, for any $k \in K^{\varphi}_{\mathcal{M}}$ there exist $f$ and $g \in \mathcal{M}$ such that $\varphi f = g + \bar{k}$ by the definition of $K^{\varphi}_{\mathcal{M}}$. In particular, we can write $f = f_1 + f_2$, where $f_1 \in \mathcal{M}_{\varphi}$ and $f_2 \in \mathcal{M}^{\varphi}$. Since $\varphi f_1 \in \mathcal{M}$, we have

$$\begin{aligned}\bar{k} &= (I_{L^2} - P_{\mathcal{M}})\bar{k}\\ &= (I_{L^2} - P_{\mathcal{M}})(\varphi f - g)\\ &= (I_{L^2} - P_{\mathcal{M}})(\varphi f_1 + \varphi f_2 - g)\\ &= (I_{L^2} - P_{\mathcal{M}})\varphi f_2,\end{aligned}$$

and which implies $\overline{K^{\varphi}_{\mathcal{M}}} \subset (I_{L^2} - P_{\mathcal{M}})\varphi\mathcal{M}^{\varphi}$. Hence we have

$$(I_{L^2} - P_{\mathcal{M}})\varphi\mathcal{M}^{\varphi} = \overline{K^{\varphi}_{\mathcal{M}}}.$$

Thus we obtain (2). □

**4 Properties of** $\operatorname{Lat} T_{\Phi}$ In this section, we study properties of $\operatorname{Lat} T_{\Phi}$ for some $\Phi$ as the union of $H^{\infty}$ and some set. First we set $\Phi$ the union of $H^{\infty}$ and the complex conjugate of functions in $H^{\infty}$.

**Proposition 4.1.** *If* $\Phi = H^{\infty} \cup \overline{H^{\infty}}$, *then* $\operatorname{Lat} T_{\Phi} = \operatorname{Lat} T_{L^{\infty}}$.

*Proof.* It is obvious that $\operatorname{Lat} T_{L^{\infty}} \subset \operatorname{Lat} T_{\Phi}$. To prove the converse inclusion, suppose that $\mathcal{M} \in \operatorname{Lat} T_{\Phi}$. Then, since $T_{h_1\overline{h_2}} = T_{\overline{h_2}}T_{h_1}$ for any $h_1, h_2 \in H^{\infty}$, we see that $T_{h_1\overline{h_2}}\mathcal{M} \subset \mathcal{M}$. We note that $L^{\infty}$ is the algebra generated by $H^{\infty}$ and $\overline{H^{\infty}}$ in the $w^*$-topology. So for any $\varphi \in L^{\infty}$ we can choose a net $\{\varphi_{\alpha}\} \subset L^{\infty}$ converging in $w^*$-topology to $\varphi$, where each $\varphi_{\alpha}$ is a linear combination of products of functions in $H^{\infty}$ and $\overline{H^{\infty}}$ and satisfies $T_{\varphi_{\alpha}}\mathcal{M} \subset \mathcal{M}$. For any $f$ and $g \in H^2$ we have

$$\lim_{\alpha\in A}\langle T_{\varphi_{\alpha}}f, g\rangle = \lim_{\alpha\in A}\int_{\Gamma^2}\varphi_{\alpha}f\overline{g}d\mu = \int_{\Gamma^2}\varphi f\overline{g}d\mu = \langle T_{\varphi}f, g\rangle.$$

In particular, for any $f \in \mathcal{M}$ and $g \in H^2 \ominus \mathcal{M}$ we see that

$$\langle T_{\varphi}f, g\rangle = \lim_{\alpha\in A}\langle T_{\varphi_{\alpha}}f, g\rangle = 0.$$

Hence $T_{\varphi}f$ is in $\mathcal{M}$. Therefore we have $T_{\varphi}\mathcal{M} \subset \mathcal{M}$ and so we conclude that $\operatorname{Lat} T_{\Phi} \subset \operatorname{Lat} T_{L^{\infty}}$. □

**Proposition 4.2.** *Suppose that $F$ is a non-constant function in $H^{\infty} \cap q\overline{H^{\infty}}$ for some inner function $q$. Let $\Phi = H^{\infty} \cup \{\overline{F}\}$. If $\mathcal{M}$ is in* $\operatorname{Lat} T_{\Phi}$, *then $\mathcal{M}_{\Phi} = \mathcal{M}_{\overline{F}} \supseteq q\mathcal{M}$.*

*Proof.* If $F \in H^{\infty} \cap q\overline{H^{\infty}}$ then there exists $f \in H^{\infty}$ such that $F = q\overline{f}$. Hence $\overline{F}q\mathcal{M} = f\mathcal{M} \subset \mathcal{M}$, and trivially, $q\mathcal{M} \subset \mathcal{M}$. Therefore we have that $q\mathcal{M} \subset \mathcal{M}_{\overline{F}}$. □

Next, we consider examples when $\Phi$ consists of all functions in $H^{\infty}$ and the complex conjugate of an inner function.

**Theorem 4.3.** *Let $\Phi = H^{\infty} \cup \{\overline{q}\}$ for some non-constant inner function $q$. Suppose that $\mathcal{M} \in$* $\operatorname{Lat} T_{\Phi}$. *Then the following statements hold.*

1. *$\mathcal{M}_{\Phi} = q\mathcal{M}$ and $\mathcal{M}^{\Phi} = \mathcal{M} \ominus q\mathcal{M}$.*
2. *$\mathcal{M}_{\Phi} \subset (H^2)_{\Phi}$ and $\mathcal{M}^{\Phi} \subset (H^2)^{\Phi}$.*

(3) $\overline{K^{\Phi}_{\mathcal{M}}} = \overline{q}(\mathcal{M} \ominus q\mathcal{M})$.

*Proof.* (1) It is sufficient to prove $\mathcal{M}_{\overline{q}} = q\mathcal{M}$ since $\mathcal{M}_{\Phi} = \mathcal{M}_{\overline{q}}$. If $f \in \mathcal{M}_{\overline{q}}$, then $\overline{q}f \in \mathcal{M}$ from the definition of $\mathcal{M}_{\overline{q}}$. The assumption that $q$ is an inner function implies that $f \in q\mathcal{M}$, and hence we see that $\mathcal{M}_{\overline{q}} \subset q\mathcal{M}$. Conversely, if $f \in q\mathcal{M}$, then $f \in \mathcal{M}$ since $q\mathcal{M} \subset \mathcal{M}$. Moreover, that $q$ is inner implies that $\overline{q}f \in \mathcal{M}$. Therefore we see that $q\mathcal{M} \subset \mathcal{M}_{\overline{q}}$, which implies that the first statement. The second statement follows from the first statement.
(2) The first statement follows from the definition of $\mathcal{M}_{\Phi}$ and $(H^2)_{\Phi}$. To show the second statement, suppose that $f \in \mathcal{M}^{\Phi}$. By (1) we have $f \in \mathcal{M}$ and $f \perp q\mathcal{M}$. Moreover, since $\mathcal{M}$ is invariant under $T_{\overline{q}}$, we see that $T_q(H^2 \ominus \mathcal{M}) \subset H^2 \ominus \mathcal{M}$, that is, $q(H^2 \ominus \mathcal{M}) \subset H^2 \ominus \mathcal{M}$. This implies that $\mathcal{M} \perp q(H^2 \ominus \mathcal{M})$. For any $g \in H^2$, there exist $g_1 \in \mathcal{M}$ and $g_2 \in H^2 \ominus \mathcal{M}$ such that $g = g_1 + g_2$. Then we have

$$\begin{aligned} \langle f, qg \rangle &= \langle f, qg_1 + qg_2 \rangle \\ &= \langle f, qg_1 \rangle + \langle f, qg_2 \rangle \\ &= 0 \end{aligned}$$

since $f \perp q\mathcal{M}$ and $\mathcal{M} \perp q(H^2 \ominus \mathcal{M})$. Therefore we see that $f \perp qH^2$, that is, $f \in (H^2)^{\Phi}$. Hence the second statement holds.
(3) By (2) of Lemma 3.6, it is obvious that

$$\overline{q}(\mathcal{M} \ominus q\mathcal{M}) \supset (I_{L^2} - P_{\mathcal{M}})\overline{q}(\mathcal{M} \ominus q\mathcal{M}) = \overline{K^{q}_{\mathcal{M}}}.$$

Next, we will show the converse inclusion. For any $f \in \mathcal{M} \ominus q\mathcal{M}$, there exist $g \in \mathcal{M}$ and $k \in K^{q}_{\mathcal{M}}$ such that $\overline{q}f = g + \overline{k}$ by (1) of Lemma 3.6. Then we have

$$\begin{aligned} \|g\|^2 &= \langle g, g \rangle \\ &= \langle \overline{q}f - \overline{k}, g \rangle \\ &= \langle \overline{q}f, g \rangle - \langle \overline{k}, g \rangle \\ &= \langle f, qg \rangle - \langle \overline{k}, g \rangle \\ &= 0, \end{aligned}$$

since $f \perp q\mathcal{M}$ and $g \perp \overline{K^{q}_{\mathcal{M}}}$. So we see that $g = 0$, which implies that $\overline{q}f = \overline{k} \in \overline{K^{q}_{\mathcal{M}}}$. Therefore we have $\overline{q}(\mathcal{M} \ominus q\mathcal{M}) \subset \overline{K^{q}_{\mathcal{M}}}$. Hence we obtain

$$\overline{q}(\mathcal{M} \ominus q\mathcal{M}) = (I_{L^2} - P_{\mathcal{M}})\overline{q}(\mathcal{M} \ominus q\mathcal{M}) = \overline{K^{q}_{\mathcal{M}}}$$

Since $\overline{K^{\Phi}_{\mathcal{M}}} = \overline{K^{q}_{\mathcal{M}}}$, the statement holds. □

More generally, we are able to consider the case when $\Phi$ is the union of $H^{\infty}$ and a set of the complex conjugate of inner functions. In Corollary 4.4, we denote by $\Lambda$ a subset of $\mathbb{R}$.

**Corollary 4.4.** *Let* $\Phi = H^{\infty} \cup \{\overline{q_{\alpha}} : q_{\alpha} \text{ is inner}, \alpha \in \Lambda\}$. *Suppose that* $\mathcal{M} \in \operatorname{Lat} T_{\Phi}$. *Then the following statements hold.*

(1) $\mathcal{M}_{\Phi} = \bigcap_{\alpha \in \Lambda} q_{\alpha}\mathcal{M}$ *and* $\mathcal{M}^{\Phi} = \mathcal{M} \ominus \bigcap_{\alpha \in \Lambda} q_{\alpha}\mathcal{M}$.

(2) $\mathcal{M}_{\Phi} \subset (H^2)_{\Phi}$ *and* $\mathcal{M}^{\Phi} \subset (H^2)^{\Phi}$.

(3) $\overline{K^{\Phi}_{\mathcal{M}}} = \bigcup_{\alpha \in \Lambda} \overline{q_{\alpha}}(\mathcal{M} \ominus q_{\alpha}\mathcal{M})$.

*Proof.* (1) These statements follow from (1) of Theorem 4.3 and the definitions of $\mathcal{M}_\Phi$ and $\mathcal{M}^\Phi$.
(2) It is clear that $q_\alpha\mathcal{M} \subset q_\alpha H^2$ for all $\alpha \in \Lambda$. Hence we have

$$\mathcal{M}_\Phi = \bigcap_{\alpha\in\Lambda} q_\alpha\mathcal{M} \subset \bigcap_{\alpha\in\Lambda} q_\alpha H^2 = (H^2)_\Phi.$$

Moreover by (2) of Theorem 4.3, we see that if $f$ is in $\mathcal{M} \ominus q_\alpha\mathcal{M}$, then $f \perp q_\alpha H^2$ for all $\alpha \in \Lambda$. Therefore the second statement holds.
(3) The statement follows from (3) of Theorem 4.3 and the definition of $K^\Phi_\mathcal{M}$. □

We will use Proposition 4.5 to determine $\operatorname{Lat} T_\Phi$ in some concrete case.

**Proposition 4.5.** *Let $q$ be a non-constant inner function and $\psi = \frac{q-a}{1-\overline{a}q}$ for some $a \in \mathbb{C}$ with $|a| < 1$. If $\Phi = H^\infty \cup \{\overline{q}\}$ and $\Psi = H^\infty \cup \{\overline{\psi}\}$, then $\operatorname{Lat} T_\Phi = \operatorname{Lat} T_\Psi$.*

*Proof.* Suppose that $\mathcal{M} \in \operatorname{Lat} T_\Phi$. Since $\mathcal{M}$ is invariant under $T_{\overline{q}}$, we see that $T_q\mathcal{N} \subset \mathcal{N}$ where $\mathcal{N} = H^2 \ominus \mathcal{M}$. In particular, we have

$$q\mathcal{N} \subset \mathcal{N}.$$

Note that $\mathcal{N}$ is a closed subspace in $H^2$. We obtain

$$(q-a)\mathcal{N} \subset \mathcal{N} \quad \text{and} \quad (1-\overline{a}q)^{-1}\mathcal{N} \subset \mathcal{N}$$

for $|a| < 1$. Thus $T_\psi\mathcal{N} \subset \mathcal{N}$ and so $T_{\overline{\psi}}\mathcal{M} \subset \mathcal{M}$. This shows that $\operatorname{Lat} T_\Phi \subset \operatorname{Lat} T_\Psi$. Since $q = \frac{\psi+a}{1+\overline{a}\psi}$, we can prove the converse inclusion similarly. □

**5 Examples** In this section, we will describe $\operatorname{Lat} T_\Phi$ for some concrete $\Phi$. To begin with, in Corollary 5.3, we will show the case that $\operatorname{Lat} T_\Phi$ is trivial. To show this, we consider when $\Phi$ is the union of $H^\infty$ and $\{\overline{q}\}$ for a one variable inner function $q = q(z)$.

**Theorem 5.1.** *Let $\Phi = H^\infty \cup \{\overline{q(z)}\}$ for a one variable non-constant inner function $q = q(z)$. If $\mathcal{M} \in \operatorname{Lat} T_\Phi$, then there exists some one variable inner function $Q = Q(w)$ such that $\mathcal{M} = Q(w)H^2$.*

*Proof.* Since $q = q(z)$ is a one variable non-constant inner function, there exist some $a, b \in \mathbb{C}$ such that $q(b) = a$ and $|a| < 1, |b| < 1$. Put $\psi = \frac{q-a}{1-\overline{a}q}$. Since $\psi(b) = 0$, we write $\psi = q_0q_1$ where $q_0 = \frac{z-b}{1-\overline{b}z}$ and $q_1(z)$ is inner. If we put $\Psi = H^\infty \cup \{\overline{\psi}\}$, then $\operatorname{Lat} T_\Phi = \operatorname{Lat} T_\Psi$ by Proposition 4.5. This implies that $\mathcal{M}$ is invariant under $T_{\overline{\psi}} = T_{\overline{q_0q_1}}$. So we have that

$$T_{\overline{q_0}}\mathcal{M} = T_{\overline{q_0q_1}}q_1\mathcal{M} \subset T_{\overline{q_0q_1}}\mathcal{M} \subset \mathcal{M}.$$

Therefore we obtain $T_{\overline{q_0}}\mathcal{M} \subset \mathcal{M}$. So if we put $\Omega = H^\infty \cup \{\overline{q_0}\}$, then $\operatorname{Lat} T_\Psi \subset \operatorname{Lat} T_\Omega$. Moreover, by Proposition 4.5, we obtain $\operatorname{Lat} T_\Omega = \operatorname{Lat} T_{\Omega'}$, where $\Omega' = H^\infty \cup \{\overline{z}\}$. Hence we have $T_{\overline{z}}\mathcal{M} \subset \mathcal{M}$. By (2) of Theorem 4.3, we see that

$$\mathcal{M} \ominus z\mathcal{M} \subset H^2 \ominus zH^2 = H^2(\Gamma_w)$$

and so $w(\mathcal{M} \ominus z\mathcal{M}) \subset \mathcal{M} \ominus z\mathcal{M} \subset H^2(\Gamma_w)$. The Beurling theorem implies that $\mathcal{M} \ominus z\mathcal{M} = QH^2(\Gamma_w)$, where $Q = Q(w)$. Thus we have $\mathcal{M} = Q(w)H^2$. □

*Remark* 5.2. Let $\Phi = H^\infty \cup \{\overline{q(w)}\}$ for a one variable non-constant inner function $q = q(w)$. Making the same argument for Theorem 5.1, we can show that if $\mathcal{M} \in \operatorname{Lat} T_\Phi$, then there exists some one variable inner function $Q = Q(z)$ such that $\mathcal{M} = Q(z)H^2$.

**Corollary 5.3.** *If* $\Phi = H^\infty \cup \{\overline{q_1(z)q_2(w)}\}$ *for one variable non-constant inner functions* $q_1 = q_1(z)$ *and* $q_2 = q_2(w)$, *then* $\operatorname{Lat} T_\Phi = \{\langle 0\rangle, H^2\}$.

*Proof.* If $\mathcal{M} \in \operatorname{Lat} T_\Phi$, then we have that

$$T_{\overline{q_1}}\mathcal{M} = T_{\overline{q_1q_2}}(q_2\mathcal{M}) \subset T_{\overline{q_1q_2}}\mathcal{M} \subset \mathcal{M}.$$

Hence by Theorem 5.1, there exists some one variable inner function $Q_2 = Q_2(w)$ such that $\mathcal{M} = Q_2(w)H^2$. Similarly we have $T_{\overline{q}_2}\mathcal{M} \subset \mathcal{M}$ and so $\mathcal{M} = Q_1(z)H^2$ for some one variable inner function $Q_1 = Q_1(z)$. This happens only when $Q_1$ and $Q_2$ are constant. Therefore we obtain the corollary. □

Next, we will show the case that $\operatorname{Lat} T_\Phi$ is nontrivial. Now we study the case of $\Phi = H^\infty \cup \{\overline{q_1}q_2, q_1\overline{q_2}\}$ for some non-constant inner functions $q_1 = q_1(z)$ and $q_2 = q_2(w)$. We note that if $\mathcal{M} = \sum_{k=0}^n q_1^{n-k}q_2^kH^2$, then it is clear that $\mathcal{M}$ is in $\operatorname{Lat} T_\Phi$. Theorem 5.4 shows properties of $\operatorname{Lat} T_\Phi$.

**Theorem 5.4.** *Let* $\Phi = H^\infty \cup \{\overline{q_1}q_2, q_1\overline{q_2}\}$ *for some non-constant one variable inner functions* $q_1 = q_1(z)$ *and* $q_2 = q_2(w)$. *Suppose that* $\mathcal{M} \in \operatorname{Lat} T_\Phi$. *Then the following statements hold.*

(1) $q_1\mathcal{M} \subset q_2\mathcal{M} + H^2 \ominus q_2H^2$ *and* $q_2\mathcal{M} \subset q_1\mathcal{M} + H^2 \ominus q_1H^2$.

(2) *If there exists some natural number* $n$ *such that* $q_1^n \in \mathcal{M}$ *and* $q_1^{n-1} \notin \mathcal{M}$, *then we have* $q_1^lq_2^m \notin \mathcal{M}$ *for* $l \geq 0, m \geq 0$ *and* $l + m < n$.

(3) *If there exists some natural number* $n$ *such that* $q_1^n \in \mathcal{M}$, *then we have* $\mathcal{M} \supset \sum_{k=0}^n q_1^{n-k}q_2^kH^2$.

*Proof.* (1) By (1) of Lemma 3.6,

$$q_1\overline{q_2}\mathcal{M} \subset \mathcal{M} + \overline{K_{\mathcal{M}}^{\Phi}}.$$

Then we have

$$q_1\mathcal{M} \subset q_2\mathcal{M} + q_2\overline{K_{\mathcal{M}}^{\Phi}} \subset q_2\mathcal{M} + q_2\overline{K}$$

since $\overline{K_{\mathcal{M}}^{\Phi}}$ is a subset of $\overline{K}$. Hence $q_1\mathcal{M} \subset q_2\mathcal{M} + q_2\overline{K} \cap H^2$. Moreover from the definition of $\overline{K}$, it is clear that $q_2\overline{K} \cap H^2 \subset H^2 \ominus q_2H^2$. Therefore we obtain

$$q_1\mathcal{M} \subset q_2\mathcal{M} + H^2 \ominus q_2H^2.$$

The same argument shows that $q_2\mathcal{M} \subset q_1\mathcal{M} + H^2 \ominus q_1H^2$.
(2) If $q_1^lq_2^m$ were in $\mathcal{M}$, then we would have

$$T_{q_1}^{n-1-m-l}T_{q_1\overline{q_2}}^m(q_1^lq_2^m) = T_{q_1}^{n-1-m-l}(q_1^{m+l}) = q_1^{n-1} \in \mathcal{M}.$$

This contradicts that $q_1^{n-1} \notin \mathcal{M}$. Hence we conclude that $q_1^lq_2^m \notin \mathcal{M}$ for $l \geq 0, m \geq 0$ and $l + m < n$.
(3) Since $q_1^n$ is in $\mathcal{M}$, we have $T_{\overline{q_1}q_2}^j(q_1^n) = q_1^{n-j}q_2^j \in \mathcal{M}$ for $0 \leq j \leq n$. Let $\mathcal{P}_+$ be the set of analytic trigonometric polynomials. Then we see that $\sum_{j=0}^n q_1^{n-j}q_2^j\mathcal{P}_+ \subset \mathcal{M}$. Since $H^2$ is the closure in the $L^2$-norm of $\mathcal{P}_+$ and the multiplication by an inner function is continuous, we have

$$\sum_{j=0}^n q_1^{n-j}q_2^jH^2 \subset \mathcal{M}.$$

□

In [3], the first author studied $\operatorname{Lat} T_\Psi$ for $\Psi = \{z^n\overline{w}, \overline{z}^n w\}$ for a fixed natural number $n$. In this context, we consider the case when $\Phi = H^\infty \cup \{\overline{z}w, z\overline{w}\}$. In Theorem 5.5, we describe $\operatorname{Lat} T_\Phi$ completely and show that $\operatorname{Lat} T_\Phi$ is nontrivial. Moreover we provide a concrete example of invariant subspaces of $T_z$ and $T_w$. We recall that $H^2(\Gamma_z)$ or $H^2(\Gamma_w)$ denotes a one variable Hardy space on the unit circle $\Gamma = \Gamma_z$ or $\Gamma_w$ respectively.

**Theorem 5.5.** *Let $\Phi = H^\infty \cup \{\overline{z}w, z\overline{w}\}$. Then the following statements hold.*

(1) *If $\mathcal{M} \in \operatorname{Lat} T_\Phi$, then*

$$z\mathcal{M} \subset w\mathcal{M} + H^2(\Gamma_z) \quad \text{and} \quad w\mathcal{M} \subset z\mathcal{M} + H^2(\Gamma_w).$$

(2) *A closed subspace $\mathcal{M}$ is in $\operatorname{Lat} T_\Phi$ if and only if there exists the smallest natural number $N$ such that $z^N$ and $w^N$ belong to $\mathcal{M}$ and $\mathcal{M} = \sum_{j=0}^N z^{N-j}w^jH^2$.*

*Proof.* (1) We note that equalities

$$H^2 \ominus zH^2 = H^2(\Gamma_w) \quad \text{and} \quad H^2 \ominus wH^2 = H^2(\Gamma_z)$$

hold. Applying (1) of Theorem 5.4, we obtain the conclusion.
(2) The ‘if’ part is not hard to prove. Now we show the ‘only if’ part. Assume that $\mathcal{M} \in \operatorname{Lat} T_\Phi$. It is clear that there exists the smallest natural number $N$ satisfying the following condition; there exists $f \in \mathcal{M}$ such that $\frac{\partial^N}{\partial z^N} f(0,0) \neq 0$ but $\frac{\partial^k}{\partial z^k} g(0,0) = 0$ for all $g \in \mathcal{M}$ if $k < N$. In order to show that $z^N \in \mathcal{M}$, we consider the extremal problem

$$\sup\{\operatorname{Re}\frac{\partial^N}{\partial z^N} f(0,0); f \in \mathcal{M}, \|f\| \leq 1\}.$$

Note that the mapping $f \mapsto \frac{\partial^N}{\partial z^N} f(0,0)$ is a bounded linear functional on $H^2$. By the Riesz representation theorem, this extremal problem has a unique solution $G \in \mathcal{M}$ with $\|G\| = 1$ and $\frac{\partial^N}{\partial z^N} G(0,0) > 0$. We will see that $G = z^N$. Put

$$g_f = \frac{G + T_{z\overline{w}}^{N+1} f}{\|G + T_{z\overline{w}}^{N+1} f\|}$$

for each $f \in \mathcal{M}$. Since $\operatorname{Re}\frac{\partial^N}{\partial z^N} g_f(0,0) \leq \frac{\partial^N}{\partial z^N} G(0,0)$, it is easy to see that $\|G + T_{z\overline{w}}^{N+1} f\| \geq 1$ for any $f \in \mathcal{M}$. From this inequality, we obtain $G \perp T_{z\overline{w}}^{N+1} f$. Hence we have $T_{\overline{z}w}^{N+1} G = 0$. Similarly we have $T_{z\overline{w}} G = 0$. From these equalities, we obtain $G = z^N$. It is obvious that $w^N = T_{\overline{z}w}^N z^N$ is in $\mathcal{M}$.

By (3) of Theorem 5.4, we obtain $\mathcal{M} \supset \sum_{j=0}^N z^{N-j}w^jH^2$. Moreover, by (2) of Theorem 5.4, we see that $z^{k_1}w^{k_2} \notin \mathcal{M}$ for $0 \leq k_1 + k_2 < N$, which shows the converse inclusion. $\square$

Corollary 5.6 shows that each $\mathcal{M}$ in $\operatorname{Lat} T_\Phi$ contains an invariant subspace $z^NH^2 + w^NH^2$ for some natural number $N$.

**Corollary 5.6.** *Let $\Phi = H^\infty \cup \{\overline{z}w, z\overline{w}\}$. If $\mathcal{M} \in \operatorname{Lat} T_\Phi$, then there exists some natural number $N$ such that*

$$\mathcal{M} \supset z^NH^2 + w^NH^2.$$

*Proof.* By (2) of Theorem 5.5, there exists some natural number $N$ such that

$$\mathcal{M} = \sum_{j=0}^N z^j w^{N-j} H^2.$$

Then we obtain

$$z^N H^2 + w^N H^2 \subset \sum_{j=0}^{N} z^j w^{N-j} H^2 = \mathcal{M}.$$

Hence the statement is clear. □

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As students of Nakazi, we were attracted by his mathematics, and remember that he always started on mathematics with his unique observation about elementary examples. We would like to express our affection and respect for his life devoted to mathematics.

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# PATTERN FORMATION FOR SELF-REGULATING HOMEOSTASIS MODEL IN A RECTANGLE

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Abstract. We continue the study on two-dimensional self-regulating homeostasis models. In the previous paper [4], after introducing a homeostasis model on a sphere, we showed global existence of solutions and constructed exponential attractors for the dynamical system generated by the model. We furthermore showed by numerical computations that white daisy and black daisy perform very clear segregation patterns on the sphere.

This paper is then devoted to investigating more on this pattern formation in a rectangular domain. We show that the competition of white and black daisies and the interaction with temperature create several types of segregation patterns and bring homeostasis of the global temperature to the planet.

**1 Introduction** We continue the study on two-dimensional self-regulating homeostasis models. In the previous paper [4], after introducing a homeostasis model on a sphere on the basis of the classical work Watson-Lovelock [6], we showed global existence of solutions and constructed exponential attractors for the dynamical system generated by the model. We furthermore showed by numerical computations that white daisy and black daisy perform very clear segregation patterns on the sphere. This paper is then devoted to investigating more on this pattern formation.

We consider the following reaction diffusion system

$$\begin{cases} \dfrac{\partial u}{\partial t} = d\Delta u + [(1-u-v)\Phi(u,v,w) - f]\,u & \text{in} \quad \Omega\times(0,\infty), \\ \dfrac{\partial v}{\partial t} = d\Delta v + [(1-u-v)\Psi(u,v,w) - f]\,v & \text{in} \quad \Omega\times(0,\infty), \\ \dfrac{\partial w}{\partial t} = D\Delta w + [1-g(u,v)]R - \sigma w^4 & \text{in} \quad \Omega\times(0,\infty), \end{cases} \tag{1.1}$$

in a rectangular domain $\Omega = (-\ell_x, \ell_x)\times(0,\ell_y)$, where $0 < \ell_x,\, \ell_y < \infty$. As in [4], the variables $u = u(x,y,t)$ and $v = v(x,y,t)$ denote the coverage rate of white and black daisy, respectively, at position $(x,y)\in\Omega$ and time $t$. Therefore, $u \geq 0$, $v \geq 0$ and $u+v \leq 1$ at any $(x,y,t)$, and $1-u-v$ denotes a rate of uncovered ground. The third state variable $w = w(x,y,t)$ denotes a surface temperature. We assume that $u$ and $v$ satisfy a diffusion equation on $\Omega$ with diffusion rate $d > 0$. It is the same for $w$ with diffusion rate $D > 0$. The function $g(u,v)$ stands for an averaged albedo of the surface that is given at each point as a function of $u$, $v$ in the form

$$g(u,v) = a_w u + a_b v + a_g(1-u-v) = (a_w - a_g)u + (a_b - a_g)v + a_g, \tag{1.2}$$

where $a_w$, $a_b$ and $a_g$ denote the proper albedo of white daisy, black daisy and bare ground, respectively. In general, we have $0 < a_b < a_g < a_w < 1$; as a consequence, it is always the case that

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$a_b \le g(u,v) \le a_w$. Furthermore, $\Phi(u,v,w)$ and $\Psi(u,v,w)$ denote a growth rate of white and black daisy, respectively. According as [6], we set

$$\Phi(u,v,w) = \{1-\delta(\overline{w}-w-q[g(u,v)-a_w])^2\}_+,$$
$$\Psi(u,v,w) = \{1-\delta(\overline{w}-w-q[g(u,v)-a_b])^2\}_+.$$

Here, $\overline{w}$ is a fixed optimal temperature for growing for both white daisy and black daisy. The term $q[g(u,v)-a_w]$ (resp. $q[g(u,v)-a_b]$) means some suitable adjustment on a local temperature to the global one $w$ at any position where white daisy (resp. black daisy) grows, $q>0$ being some coefficient. Since $g(u,v) \le a_w$ (resp. $g(u,v) \ge a_b$), we see that $w$ is always adjusted negatively (resp. positively) where white daisy (resp. black daisy) grows. The notation $\{w\}_+ = \max\{w,0\}$ denotes a positive cutoff of the function $w$ for $-\infty < w < \infty$; consequently, $\{1-\delta(\overline{w}-w)^2\}_+$ is a positive cutoff of the square function $1-\delta(\overline{w}-w)^2$ for $-\infty < w < \infty$, $\delta > 0$ being some coefficient. Both white daisy and black daisy die at a rate $f>0$. Finally, the term $[1-g(u,v)]R$ denotes an increasing rate of the global temperature which is determined by the averaged albedo $g(u,v)$ mentioned above and the incoming energy $R$ from the sun which is assumed to be constant in $\Omega$. And, the term $-\sigma w^4$ denotes a decaying rate of the temperature due to the Stefan-Boltzmann law, $\sigma > 0$ being the Stefan-Boltzmann constant of the surface.

We impose, as boundary conditions, the periodic conditions in $x$-variable and the homogeneous Neumann conditions in $y$-variable for all of $u$, $v$ and $w$. That is,

$$\begin{cases} \zeta(-\ell_x,y,t) = \zeta(\ell_x,y,t) \quad \text{and} \quad \zeta_x(-\ell_x,y,t) = \zeta_x(\ell_x,y,t) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \{-\ell_x,\ell_x\} \times (0,\ell_y) \times (0,\infty), \\ \zeta_y(x,0,t) = \zeta_y(x,\ell_y,t) = 0, \qquad \text{on } (-\ell_x,\ell_x) \times \{0,\ell_y\} \times (0,\infty), \end{cases} \tag{1.3}$$

where $\zeta$ stands for $u$, $v$ and $w$. Finally, the initial conditions are set as

$$u(x,y,0) = u_0(x,y),\ \ v(x,y,0) = v_0(x,y) \ \text{ and } \ w(x,y,0) = w_0(x,y) \quad \text{in} \quad \Omega. \tag{1.4}$$

The main interest of the present paper is as mentioned above to investigate when homogeneous distribution of white and black daisies becomes unstable and how segregation patterns are created by the competition of two daisies and the interaction with global temperature. For this purpose we want to consider the case where (1.1) has a stationary solution which is homogeneous in the spatial variables $(x,y)$. This is reason why we assume that the incoming energy $R$ is constant with respect to the variables $(x,y)$. (In [4], $R$ depends on the latitude.) In addition, for simplicity, we want to consider (1.1) on the cylindrical surface instead of on the sphere. This is reason why we handle (1.1) in $(-\ell_x,\ell_x)\times(0,\ell_y)$ under the periodic-Neumann boundary conditions (1.3) on $u$, $v$ and $w$. If $R$ is constant, then similar results will be obtained for the problem (1.1) and (1.4) on the sphere.

Global solutions are constructed as in [4], although we have to prepare and use the Proposition 2.1 which may not be so standard. Construction of the dynamical system and its exponential attractors can be carried out in a quite analogous way as in [4]. In order to investigate stability and instability of the homogeneous stationary solutions, we will restrict our interest only to a typical case where the parameters in (1.1) are fixed as

$$a_b = \frac{1}{4},\ a_g = \frac{1}{2},\ a_w = \frac{3}{4},\ q = 20,\ \delta = 3.265 \times 10^{-3},$$
$$f = 0.3,\ \overline{w} = 295.5 \ \text{ and } \ \sigma = 5.67 \times 10^{-8},$$

except $R$ that is treated as a tuning parameter. Such a setting is suggested by [6]. Then, it is proved that there is an interval $(R_*, R^*)$ for $R$ such that if $R \notin [R_*, R^*]$ there is no positive homogeneous stationary solution, meanwhile if $R \in (R_*, R^*)$ there is a unique one $U_* = {}^t(u_*, v_*, w_*)$. Furthermore, for

$$\varphi(u, v, w) = [(1 - u - v)\Phi(u, v, w) - f]\, u,$$
$$\psi(u, v, w) = [(1 - u - v)\Psi(u, v, w) - f]\, v,$$

it is proved that, if $(u_*, v_*, w_*)$ satisfies

$$\varphi_u(u_*, v_*, w_*)\psi_v(u_*, v_*, w_*) \geq \varphi_v(u_*, v_*, w_*)\psi_u(u_*, v_*, w_*),$$

then $U_*$ is stable, meanwhile if $(u_*, v_*, w_*)$ satisfies

$$\varphi_u(u_*, v_*, w_*)\psi_v(u_*, v_*, w_*) < \varphi_v(u_*, v_*, w_*)\psi_u(u_*, v_*, w_*),$$

and if the diffusion coefficient $D$ is sufficiently large with respect to the other $d$, then $U_*$ becomes unstable. Roughly speaking, if the intra-species competition is stronger than the inter-species one at $U_*$, then $U_*$ is stable. Meanwhile, if the intra-species competition is weaker than the inter-species one at $U_*$ and if global temperature diffuses much faster than daisies, $U_*$ loses its stability, that is, the diffusion driven instability takes place.

As the dynamical system possesses a finite-dimensional attractor, when $U_*$ is unstable, the trajectories are attracted to some states of a finite number of freedoms which does not include the homogeneous state. This fact then suggests that some pattern might be created spontaneously by the white and black daisies. As a matter of fact, we find by numerical computations under suitably fixed diffusion coefficients $d$ and $D$ that some segregation patterns emerge and they change their types from homogeneous, spot, island and to labyrinth as $R$ changes. On the other hand, the mean of the global temperature, i.e.,

$$W(\infty) = \frac{1}{|\Omega|} \iint_\Omega w(x, y, \infty)\, dx dy,$$

is observed to be stable during $R$ changes in this range. In this way, the competition between two daisies and the interaction with global temperature create several types of segregation patterns of daisies, and simultaneously they bring the homeostasis of global temperature to the planet.

The mechanism of self-regulating homeostasis has already been studied by using zero and one-dimensional Daisyworld models. For a survey, we refer the reader to [4, Introduction].

## 2 Local Solutions

**2.1 Laplacian under periodic-Neumann boundary conditions** In order to formulate (1.1)-(1.3) in the space $L_2(\Omega)$, we have to define $\varDelta$ as a linear operator of $L_2(\Omega)$ under the boundary conditions stated in (1.3).

For this purpose, we consider the sesquilinear form

$$a(u, v) = a \int_\Omega \nabla u \cdot \nabla \overline{v}\, dx + c \int_\Omega u\overline{v}\, dx, \qquad u,\, v \in V, \tag{2.1}$$

where $a$ and $c$ are positive constants, on the space

$$H^1_{\mathrm{per}}(\Omega) = \{u \in H^1(\Omega);\ u(-\ell_x, y) = u(\ell_x, y) \text{ in the interval } (0, \ell_y)\}. \tag{2.2}$$

As $u \in H^1(\Omega)$ implies $u_{|\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega) \subset L_2(\partial\Omega)$, the coincidence $u(-\ell_x, y) = u(\ell_x, y)$ is meaningful as a function of $L_2(0, \ell_y)$. Thereby, $H^1_{\mathrm{per}}(\Omega)$ is a closed subspace of $H^1(\Omega)$ and becomes a Hilbert space with the $H^1$-inner product. Of course, $H^1_{\mathrm{per}}(\Omega)$ is dense in $L_2(\Omega)$. Therefore,

$$H^1_{\mathrm{per}}(\Omega) \subset L_2(\Omega) \subset H^1_{\mathrm{per}}(\Omega)^*$$

defines a triplet of spaces. In the meantime, $a(u, v)$ given by (2.1) is continuous and coercive on $H^1_{\mathrm{per}}(\Omega)$. By the theory of variation (see Dautray-Lions [2]), $a(u, v)$ then determines a linear operator $\mathcal{A}$ by the formula $a(u, v) = \langle \mathcal{A}u, v\rangle_{H^1_{\mathrm{per}}{}^* \times H^1_{\mathrm{per}}}$ for all $u$, $v \in H^1_{\mathrm{per}}(\Omega)$. The operator $\mathcal{A}$ is seen to be a sectorial operator of $H^1_{\mathrm{per}}(\Omega)^*$ with the domain $\mathcal{D}(\mathcal{A}) = H^1_{\mathrm{per}}(\Omega)$ and is therefore regarded as a realization of $-a\Delta + c$ in the space $H^1_{\mathrm{per}}(\Omega)^*$.

The part of $\mathcal{A}$ in the space $L_2(\Omega)$ is defined by

$$\begin{cases} \mathcal{D}(A) = \{u \in H^1_{\mathrm{per}}(\Omega);\ \mathcal{A}u \in L_2(\Omega)\}, \\ Au = \mathcal{A}u. \end{cases}$$

In other words, $u \in \mathcal{D}(A)$ if and only if $a(u, v) = (f, v)$ for all $v \in H^1_{\mathrm{per}}(\Omega)$ with some $f \in L_2(\Omega)$. By the theory of variation, again, $A$ is a densely defined linear operator of $L_2(\Omega)$. As $a(u, v)$ is symmetric, $A$ is a positive definite self-adjoint operator of $L_2(\Omega)$. In the present case, we can characterize the domain $\mathcal{D}(A)$ as follows.

**Proposition 2.1.** *The domain $\mathcal{D}(A)$ is given by*

$$\mathcal{D}(A) = \{u \in H^2(\Omega);\ u \textit{ satisfies the conditions on } \partial\Omega \textit{ stated in } (1.3)\}. \tag{2.3}$$

*Moreover, it holds true that*

$$\|u\|_{H^2} \le C\|Au\|_{L_2}, \qquad u \in \mathcal{D}(A). \tag{2.4}$$

*Proof.* Let $u \in H^2(\Omega)$ satisfy (1.3) and let $v \in H^1_{\mathrm{per}}(\Omega)$ be any function. By integration by parts,

$$\iint_\Omega u_x \overline{v}_x\, dxdy = \int_0^{\ell_y} dy \int_{-\ell_x}^{\ell_x} u_x \overline{v}_x\, dx = \int_0^{\ell_y} dy \left\{ [u_x\overline{v}]_{x=-\ell_x}^{x=\ell_x} - \int_{-\ell_x}^{\ell_x} u_{xx}\overline{v}\, dx \right\}.$$

Here, the periodic conditions on $u$ yield that

$$[u_x\overline{v}]_{x=-\ell_x}^{x=\ell_x} = u_x(\ell_x, y)\overline{v}(\ell_x, y) - u_x(-\ell_x, y)\overline{v}(-\ell_x, y) = 0 \quad \text{for a.e. } y \in (0, \ell_y).$$

Therefore, $\iint_\Omega u_x\overline{v}_x\, dxdy = -\iint_\Omega u_{xx}\overline{v}\, dxdy$. By the similar arguments, we have

$$\iint_\Omega u_y\overline{v}_y\, dxdy = \int_{-\ell_x}^{\ell_x} dx \left\{ [u_y\overline{v}]_{y=0}^{y=\ell_y} - \int_0^{\ell_y} u_{yy}\overline{v}\, dy \right\} = -\iint_\Omega u_{yy}\overline{v}\, dxdy.$$

In this way, we observe that $(\nabla u, \nabla v) = (-\Delta u, v)$. In view of (2.1), this in fact shows that $a(u, v) = (-a\Delta u + cu, v)$, hence $u \in \mathcal{D}(A)$ and $Au = -a\Delta u + cu$.

In order to prove that $u \in \mathcal{D}(A)$ implies $u \in H^2(\Omega)$, we will use a double Fourier expansion for the functions of $L_2(\Omega)$. For the variable $x \in (-\ell_x, \ell_x)$, we use an expansion by the base functions

$\cos\frac{m\pi}{\ell_x}x$ and $\sin\frac{m\pi}{\ell_x}x$ for $m = 0, 1, 2, \ldots$; for the variable $y \in (0, \ell_y)$, an expansion by the base functions $\cos\frac{n\pi}{\ell_y}y$ for $n = 0, 1, 2, \ldots$. Then, $u$ can be expressed by the series

$$u = \sum_{m,n=0}^{\infty} \left[u_{mn} \cos\frac{m\pi}{\ell_x}x + v_{mn} \sin\frac{m\pi}{\ell_x}x\right] \cos\frac{n\pi}{\ell_y}y$$

with Fourier coefficients $u_{mn}$ and $v_{mn}$ determined by the base functions. And they satisfy $\sum_{m,n}|u_{mn}|^2 < \infty$ and $\sum_{m,n}|v_{mn}|^2 < \infty$. In the distribution sense, we observe that

$$-\varDelta u = \sum_{m,n=0}^{\infty} \left[\left(\frac{m\pi}{\ell_x}\right)^2 + \left(\frac{n\pi}{\ell_y}\right)^2\right] \left[u_{mn} \cos\frac{m\pi}{\ell_x}x + v_{mn} \sin\frac{m\pi}{\ell_x}x\right] \cos\frac{n\pi}{\ell_y}y.$$

If $u \in \mathcal{D}(A)$, then, since $\mathcal{C}_0^\infty(\Omega) \subset H^1_{\mathrm{per}}(\Omega)$, the condition $\mathcal{A}u \in L_2(\Omega)$ implies that $-\varDelta u = f \in L_2(\Omega)$. So, if $f_{mn}$ and $g_{mn}$ are the Fourier coefficients of $f$, then it follows that

$$u_{mn} = \left[\left(\frac{m\pi}{\ell_x}\right)^2 + \left(\frac{n\pi}{\ell_y}\right)^2\right]^{-1} f_{mn} \quad \text{and} \quad v_{mn} = \left[\left(\frac{m\pi}{\ell_x}\right)^2 + \left(\frac{n\pi}{\ell_y}\right)^2\right]^{-1} g_{mn}$$

for every $(m, n) \neq (0, 0)$. Furthermore, it follows that

$$\|u_{xx}\|^2_{L_2} + \|u_{xy}\|^2_{L_2} + \|u_{yy}\|^2_{L_2} \le C\left(\sum_{m,n}|f_{mn}|^2 + \sum_{m,n}|g_{mn}|^2\right) \le C\|f\|^2_{L_2}. \tag{2.5}$$

Hence, $u$ belongs to $H^2(\Omega)$.

Knowing that $u \in \mathcal{D}(A)$ implies $u \in H^2(\Omega)$, we can repeat the arguments above to conclude that the two integrals

$$\int_0^{\ell_y} [u_x\overline{v}]_{x=-\ell_x}^{x=\ell_x}\, dy = \int_0^{\ell_y} [u_x(\ell_x, y)\overline{v}(\ell_x, y) - u_x(-\ell_x, y)\overline{v}(-\ell_x, y)]dy,$$

$$\int_{-\ell_x}^{\ell_x} [u_y\overline{v}]_{y=0}^{y=\ell_y}\, dx = \int_{-\ell_x}^{\ell_x} [u_y(x, \ell_y)\overline{v}(x, \ell_y) - u_y(x, 0)\overline{v}(x, 0)]dx$$

must vanish for all $v \in H^1_{\mathrm{per}}(\Omega)$. Remembering the definition (2.2), we verify that $u_x(\ell_x, y) - u_x(-\ell_x, y) = 0$ for a.e. $y \in (0, \ell_y)$ and $u_y(x, \ell_y) = u_y(x, 0) = 0$ for a.e. $x \in (-\ell_x, \ell_x)$, that is, $u$ satisfies the boundary conditions of (1.3).

Finally, since $\|u\|_{H^1} \le C\|Au\|_{L_2}$ is already known, (2.4) is immediately verified from (2.5). □

We have thus shown that $A$ is a realization of $-a\varDelta + c$ in $L_2(\Omega)$ under the periodic-Neumann boundary conditions stated in (1.3).

**2.2 Abstract formulation** Let us formulate the problems (1.1)-(1.4) as the Cauchy problem for an abstract evolution equation

$$\begin{cases} \dfrac{dU}{dt} + AU = F(U), & 0 < t < \infty, \\ U(0) = U_0, \end{cases} \tag{2.6}$$

in a Banach space $X$. As $X$ we set the product $L_2$-space, i.e.,

$$X = \left\{ U = \begin{pmatrix} u \\ v \\ w \end{pmatrix} ;\ u \in L_2(\Omega),\ v \in L_2(\Omega),\ w \in L_2(\Omega) \right\}.$$

The operator $A$ denotes an operator matrix acting in $X$ given by $\operatorname{diag}\{A_d, A_d, A_D\}$, where $A_d$ (resp. $A_D$) is the realization of $-d\Delta + f$ (resp. $-D\Delta + 1$) in $L_2(\Omega)$ under the boundary conditions stated in (1.3). Then, $A$ is a positive definite self-adjoint operator of $X$. Of course the domain $\mathcal{D}(A)$ is characterized by (2.3).

The nonlinear operator $F(U)$ is defined from the reaction terms including in (1.1). However, in view of our modeling, we expect that the solutions must exist in the ranges of $u \geq 0$, $v \geq 0$, $u+v \leq 1$ and $0 \leq w \leq (R/\sigma)^{\frac{1}{4}}$. On account of this expectation on the ranges, we will define $F(U)$ as follows:

$$F(U) = \begin{pmatrix} H_1(1 - \operatorname{Re} u - \operatorname{Re} v)\, \Phi\big(H_1(\operatorname{Re} u), H_1(\operatorname{Re} v), H_2(\operatorname{Re} w)\big)\, H_1(\operatorname{Re} u) \\ H_1(1 - \operatorname{Re} u - \operatorname{Re} v)\, \Psi\big(H_1(\operatorname{Re} u), H_1(\operatorname{Re} v), H_2(\operatorname{Re} w)\big)\, H_1(\operatorname{Re} v) \\ [1 - g\big(H_1(\operatorname{Re} u), H_1(\operatorname{Re} v)\big)]R - \sigma H_2(\operatorname{Re} w)^4 + H_2(\operatorname{Re} w) \end{pmatrix}.$$

Here, $H_1(\xi)$ and $H_2(\xi)$ are cutoff functions defined by

$$H_1(\xi) = \begin{cases} 0, & -\infty < \xi \leq 0, \\ \xi, & 0 < \xi \leq 1, \\ 1, & 1 < \xi < \infty, \end{cases} \qquad H_2(\xi) = \begin{cases} 0, & -\infty < \xi \leq 0, \\ \xi & 0 < \xi \leq (R/\sigma)^{\frac{1}{4}}, \\ (R/\sigma)^{\frac{1}{4}}, & (R/\sigma)^{\frac{1}{4}} < \xi < \infty, \end{cases}$$

respectively.

Finally, the initial value $U_0$ is taken from the space

$$K = \left\{ U_0 = \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix} \in X;\ u_0 \geq 0,\ v_0 \geq 0,\ u_0 + v_0 \leq 1, 0 \leq w_0 \leq \left(\frac{R}{\sigma}\right)^{\frac{1}{4}} \right\}, \tag{2.7}$$

$K$ being thus the space of initial values.

**2.3 Construction of local solutions** Construction of the local solution to (2.6) is easily carried out if we employ the theory of abstract parabolic evolution equations.

In fact, it is clear that $H_1(\xi)$ and $H_2(\xi)$ are uniformly bounded and globally Lipschitz continuous functions for $-\infty < \xi < \infty$. Consequently, $\Phi(H_1(\operatorname{Re} u), H_1(\operatorname{Re} v), H_2(\operatorname{Re} w))$ and $\Psi(H_1(\operatorname{Re} u), H_1(\operatorname{Re} v), H_2(\operatorname{Re} w))$ are uniformly bounded and globally Lipschitz continuous functions for $(u, v, w) \in \mathbb{C}^3$. Therefore, it is easily verified that $F(U)$ is a bounded operator on $X$ and satisfies the Lipschitz condition, i.e.,

$$\|F(U)\|_X \leq C_1, \qquad U \in X,$$
$$\|F(U) - F(V)\|_X \leq C_2 \|U - V\|_X, \qquad U, V \in X,$$

with suitable constants $C_i > 0\,(i = 1, 2)$.

It is then possible to apply [7, Theorem 4.4] to obtain that for any $U_0 \in X$, there exists a unique local solution to (2.6) in the function space:

$$U \in \mathcal{C}([0, T_{U_0}]; X) \cap \mathcal{C}^1((0, T_{U_0}]; X) \cap \mathcal{C}((0, T_{U_0}]; \mathcal{D}(A)).$$

In addition, the solution $U(t)$ satisfies the norm estimate

$$\|U(t)\|_X + t\|AU(t)\|_X \le C_{U_0}, \qquad 0 < t \le T_{U_0}. \tag{2.8}$$

Here, the constant $C_{U_0}$ and the time $T_{U_0} > 0$ are determined by the norm $\|U_0\|_X$ alone.

Let us next prove that, if $U_0$ is in $K$, then the local solution $U(t)$ also takes values in $K$ for every $0 < t \le T_{U_0}$.

**Proposition 2.2.** *If* $U_0 \in K$*, then* $U(t) \in K$ *for every* $0 < t \le T_{U_0}$.

*Proof.* It is easy to verify that the complex conjugate $\overline{U(t)}$ of $U(t)$ is also a local solution to (2.6). Uniqueness of solution yields that $U(t) = \overline{U(t)}$ for every $0 < t \le T_{U_0}$, hence $U(t)$ is real valued.

First, let us see that $u(t) \ge 0$. For this purpose, we use the cutoff function given by $H(u) = \frac{1}{2}u^2$ for $-\infty < u < 0$ and $H(u) = 0$ for $0 \le u < \infty$. Put $g(t) = \iint_\Omega H(u(x,y,t))dxdy$. Then, for $0 < t \le T_{U_0}$,

$$\begin{aligned}\frac{dg}{dt}(t) = \iint_\Omega H'(u)\frac{\partial u}{\partial t}\,dxdy = d\iint_\Omega H'(u)\Delta u\,dxdy\\ + \iint_\Omega H'(u)\left[H_1(1-u-v)\Phi\big(H_1(u),H_1(v),H_2(w)\big)H_1(u) - fu\right]dxdy.\end{aligned}$$

Here, on account of $H'(u) \in H^1_{\mathrm{per}}(\Omega)$, we observe that

$$\iint_\Omega H'(u)\Delta u\,dxdy = -\iint_\Omega \nabla H'(u)\cdot\nabla u\,dxdy = -\iint_\Omega H''(u)|\nabla u|^2 dxdy \le 0.$$

Meanwhile, since $H'(u)H_1(u) = 0$ and $-H'(u)u \le 0$ for all $-\infty < u < \infty$, it follows that $\frac{dg}{dt}(t) \le 0$, i.e., $g(t) \le g(0) = 0$. This means that $u(t) \ge 0$ for every $0 < t \le T_{U_0}$

The same arguments for $v(t)$ conclude that $v(t) \ge 0$ for every $0 < t \le T_{U_0}$.

Second, in order to see that $u(t) + v(t) \le 1$, we notice that $z(t) = 1 - u(t) - v(t)$ is regarded as a solution to

$$\frac{\partial z}{\partial t} = d\Delta z - \left[\Phi\big(H_1(u),H_1(v),H_2(w)\big) + \Psi\big(H_1(u),H_1(v),H_2(w)\big)\right]H_1(z) + f[u+v].$$

We can then repeat the same arguments for $z(t)$ to conclude that $z(t) \ge 0$, i.e., $u(t) + v(t) \le 1$ for every $0 < t \le T_{U_0}$.

Third, let us observe that $0 \le w(t) \le (R/\sigma)^{\frac14}$. But observation of the non negativity $w(t) \ge 0$ is the same as for $u(t)$ and $v(t)$. Putting $w_1(t) = (R/\sigma)^{\frac14} - w(t)$, we notice that $w_1(t)$ is a solution to

$$\frac{\partial w_1}{\partial t} = D\Delta w_1 - \sigma[R/\sigma - H_2(w)^4] + Rg(u,v) + [w - H_2(w)].$$

Then, consider the function $h(t) = \iint_\Omega H(w_1(x,y,t))dxdy$ and differentiate it. Since $H'((R/\sigma)^{\frac14} - w)[R/\sigma - H_2(w)^4] = 0$ and $H'((R/\sigma)^{\frac14} - w)[w - H_2(w)] \le 0$ for all $-\infty < w < \infty$, it follows that $\frac{dh}{dt}(t) \le 0$, i.e., $h(t) \le h(0) = 0$. Hence, $(R/\sigma)^{\frac14} - w(t) \ge 0$ for every $0 < t \le T_{U_0}$.

We have thus verified all the conditions in order that $U(t)$ lies in $K$. □

Once $U(t) \in K$, $U(t)$ actually satisfies that $H_1(u(t)) = u(t)$, $H_1(v(t)) = v(t)$, $H_1(1 - u(t) - v(t)) = 1 - u(t) - v(t)$ and $H_2(w(t)) = w(t)$. This means that the local solution $U(t)$ to (2.6) constructed above can be regard as a local solution to the original problem (1.1), (1.3) and (1.4), too.

**3 Global Solutions and Dynamical System** This section is devoted to constructing global solutions, a dynamical system generated by (2.6) and its exponential attractors. But the similar techniques used in [4] are available equally to the present problem.

**3.1 Construction of global solutions** It is immediate to construct a unique global solution to (2.6) for any initial value in $K$. In fact, let $U_0 \in K$. Then, Proposition 2.2 provides that the norm $\|U(t)\|_X$ remains uniformly bounded on the interval $[0, T_{U_0}]$. This then means that we can extend this local solution over some time interval $[0, T_{U_0} + \tau]$, $\tau > 0$ being determined by the norm $\|U(T_{U_0})\|_X$ alone. It is then possible to repeat such an extension, for any local solution of (2.6) (with this initial value $U_0$) takes its values in $K$ for every $t$ and the extended time interval $\tau > 0$ is taken uniformly.

Therefore, we obtain the following existence theorem.

**Theorem 3.1.** *For any $U_0 \in K$, (2.6) possesses a unique global solution lying in*

$$U \in \mathcal{C}([0,\infty); X) \cap \mathcal{C}^1((0,\infty); X) \cap \mathcal{C}((0,\infty); \mathcal{D}(A)).$$

*The solution $U(t)$ takes its values in $K$ for every $0 < t < \infty$ and satisfies the estimate*

$$\|U(t)\|_X + t(1+t)^{-1}\|AU(t)\|_X \leq C_3, \qquad 0 < t < \infty, \tag{3.1}$$

*with some constant $C_3 > 0$ which is uniform for the initial values from $K$.*

*Proof.* It suffices to prove the estimate (3.1). We already know that (3.1) holds true locally in the interval $(0, \tau]$, where $\tau$ is the time interval mentioned above. We then reset an initial value $U_1 = U(\frac{\tau}{2}) \in K$ and apply (2.8) to this local solution. Then,

$$\|U(t)\|_X + (t - \tfrac{\tau}{2})\|AU(t)\|_X \leq C, \qquad \tau \leq t \leq \tfrac{3\tau}{2}.$$

This shows that (3.1) holds true in the extended interval $(0, \frac{3\tau}{2}]$. Repeating this procedure, we obtain (3.1) on the whole interval $(0, \infty)$. ☐

It is also verified that the global solution is Lipschitz continuous with respect to the initial value in $K$. But, as the proof is quite analogous to that of [4, Theorem 3.3], we state the following theorem without its proof.

**Theorem 3.2.** *Let $U_0$, $V_0 \in K$ and let $U(t)$ and $V(t)$ be the global solutions of (2.6) with initial values $U_0$ and $V_0$, respectively. Then,*

$$\|U(t) - V(t)\|_X \leq C_4 e^{\beta t}\|U_0 - V_0\|_X, \qquad 0 \leq t < \infty, \tag{3.2}$$

$$\sqrt{t}\|\nabla[U(t) - V(t)]\|_X \leq C_4 e^{\beta t}\|U_0 - V_0\|_X, \qquad 0 < t < \infty, \tag{3.3}$$

*with some exponent $\beta > 0$ and some constant $C_4 > 0$ which are both uniform for the initial values from $K$.*

**3.2 Dynamical system** By utilizing the theory of dynamical systems for semilinear abstract parabolic evolution equations (see [7, Section 6.5]), it is immediate to construct a dynamical system generated by (2.6) in the space $X$.

For $U_0 \in K$, let $U(t; U_0)$ denote the global solution of (2.6) and set

$$S(t)U_0 = U(t; U_0), \qquad 0 \leq t < \infty.$$

Then, $S(t)$ is a nonlinear semigroup acting on $K$, i.e., $S(0) = I$ and $S(t+s) = S(t)S(s)$ for $0 \leq s, t < \infty$. Furthermore, $S(t)$ is seen to be continuous in the sense that $(t, U_0) \mapsto S(t)U_0$ is continuous from $[0,\infty) \times K$ into $K$. Indeed, due to (3.2), we have

$$\begin{aligned}\|S(s)V_0 - S(t)U_0\|_X &\leq \|S(s)V_0 - S(s)U_0\|_X + \|S(s)U_0 - S(t)U_0\|_X \\ &\leq e^{\beta s}\|V_0 - U_0\|_X + \|S(s)U_0 - S(t)U_0\|_X.\end{aligned}$$

Then, $(s, V_0) \to (t, U_0)$ implies $S(s)V_0 \to S(t)U_0$ in $X$.

The nonlinear semigroup $S(t)$ thus defines a dynamical system in the space $X$, which is denoted by $(S(t), K, X)$. The phase space $K$ presented by (2.7) is a bounded, closed subset of $X$.

As well known (see Babin-Vishik [1] and Temam [5]), the dissipative estimate provides existence of the global attractor. Consider a subset $B$ of $K$ defined by

$$B = K \cap \{U \in \mathcal{D}(A); \|AU\|_X \leq C_3 + 1\}.$$

Then, (3.1) means that there is a time $T$ such that $S(t)K \subset B$ for every $t \geq T$, i.e., $B$ is an absorbing set. In addition, $B$ is a compact set of $X$. Thereby, $B$ is a compact absorbing set of $(S(t), K, X)$. In view of the fact that $S(T)B \subset S(T)K \subset B$, we reset a phase space as

$$\mathcal{K} \equiv \bigcup_{0 \leq t \leq T} S(t)B \quad \subset \quad K.$$

It is obvious that $S(t)\mathcal{K} \subset \mathcal{K}$ for every $t > 0$, i.e., $\mathcal{K}$ is an invariant set. Therefore, $\mathcal{K}$ is not only compact and absorbing but also invariant. This means that the asymptotic behavior of trajectories of $(S(t), K, X)$ can be reduced to a sub dynamical system $(S(t), \mathcal{K}, X)$ in which the phase space $\mathcal{K}$ is a compact set of $X$. By the usual arguments, it is then seen that $\mathcal{B} = \bigcap_{0 \leq t < \infty} S(t)\mathcal{K}$ becomes a global attractor of $(S(t), \mathcal{K}, X)$.

Furthermore, thanks to the estimate (3.3), we can construct the exponential attractors. Remember (see Eden-Foias-Nicolaenko-Temam [3]) that a subset $\mathcal{M} \subset \mathcal{K}$ satisfying the following conditions is called the exponential attractor of $(S(t), \mathcal{K}, X)$:

1. $\mathcal{M}$ is a compact subset of $X$ with finite fractal dimension.
2. $\mathcal{M}$ includes the global attractor $\mathcal{B}$.
3. $\mathcal{M}$ is an invariant set, i.e., $S(t)\mathcal{M} \subset \mathcal{M}$ for every $t > 0$.
4. There exists an exponent $k > 0$ such that

$$h(S(t)\mathcal{K}, \mathcal{M}) \leq C_5 e^{-kt}, \qquad 0 < t < \infty,$$

with a constant $C_5 > 0$.

Here, $h(K_1, K_2) = \sup_{F \in K_1} \inf_{G \in K_2} \|F - G\|_X$ is a semi-distance of two subsets $K_1$ and $K_2$ of $\mathcal{K}$.

As explained in [7, Section 6.4], the compact smoothing property

$$\|S(t^*)U_0 - S(t^*)V_0\|_{H^1(\Omega)} \leq C_6\|U_0 - V_0\|_X, \qquad U_0, V_0 \in \mathcal{K},$$

of $S(t^*)$ with any fixed time $t^* > 0$ provides existence of exponential attractors. But, in the present case, this property is nothing more than the estimate (3.3). In this way, we obtain the following theorem.

**Theorem 3.3.** *The dynamical system* $(S(t), K, X)$ *possesses exponential attractors.*

*Proof.* As noticed above, we already know that there exists an exponential attractor $\mathcal{M}$ for $(S(t), \mathcal{K}, X)$. Then, as $S(T)K \subset B \subset \mathcal{K}$, it is readily verified that $\mathcal{M}$ is an exponential attractor for $(S(t), K, X)$, too. □

**4 Homogeneous Stationary Solutions** Consider the system of equations for $u, v$ and $w$:

$$\varphi(u,v,w) \equiv \big[(1-u-v)\{1-\delta(\overline{w}-w-q[g(u,v)-a_w])^2\}-f\big]u = 0, \tag{4.1}$$

$$\psi(u,v,w) \equiv \big[(1-u-v)\{1-\delta(\overline{w}-w-q[g(u,v)-a_b])^2\}-f\big]v = 0, \tag{4.2}$$

$$\chi(u,v,w) \equiv [1-g(u,v)]R - \sigma w^4 = 0, \tag{4.3}$$

where $g(u, v)$ is the function given by (1.2). Here, according to [6], we want to handle a typical case that the parameters are given by

$$a_b = \frac{1}{4},\ a_g = \frac{1}{2},\ a_w = \frac{3}{4},\ q = 20,\ \delta = 3.265 \times 10^{-3}, \quad f = 0.3,\ \overline{w} = 295.5 \text{ and } \sigma = 5.67 \times 10^{-8}, \tag{4.4}$$

except $R$ that is treated as a tuning parameter.

**4.1 Positive solutions** We are concerned with the solutions such that $0 < u < 1$ and $0 < v < 1$. Then, since $1 - u - v \neq 0$, it follows from (4.1) and (4.2) that

$$1-\delta(\overline{w}-w-q[g(u,v)-a_w])^2 = 1-\delta(\overline{w}-w-q[g(u,v)-a_b])^2.$$

Therefore, $2(\overline{w} - w) - q[2g(u, v) - a_w - a_b] = 0$. In view of $a_b + a_w = 1$, we have

$$g(u,v) = \frac{1}{q}(\overline{w}-w) + \frac{1}{2}. \tag{4.5}$$

It then follows from (1.2) that

$$u - v = \frac{4}{q}(\overline{w}-w). \tag{4.6}$$

Meanwhile, (4.5) together with (4.3) yields the 4-th order equation

$$w^4 - \rho(w - w_0) = 0 \tag{4.7}$$

for $w$, where $\rho = \frac{R}{q\sigma}$ and $w_0 \equiv \overline{w} - \frac{q}{2} > 0$. On the other hand, (4.5) together with (4.1) yields the equation

$$u + v = 1 - \frac{f}{1-(q/4)^2\delta}. \tag{4.8}$$

In this way, we have observed that the equations (4.1)-(4.3) reduced to (4.6)-(4.8).

Let us next solve the equations (4.6)-(4.8). We first observe that when $\rho = \frac{4^4}{3^3}w_0^3$, i.e., $R = R_0 \equiv \frac{4^4}{3^3}q\sigma w_0^3$, (4.7) has a unique solution $w = \frac{4}{3}w_0$. Consequently, when $R > R_0$, (4.7) has two solutions $w_* < w^*$ such that $w_0 < w_* < \frac{4}{3}w_0 < w^*$. But, here, we easily see for $w^*$ that the equations (4.6) and (4.8) cannot have positive solutions. Meanwhile, there is a range for $w_*$ in which (4.6) and (4.8) admit a unique positive solution. As $R > R_0$ increases, $w_*$ monotonously decreases in the range $\frac{4}{3}w_0 > w_* > w_0$. Therefore, we verify the following result.

**Proposition 4.1.** *There is a range* $(R_*, R^*)$ *of* $R$ *for which* (4.6)-(4.8) *have a unique positive solution* $(u_*, v_*, w_*)$.

Moreover, under (4.4) it is easy to see that

$$1-\delta(\overline{w}-w_*-q[g(u_*,v_*)-a_i])^2 \geq 0$$

for $i = w, b$. This shows that for $R_* < R < R^*$, $U_* = (u_*, v_*, w_*)$ gives a unique positive homogeneous stationary solution of (2.6).

**4.2 Stability and instability of** $U_*$ We investigate stability and instability of the homogeneous positive stationary solution $U_*$ when $R_* < R < R^*$.

For this purpose we use the linearization principle. Linearizing (2.6) in a neighborhood of $U^*$, let us consider the linear problem

$$\begin{cases} \dfrac{dU}{dt} + AU = F'(U_*)U, \qquad 0 < t < \infty, \\ U(0) = U_0. \end{cases} \tag{4.9}$$

Here, $F'(U_*)$ is a multiplicative operator of $X$ by the matrix

$$F'(U_*) = \begin{pmatrix} \varphi_u^* & \varphi_v^* & \varphi_w^* \\ \psi_u^* & \psi_v^* & \psi_w^* \\ \chi_u^* & \chi_v^* & \chi_w^* \end{pmatrix} \equiv \begin{pmatrix} \varphi_u(u_*,v_*,w_*) & \varphi_v(u_*,v_*,w_*) & \varphi_w(u_*,v_*,w_*) \\ \psi_u(u_*,v_*,w_*) & \psi_v(u_*,v_*,w_*) & \psi_w(u_*,v_*,w_*) \\ \chi_u(u_*,v_*,w_*) & \chi_v(u_*,v_*,w_*) & \chi_w(u_*,v_*,w_*) \end{pmatrix}.$$

By elementary calculations, we observe that

$$\varphi_u^* = \left[\tfrac{q^2\delta}{16} + \tfrac{2fq^2\delta}{16-q^2\delta} - 1\right]u_*, \quad \varphi_v^* = \left[\tfrac{q^2\delta}{16} - \tfrac{2fq^2\delta}{16-q^2\delta} - 1\right]u_*, \quad \varphi_w^* = \tfrac{8fq\delta}{16-q^2\delta}u_*, \tag{4.10}$$

$$\psi_u^* = \left[\tfrac{q^2\delta}{16} - \tfrac{2fq^2\delta}{16-q^2\delta} - 1\right]v_*, \quad \psi_v^* = \left[\tfrac{q^2\delta}{16} + \tfrac{2fq^2\delta}{16-q^2\delta} - 1\right]v_*, \quad \psi_w^* = -\tfrac{8fq\delta}{16-q^2\delta}v_*, \tag{4.11}$$

$$\chi_u^* = -\tfrac{R}{4}, \qquad \chi_v^* = \tfrac{R}{4}, \qquad \chi_w^* = -4\sigma w_*^3. \tag{4.12}$$

We utilize again the base functions

$$\begin{Bmatrix} \cos\dfrac{m\pi}{\ell_x}x \\ \sin\dfrac{m\pi}{\ell_x}x \end{Bmatrix} \times \cos\frac{n\pi}{\ell_y}y, \qquad m,\, n = 0,1,2,\dots,$$

which have been introduced in the proof of Proposition 2.1. They compose an orthogonal basis of $L_2(\Omega)$ and are an eigenfunction of $-\Delta$ under the periodic-Neumann boundary conditions with the eigenvalue

$$\mu_{mn} = \left(\frac{m\pi}{\ell_x}\right)^2 + \left(\frac{n\pi}{\ell_y}\right)^2, \qquad m,\, n = 0,1,2,\dots, \quad \text{respectively.}$$

Consider the subspaces of $X$ which are defined by

$$X_{mn}^c = \mathrm{Span}\,\{e_1\cos\tfrac{m\pi}{\ell_x}x\cdot\cos\tfrac{n\pi}{\ell_y}y,\ e_2\cos\tfrac{m\pi}{\ell_x}x\cdot\cos\tfrac{n\pi}{\ell_y}y,\ e_3\cos\tfrac{m\pi}{\ell_x}x\cdot\cos\tfrac{n\pi}{\ell_y}y\},$$

$$X_{mn}^s = \mathrm{Span}\,\{e_1\sin\tfrac{m\pi}{\ell_x}x\cdot\cos\tfrac{n\pi}{\ell_y}y,\ e_2\sin\tfrac{m\pi}{\ell_x}x\cdot\cos\tfrac{n\pi}{\ell_y}y,\ e_3\sin\tfrac{m\pi}{\ell_x}x\cdot\cos\tfrac{n\pi}{\ell_y}y\},$$

where $e_1 = {}^t(1,0,0)$, $e_2 = {}^t(0,1,0)$, $e_3 = {}^t(0,0,1)$. Then, it is easily verified that they are all a three-dimensional subspace of $X$, are mutually orthogonal in $X$ and their Hilbert sum coincides with the space $X$, i.e.,

$$X = \sum_{0\le m,n<\infty} X^c_{mn} + \sum_{1\le m<\infty, 0\le n<\infty} X^s_{mn}.$$

Furthermore, it is verified that they are all an invariant subspace of the operator $-A + F'(U_*)$. Hence, the problem (4.9) can be decomposed into the infinite number of subproblems of (4.9) in the three-dimensional subspaces $X^c_{mn}$ and $X^s_{mn}$.

By the way, the transformation matrices of $-A + F'(U_*)$ both in $X^c_{mn}$ and $X^s_{mn}$ are given by $M_{\mu_{mn}}$, where we put

$$M_\mu = \begin{pmatrix} -d\mu + \varphi^*_u & \varphi^*_v & \varphi^*_w \\ \psi^*_u & -d\mu + \psi^*_v & \psi^*_w \\ \chi^*_u & \chi^*_v & -D\mu + \chi^*_w \end{pmatrix} \qquad \text{for } 0 \le \mu < \infty.$$

If for all $M_{\mu_{mn}}$, their eigenvalues have negative real parts, then $U_*$ is concluded to be a stable stationary solution. To the contrary, if there exists at least one $M_{\mu_{mn}}$ such that one of its eigenvalues has a positive real part, then $U_*$ is concluded to be an unstable one. The characteristic polynomial of $M_\mu$ is given by

$$P_\mu(\lambda) \equiv \det(\lambda I - M_\mu) = \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3$$

with the following coefficients:

$$\begin{aligned} p_1 &= (2d + D)\mu - (\varphi^*_u + \psi^*_v + \chi^*_w), \qquad p_3 = -\det M_\mu, \\ p_2 &= (d^2 + 2dD)\mu^2 - [(\varphi^*_u + \psi^*_v)D + (\psi^*_v + \chi^*_w)d + (\chi^*_w + \varphi^*_u)d]\mu \\ &\qquad + (\varphi^*_u\psi^*_v + \psi^*_v\chi^*_w + \chi^*_w\varphi^*_u) - (\varphi^*_v\psi^*_u + \varphi^*_w\chi^*_u + \psi^*_w\chi^*_v). \end{aligned}$$

Furthermore, $p_3$ is described as a third order polynomial of $\mu$ by

$$\begin{aligned} p_3 &= d^2D\mu^3 - [(\varphi^*_u + \psi^*_v)dD + \chi^*_w d^2]\mu^2 \\ &\quad + [(\varphi^*_u\psi^*_v - \varphi^*_v\psi^*_u)D + (\varphi^*_u\chi^*_w + \psi^*_v\chi^*_w - \psi^*_w\chi^*_v - \varphi^*_w\chi^*_u)d]\mu - \det M_0. \end{aligned}$$

Here, it is verified from (4.10)-(4.12) that $p_1 > 0$ and $p_1p_2 - p_3 > 0$. The Routh-Hurwitz theorem then provides that $P_\mu(\lambda)$ has a root of positive real part if and only if $p_3 < 0$. But we notice that

$$\begin{aligned} \varphi^*_u\psi^*_v - \varphi^*_v\psi^*_u &= \left(\left[\frac{q^2\delta}{16} + \frac{2fq^2\delta}{16 - q^2\delta} - 1\right]^2 - \left[\frac{q^2\delta}{16} - \frac{2fq^2\delta}{16 - q^2\delta} - 1\right]^2\right) u_*v_* \\ &= \frac{4fq^2\delta}{16 - q^2\delta}\left(\frac{q^2\delta}{8} - 2\right) u_*v_* < 0. \end{aligned}$$

This shows that, if the diffusion coefficient $D$ is sufficiently large with respect to the other $d$, then $p_3 < 0$ for $\mu$ varying in some interval $(\mu_*, \mu^*)$. Consequently, for $\mu \in (\mu_*, \mu^*)$, the polynomial $P_\mu(\lambda)$ has at least one positive root. As explained above, if there is some eigenvalue $\mu_{mn}$ that is included in this interval, then $U^*$ is unstable.

For example, set $R = 917$ in addition to (4.4). Then,

$$p_3 \approx k(d\mu)^3 + (5.813 + 0.582k)(d\mu)^2 + (5.032 - 0.021k)(d\mu) + 0.885,$$

where we put $D = kd$. Thereby, if

$$k > 5118.845, \tag{4.13}$$

then there exists the interval $(\mu_*, \mu^*)$ of $\mu$ in which $p_3$ takes negative values.

**5 Numerical Results** This section is devoted to showing numerical results for (2.6).

Set $\Omega = (0, 2\pi) \times (0, \pi)$, and set the parameters appearing in (1.1) as (4.4). The parameter $R$ is tuned as a control parameter. In view of (4.13), the diffusion coefficients are fixed by

$$D = 1 \quad \text{and} \quad d = 10^{-5}.$$

According to the thermal physics, the incoming energy $R$ is more precisely described by $R = S \times L$, where $S$ is a radiation energy of sunlight and $L$ is intensity of sunlight. Setting $S = 917$, we actually tune $L$ in a range

$$R = 917 \times L \qquad \text{for} \quad 0.75 \leq L \leq 1.35.$$

(Consequently, $R$ varies in $[687.75, 1237.95]$.) By the results obtained in Section 4, we know for each $L$ of this range that (2.6) has a unique positive homogeneous stationary solution $U_*$. The initial value $U_0$ is then set by a random small perturbation of this homogeneous stationary solution.

All the numerical computations are performed by using the two-dimensional ADI methods.

**5.1 Segregation patterns** We vary $L$ from 0.75 to 1.35 with step size $\varDelta L = 0.05$.

For $0.75 \leq L \leq 1.30$, the stationary solution $U_*$ is unstable. So, in these cases, the perturbation added to $U_*$ increases and the trajectory $S(t)U_0$ leaves from $U_*$ and goes far away. About $t = 6,000$, the numerical solution is almost stabilized. The trajectory $S(t)U_0$ might have been attracted by the exponential attractors. The profiles of the graphs of $u(t)$ and $v(t)$ at $t = 6,000$ are illustrated by means of the color graduation by Fig. 1 ($L = 0.75$), Fig. 2 ($L = 0.80$), Fig. 3 ($L = 0.85$), Fig. 4 ($L = 0.90$), Fig. 5 ($L = 0.95$), Fig. 6 ($L = 1.00$), Fig. 7 ($L = 1.05$), Fig. 8 ($L = 1.10$), Fig. 9 ($L = 1.15$), Fig. 10 ($L = 1.20$), Fig. 11 ($L = 1.25$) and Fig. 12 ($L = 1.30$), respectively. On the contrary, for $L = 1.35$, the stationary solution $U_*$ is stable. So, the trajectory $S(t)U_0$ goes back to $U_*$, see Fig. 13 . But, as the stability is very weak, it takes longtime ($t = 6,000$) until $S(t)U_0$ is numerically stabilized.

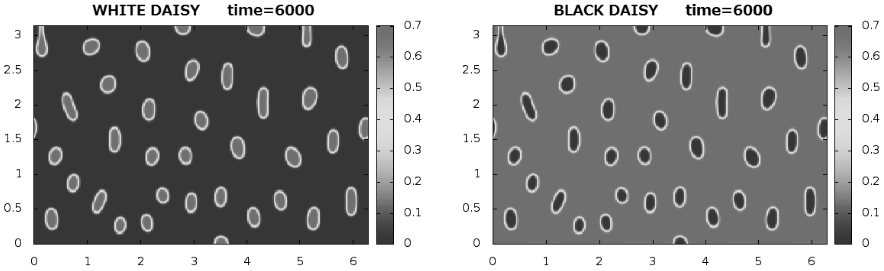

Fig. 1: $L = 0.75$.

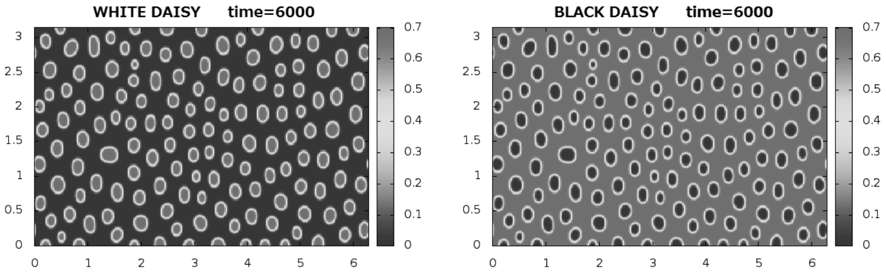

Fig. 2: $L = 0.80$.

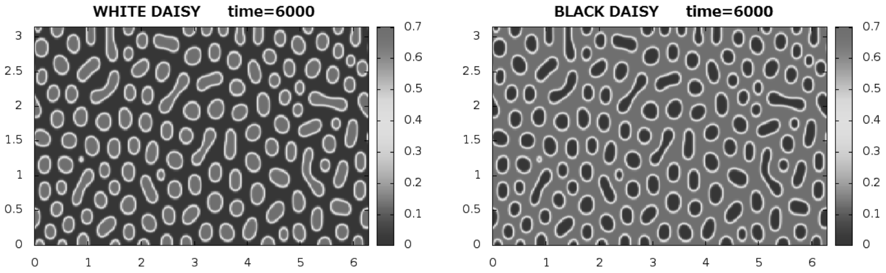

Fig. 3: $L = 0.85$.

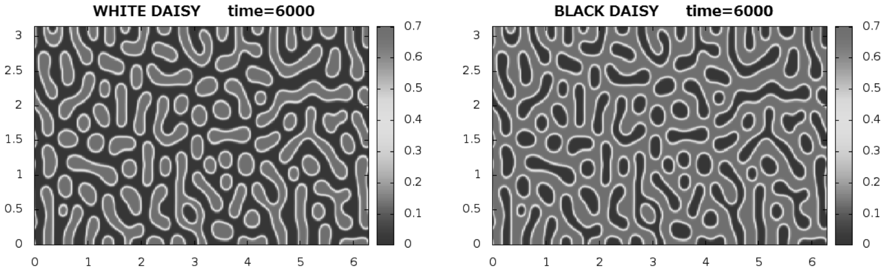

Fig. 4: $L = 0.90$.

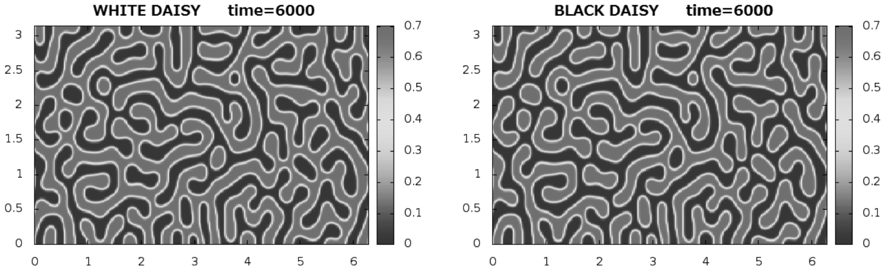

Fig. 5: $L = 0.95$.

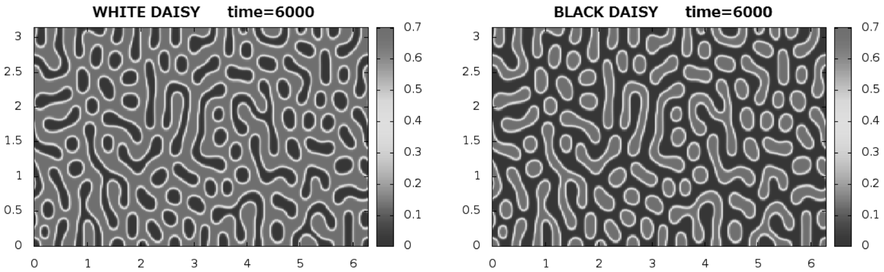

Fig. 6: $L = 1.00$.

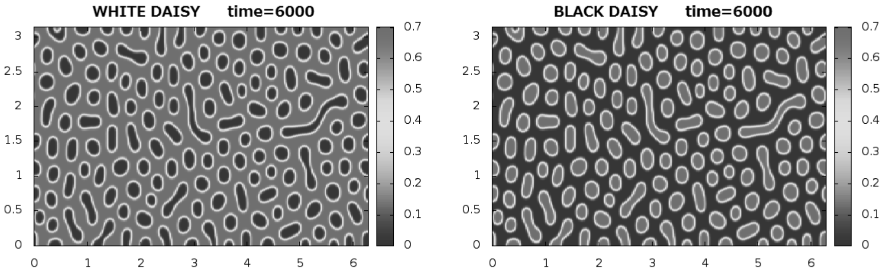

Fig. 7: $L = 1.05$.

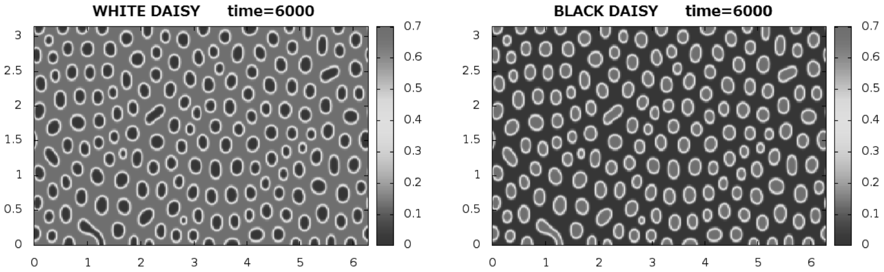

Fig. 8: $L = 1.10$.

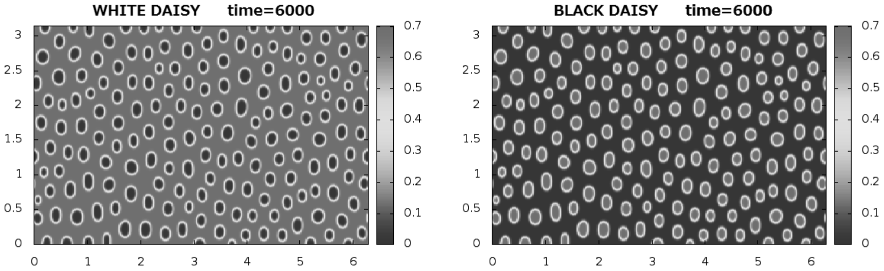

Fig. 9: $L = 1.15$.

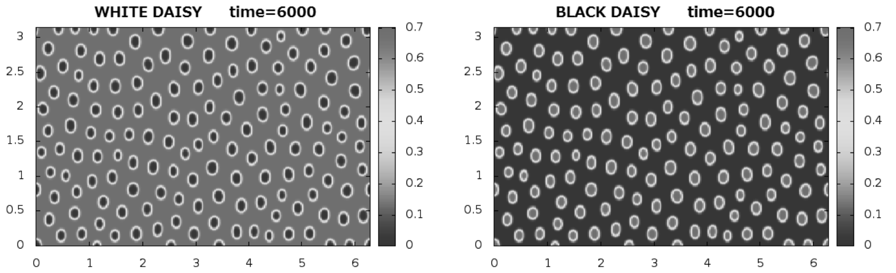

Fig. 10: $L = 1.20$.

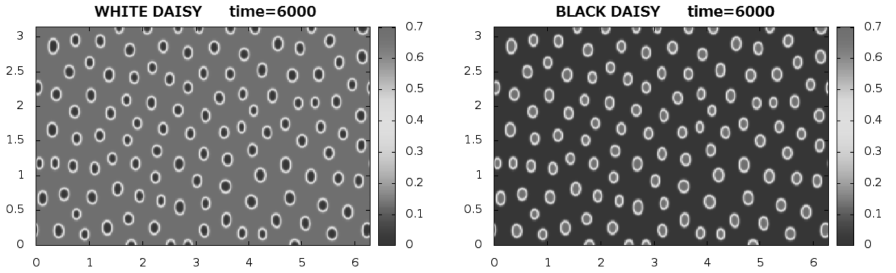

Fig. 11: $L = 1.25$.

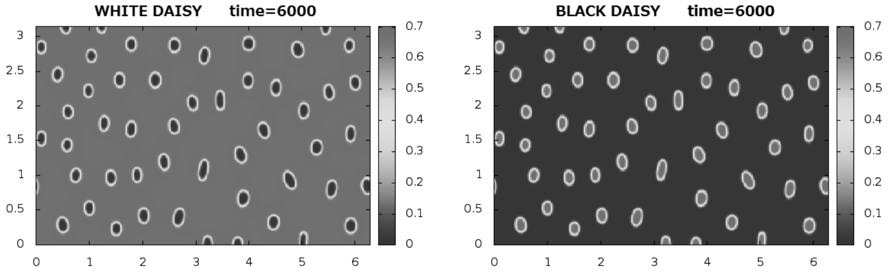

Fig. 12: $L = 1.30$.

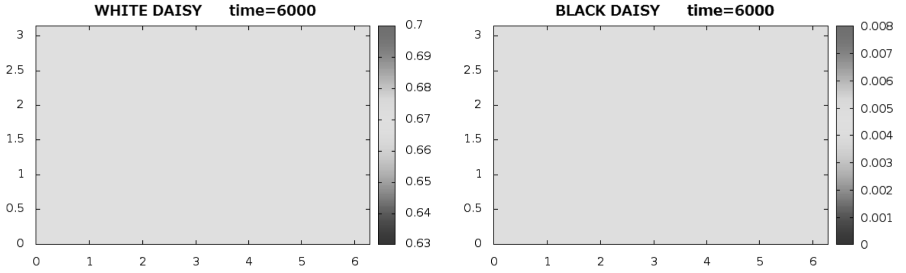

Fig. 13: $L = 1.35$.

For $0.75 \le L \le 1.30$, we find clear segregation patterns formed by the white and black daisies. At $L = 0.75$, black daisy is dominant in $\Omega$ and white daisy occurs only in a small number of spots. At $L = 0.80$, the number of spots generated by white daisy increases; but, at $L = 0.85$ and $0.90$, some of these spots are jointed to make a long island of white daisy. At $L = 0.95$, the growth of two daisies seems to balance in $\Omega$ and both of them form a labyrinth pattern. For $1.00 \le L \le 1.30$, white daisy in turn becomes dominant. As $L$ increases, the very reversed patterns of white daisy and black daisy are successively performed. At $L = 1.35$, white and black daisies coexist but two daisies are distributed homogeneously in $\Omega$.

**5.2 Mean of global temperature** For $0.75 \le L \le 1.35$, the numerical values of $w(t)$ are as well stabilized about $t = 6,000$. The profiles of the graphs of $w(t)$ at $t = 6,000$ are illustrated by means of the color graduation by Figs. 14-26. Of course, the distribution of the global temperature depends closely on those of white and black daisies. So, we want to consider the spatial mean of $w(x, y, t)$, i.e.,

$$W(t) = \frac{1}{|\Omega|} \iint_{\Omega} w(x, y, t)dxdy, \qquad 0 \le t < \infty.$$

For each $L$, an approximate value of $W(6,000)$ is computed by a numerical integration. Its graph is drawn by Fig. 27. (However, the temperature is expressed in degrees Celsius.) We find that during the interval $[0.75, 1.35]$ of $L$, the mean of the global temperature is completely stabilized.

We thus observe that the homeostasis in the global temperature is maintained in $\Omega$ with respect to a change of intensity of sunlight, although the segregation pattern of white and black daisies clearly changes its types from homogeneous, spot, island and to labyrinth.

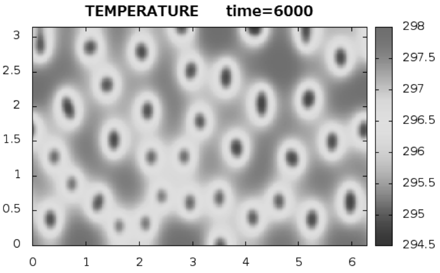

Fig. 14: $L = 0.75$.

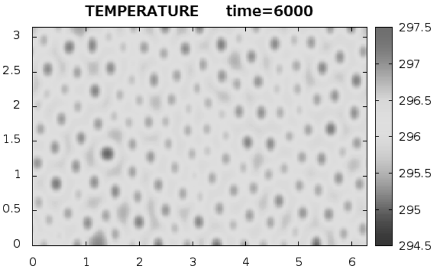

Fig. 15: $L = 0.80$.

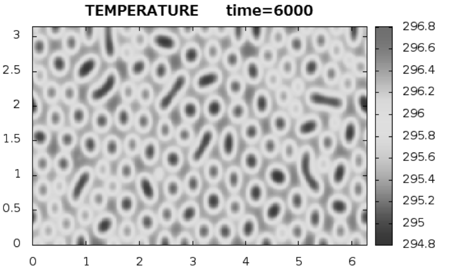

Fig. 16: $L = 0.85$.

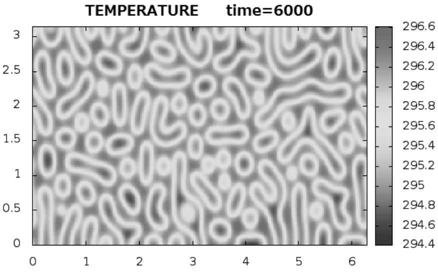

Fig. 17: $L = 0.90$.

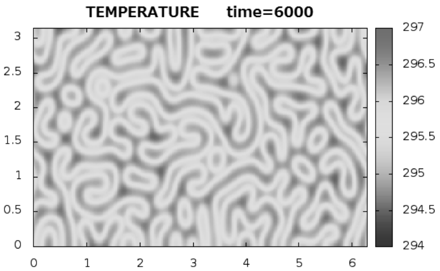

Fig. 18: $L = 0.95$.

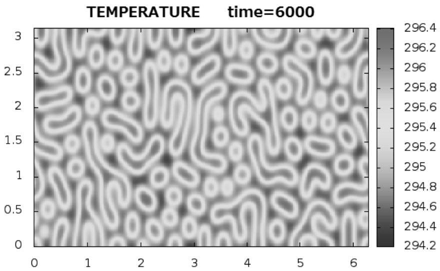

Fig. 19: $L = 1.00$.

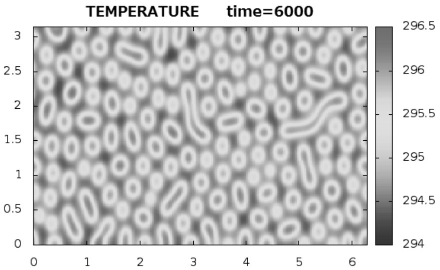

Fig. 20: $L = 1.05$.

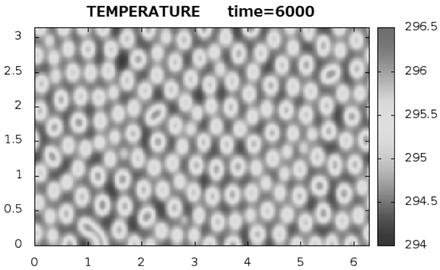

Fig. 21: $L = 1.10$.

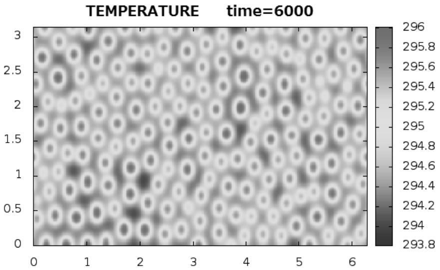

Fig. 22: $L = 1.15$.

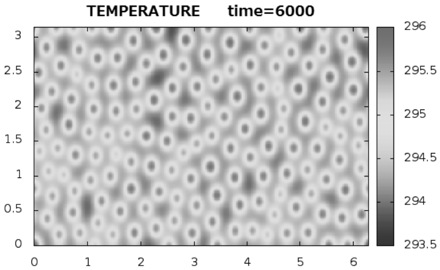

Fig. 23: $L = 1.20$.

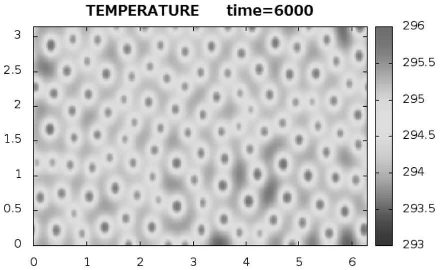

Fig. 24: $L = 1.25$.

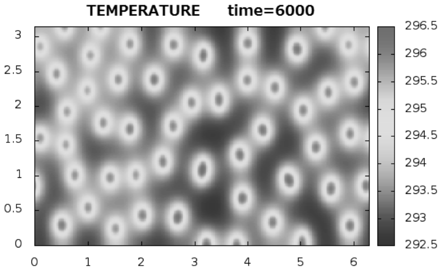

Fig. 25: $L = 1.30$.

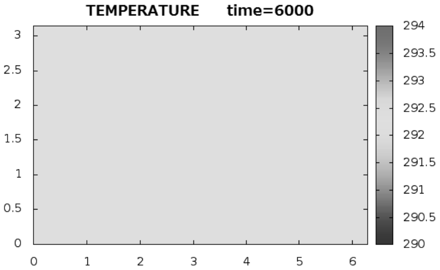

Fig. 26: $L = 1.35$.

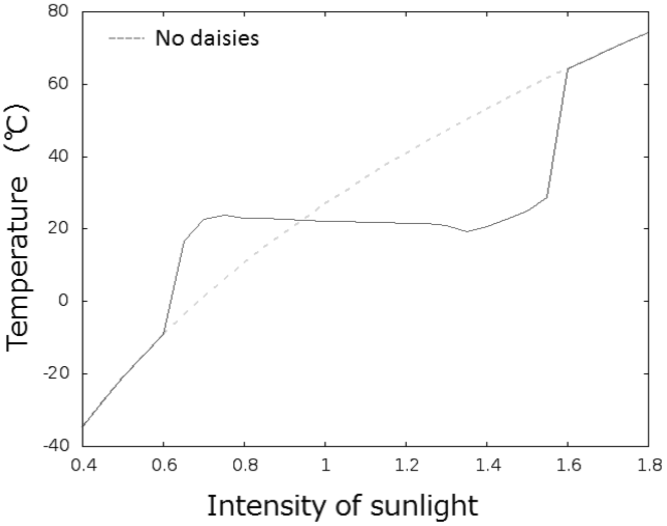

Fig. 27: The spatial mean of temperature.

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# ESTIMATE ON DIFFUSION RATE OF CONTAMINANT IN RECYCLING LINE OF FOOD-TRAYS BY FPCO'S METHODS

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Abstract. Reduction of the amount of wastes coming from food containers and packaging is one of urgent issues for the humankind. Japanese manufacturers, including F. P. Corporation, are devising their own recycling system of disposable food containers for reusing resources in containers and packagings. Without waiting the Guidelines issued by the Ministry of Health, Labour and Welfare of Japanese Government, it is indispensable to ensure food safety when the manufacturer uses such recycled materials. This paper then intends to present methods for estimating a diffusion rate of contaminant if it is contained in post-consumer food containers and enters the recycling line. Our methods will be explained by applying them to the recycling line realized by F. P. Corporation. As our methods are quite general, they may easily be applied to any other recycling lines.

**1 Introduction** It is ordinarily seen that a large amount of household wastes is occupied by those which come from food containers and packagings. In order to reduce the amount of such wastes, the Recycling Law of Food Containers and Packaging has been established in Japan in 1995 for promoting more effective use of resources in containers and packagings.

F. P. Corporation (abbreviated to FPCO), a manufacturer of disposable food containers to be used in supermarkets, convenience stores and others, has been realizing an original recycling system since 1990.

Post-consumer food containers brought to supermarkets and others are gathered by collection boxes and are brought back to the recycling plants of FPCO by utilizing returning trucks which delivered their products as explained in [1]. FPCO's recycling process of foamed polystyrene containers consists of three main steps, namely, (1) sorting/crashing, (2) washing/dehydration, and (3) extrusion/pelletizing, in order to remove contaminators from the collected polystyrene containers. Using the regenerated polystyrene pellets, recycled foamed polystyrene containers are made via sheet formations. Its schematic diagram is sketched by Figure 1. For details, see the homepage [2].

Without waiting the Guidelines [3] issued by the Ministry of Health, Labour and Welfare of Japanese Government, it is indispensable to ensure food safety when the manufacturer uses such recycled materials for reproducing food containers. Careful and sufficient considerations must be taken for preventing any recycled containers containing adventitious chemical contaminant which may migrate into foods and influence human health from being distributed to the markets.

FPCO has received a non-objection letter on recycled foamed polystyrene containers from U. S. Food and Drug Administration. In addition, constant inspections are carried out in daily production activities in accordance with Japanese Food Sanitation Act.

Meanwhile, investigations on the worst case are always required in the field of food sanitation. One of these investigations, to know scientifically how contaminants diffuse through

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the recycling process is very important and to estimate reasonably the highest possible contaminant concentration is very crucial. By these reasons, a mathematical approach is proposed by the present authors and some analytical results are described in the paper. Specifically, we assume that a tray containing a unit amount of contaminant enters FPCO's recycling line. Then, under the worst external conditions to be considered, we analyse its diffusion rate. Finally, we compute the highest contaminant concentration by means of the random variable.

As our methods of estimation are very general, it is easy to know how the response is with respect to the change of controllable internal conditions. We then hope that the methods presented in this paper would play a meaningful role in order to establish safer and more reliable recycling processes for reusing more post-consumer food containers and packaging waste.

Finally, let us review FPCO's recycling line whose schematic diagram is sketched by Figure 1. The collected trays are crashed into small fragments. After being fully washed, the fragments are melted by a heater and the polystyrene in gel is pelletized by an extruding machine to yield numbers of pellets which are a unit grain of foamed polystyrene of a uniformed size in order to reproduce new food-trays. The pellets made from the used trays are packed in big boxes and are quadrupled by adding three times virgin pellets. After being entirely blended, the quadrupled pellets are laid in a thin layer, once again melted and are sheeted by another extruding machine to make polystyrene sheets. These sheets are laminated by a virgin film and cut into a unit size of tray. By these processes, the used trays are recycled to new ones.

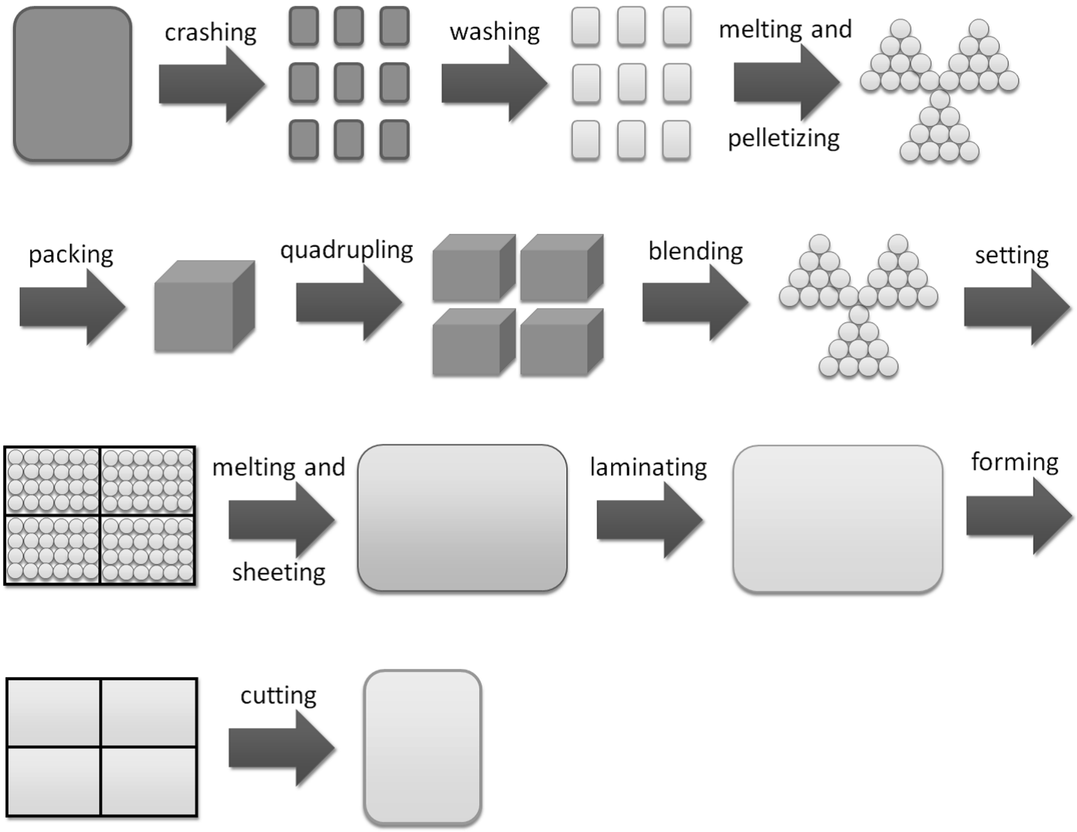

Fig. 1: FPCO's Recycling Methods

**2 Material and Methods** We first want to notice that through the recycling line sketched by Figure 1, the diffusion of contaminant consists of three independent kinds of diffusions.

First one is the temporal diffusion. Assume that one tray containing a unit amount of contaminant has entered the production line. Then, the contaminated tray is crashed into almost 250 fragments which contain as a result $4.0 \times 10^{-3}$ unit of contaminant for each. Through melting and pelletizing, the 250 fragments are processed into numbers of pellets which contain a certain unit of contaminant. And these contaminated pellets together with other clear ones are packed in several boxes. Then, how do the contaminated pellets diffuse over the packing boxes?

Second one is the diffusion caused by combination which may be called the combinatorial diffusion. Consider a box of pellets which nearly consist of $1.0 \times 10^7$ pellets and assume that some of these, say $n$ pellets, are contaminated. By addition of three boxes of virgin pellets, we have $4.0 \times 10^7$ pellets as a whole. These pellets are randomly divided into sets consisting of 100 pellets uniformly; consequently, we make $4.0 \times 10^5$ sets. Each set of pellets can yield just one new tray after melting, sheeting and cutting processes. Then, how do the $n$ contaminated pellets included in $4.0 \times 10^7$ pellets in total diffuse over the dividing sets?

Third one is the diffusion caused by melting and extruding (here and after the word extruding will be used for two meanings: pelletizing by extrusion and sheeting by extrusion). The production line has two processes of melting and extruding. Naturally, through the two processes the contaminant in contaminated fragments or in contaminated pellets diffuses in the gel of polystyrene. Then, how does the contaminant diffuse in the gel spatially?

Let us next explain how we analysed these different kinds of diffusions.

As for the temporal diffusion, we made the following experiments. A certain number of colored fragments of tray were inserted in the recycling line and the arriving time of each fragment at the first melting stage was checked. Several times this trial was repeated. Through these experiments we know how long the fragments made of a contaminated tray entered in the line diffuse temporally before arriving at the first melting stage.

The combinatorial diffusion can be analysed exactly by using the theory of probability and combinatorics (e.g., see [4, 7]). Consider a collection of $N = 4.0 \times 10^7$ pellets which includes $n$ contaminated pellets. We divide all the pellets randomly into $4.0 \times 10^5$ sets which consist uniformly of 100 pellets. Denote by $X$ the maximum of contaminated pellets included in one set throughout the $4.0 \times 10^5$ sets. Of course $X$ changes depending on how to divide, so $X$ is considered as a random variable. The most favorable case is that the $n$ contaminated pellets are completely divided into different sets, i.e., $X = 1$. On the contrary, the worst case is that the $n$ pellets are divided into a single set, i.e., $X = n$, but the probability of such a division should be negligibly small. We will devise an easy way how to compute the probability such that $X = k$ for the variable $k = 1, 2, 3, \dots, n$.

Finally, the spatial diffusion due to melting and extruding is analysed by the following experiments. A similar type of melting and sheeting machine was prepared. Among numbers of pellets, just one pellet which contains a material emitting fluorescent X-rays was put and passed through the heater and extruder. The resultant sheet was then carefully examined. How wide is the emitting material spread? What is magnitude of the X-ray in each part of sheet? Several times this experiment was repeated. Out of those data, we built a fitting function which describes the diffusion of the emitting material as a 3D graph, by using the techniques of implicit surface fitting (see [6, 10]). By these arguments we know how wide the contaminant in a pellet is spread and by what rate the contaminant diffuses through the melting and extruding processes.

It is, however, very difficult to analyse the spatial diffuses of contaminant in the first melting process, because the gel made from the fragments is immediately formed into numbers of pellets by a pelletizing extruder. So we want to introduce an imaginary process of sheeting and want to consider that the gel is once formed into sheets and then those sheets are formed into pellets.

## 3 Results

**3.1 Temporal diffusion** We inserted 50 colored fragments of tray at the end of crashing process and checked the arriving time of each fragment at the checking point which was set almost in the middle of crashing and melting stages. This trial was repeated 5 times. We could check for almost 30 fragments their arriving time for each trial. The result is graphed in Figure 2.

Here, $\Delta t = 1, 2, 3$ (min.) denotes a unit of time interval, the axis of abscissas $i = 1, 2, 3, \ldots$ denotes time $i\Delta t$ (min.), and the axis of ordinates denotes a number of fragments which arrived during the time from $(i-1)\Delta t$ to $i\Delta t$. From the data we observe that the range of arrival time is not so long and all the checked fragments arrived within 26 min. Indeed, we verify that, if the graphs in Figure 2 can be approximated by the normal distributions, then it is concluded that 95% of fragments arrive within 25 minutes (see [8]). Remembering that our cheking point is set at the middle of crashing and melting stages, we want to estimate that the temporal diffusion of contaminated fragments is about 1 hour.

After being melted and pelletized, the fragments are formed into pellets and the pellets are packed in big boxes. We know that each packing box is filled with pellets by just 1 hour. This means that the pellets made from the 250 contaminated fragments must be packed at most 2 boxes. In this way the $n$ contaminated pellets can be included in a single packing box with a high probability, which means that the temporal diffusion must be disregarded.

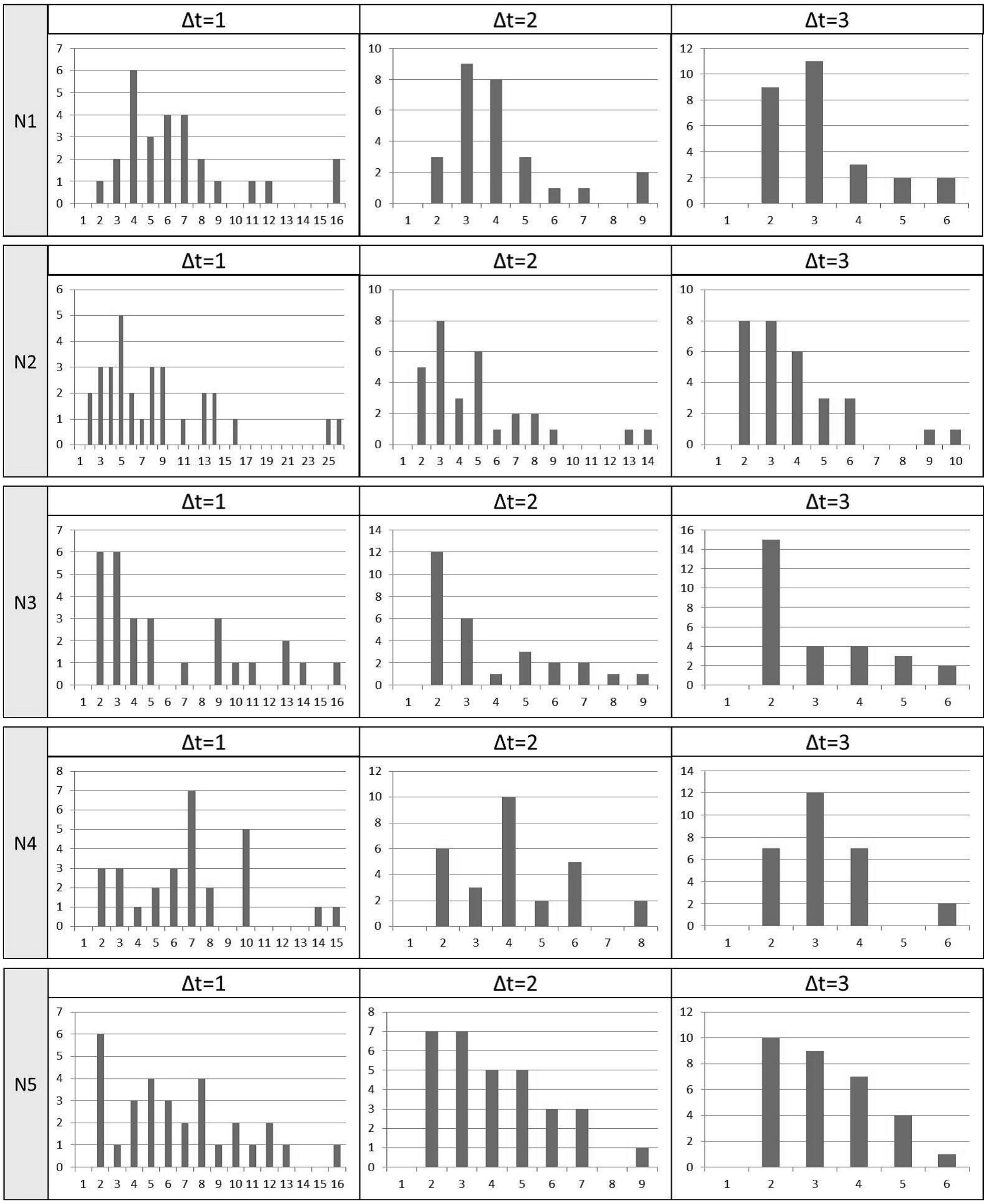

Fig. 2: Experimental Data

**3.2 Spatial diffusion** We put one pellet which contains a material emitting fluorescent X-rays in a similar type of melting and sheeting machine. Magnitude of the X-ray in each part of the resultant sheet was measured by a photometer. The data is given by Table 1.

Table 1: Data

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.027003484 | 0.031068525 | 0.024970964 | 0.022357724 | 0.019454123 | 0.012775842 | 0.019163763 | 0.008420441 | 0.007839721 | 0 | 0.008420441 | 0 | 0 | 0 | 0 |
| 2 | 0.013066202 | 0.028745645 | 0.030197445 | 0.026422764 | 0.022938444 | 0.019163763 | 0.018873403 | 0.013646922 | 0.009872242 | 0.007839721 | 0.006678281 | 0 | 0 | 0 | 0 | 0 |
| 3 | 0.007549361 | 0.034262485 | 0.042973287 | 0.033391405 | 0.022938444 | 0.014808362 | 0.012775842 | 0.008710801 | 0.010162602 | 0.007839721 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0.020325203 | 0.041521487 | 0.030487805 | 0.025551684 | 0.018583043 | 0.016260163 | 0.012485482 | 0.011614402 | 0.010162602 | 0.007259001 | 0.005807201 | 0 | 0 | 0 | 0 |
| 5 | 0.006097561 | 0 | 0.009001161 | 0.012485482 | 0.032520325 | 0.025842044 | 0.031939605 | 0.022067364 | 0.021777003 | 0.016260163 | 0.012775842 | 0.011904762 | 0.011614402 | 0.011324042 | 0 | 0 |

The resultant sheet is of width 35cm × 480cm. This area is divided into 5 × 16 parts which are uniformly of width 7cm × 30cm. The numbers in Table 1 show the magnitude of the X-ray in these parts. The total magnitude is just 1. We see that the part $(3, 3)$ has the maximum magnitude. The data can also be illustrated by a rectangular graph drew in Figure 3.

In order to use these data more conveniently, it is necessary to describe the graph by a suitable fitting surface. Several methods are known how to fit a function $f(x, y)$ to a given rectangular graph. We here use the normal distribution for the variable $x$ and the Johnson Sb distribution for the variable $y$ due to [6], that is,

$$f(x,y) = \frac{b-a}{2\pi\sigma(b-y)(y-a)} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} - \frac{1}{2}\left[\gamma + \delta \log\left(\frac{y-a}{b-y}\right)\right]^2 \right\}, \tag{3.1}$$

where $a, b, \gamma, \delta, \mu$ and $\sigma$ are parameters to be determined, see [10]. Some optimization

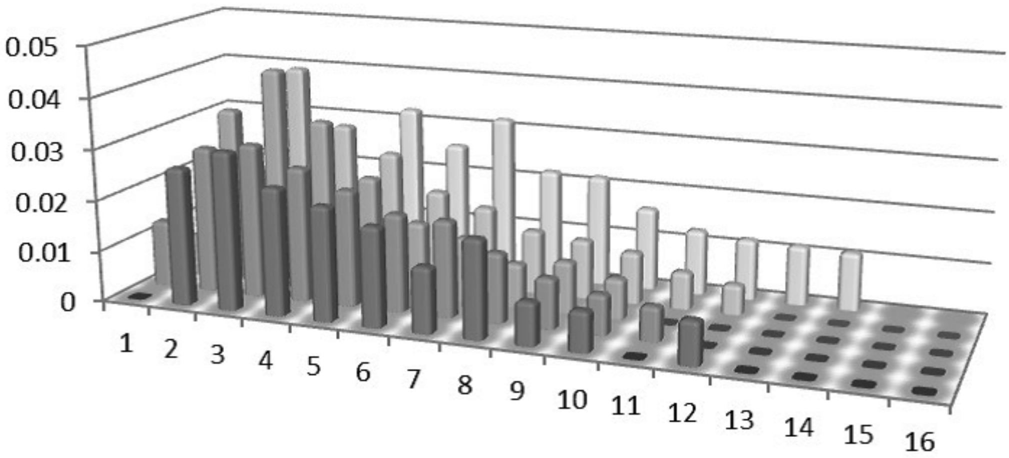

Fig. 3: Rectangular Graph

arguments owing to [5] yield that, under

$$\begin{cases} a = -10.1864 \\ b = 15.9004 \\ \gamma = 0.6660 \\ \delta = 0.6671 \\ \mu = 2.3588 \\ \sigma = 1.8366, \end{cases} \tag{3.2}$$

its fitting becomes the maximum, for the details see [9].

We also impose a condition that the numerical integral of $f(x, y)$ is nearly equal to 1. The graph of the function (3.1) with parameters (3.2) is given by Figure 4.

It is possible to derive many properties of the spatial diffusion through the melting and sheeting processes by using this fitting function.

Assume that one contaminated pellet containing, say a unit amount of, contaminant is put in the second melting process. The contaminant in the pellet diffuses, after melting and sheeting, over the sheet to be laminated and cut according to the function obtained by Figure 4. Noticing that a reproduced tray is of width 12cm × 20cm, we can compute the maximum amount of contaminant in a tray as

$$SDR = 0.037264 \tag{3.3}$$

(for the details see [9]), which is called *the Spatial Diffusion Rate.*

Let us now estimate the spatial diffusion in the first melting process. As discussed above, we should disregard the temporal diffusion of contaminated fragments. So we assume that 250 contaminated fragments are put simultaneously in the first melting stage. In addition,

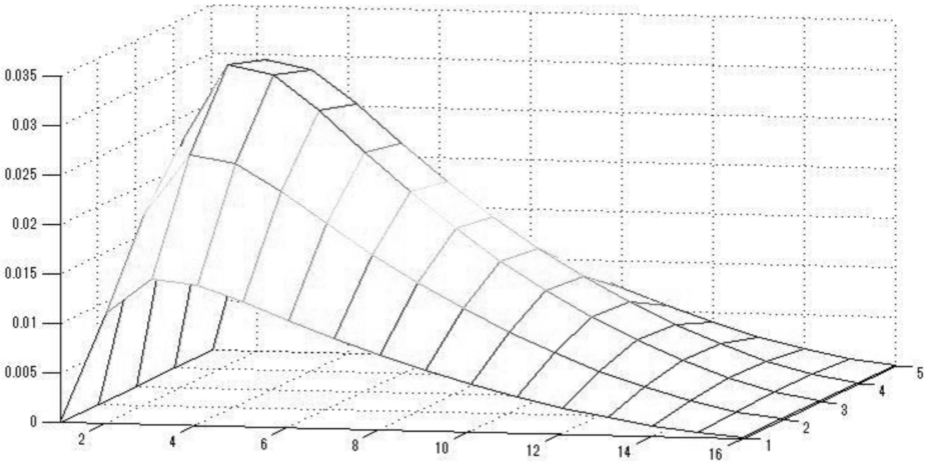

Fig. 4: Johnson Sb Distribution

we set an imaginary process of sheeting, namely, we consider that the fragments are once melted by a heater, the gel is extruded to form it into a sheet, and the sheet is processed into pellets. We therefore assume that a unit amount of contaminant is put in the melting and sheeting processes. Then its diffusion can estimated as above. The contaminant spreads over a sheet of width 35cm × 480cm and its distribution is given by the function (3.1) with parameters (3.2). Since one tray measures 12cm × 20cm and consists of almost 100 pellets, this sheet yields 70 trays, i.e., $7.0 \times 10^3$ pellets which are contaminated. In this way, a unit amount of contaminant diffuses over $7 \times 10^3$ pellets with some rate which depends on each pellet. It is, however, very difficult to estimate a distribution of rates over such a large number of pellets. So, considering the fact that the gel of polystyrene is stirred harder by the pelletizing extruder, we want to take a homogeneous distribution but over a little bit smaller number of pellets. In this paper, we set $6.0 \times 10^3$ contaminated pellets which contain a uniform amount of contaminant, namely,

$$n = 6.0 \times 10^3 \tag{3.4}$$

and all these pellets contain uniformly a $1/[6.0 \times 10^3]$ unit of contaminant.

**3.3 Combinatorial diffusion** Consider a collection of $N = 4.0 \times 10^7$ pellets which includes, according to (3.4), $n = 6.0 \times 10^3$ contaminated pellets. We divide these pellets randomly into $q = 4.0 \times 10^5$ sets of pellets which consist uniformly of $p = 100$ pellets.

More precisely, we study dispositions of the $N$ pellets into the $q \times p$ sites described by Figure 5. Let $X$ be a random variable which is defined as the maximum number of contaminated pellets through the all dividing sets for each disposition. That is, $X$ is a random variable defined on the sample space

$$\Omega = \{\text{all the permutations of the } N \text{ pellets into the } q \times p \text{ sites}\}.$$

The probability such that $X = k$, where $k = 1, 2, 3, \ldots, n$, can be computed by the following methods.

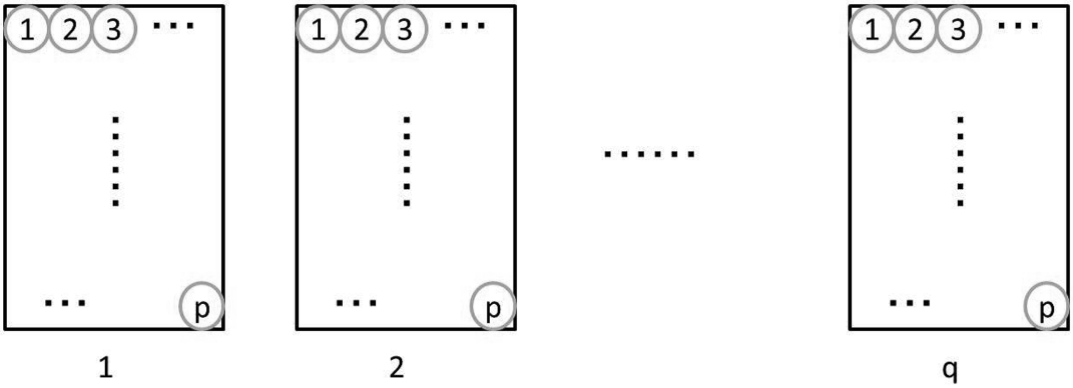

Fig. 5: Division

I. *Probability of* $X = 1$. The total number of elements of $\Omega$, namely, the total number of permutations of $N$ pellets is of course $N!$.

In the meantime, the number of permutations such that $X = 1$, namely, the number of permutations in which the $n$ contaminated pellets are completely disposed in different sets is computed by the following procedure:

1. First, we count the number of choice of $n$ sites for contaminated pellets. As for sets, we have ${}_qC_n$. For such a choice, each set has ${}_pC_1$ sites for a contaminated pellet. Therefore, it counts ${}_qC_n[{}_pC_1]^n$.
2. Let the $n$ sites for contaminated pellets be fixed as (1). Then there are $n!$ permutations of the contaminated pellets.
3. Let the sites for contaminated pellets be fixed as (1) and let the contaminated pellets be disposed as (2). Then the non-contaminated pellets are disposed by $(N-n)!$ ways.

We therefore conclude that

$$P(X=1) = \frac{{}_qC_n \cdot [{}_pC_1]^n \cdot n! \cdot (N-n)!}{N!} = \frac{p^n \cdot q! \cdot (N-n)!}{(q-n)! \cdot N!}. \tag{3.5}$$

By some calculations,

$$P(X=1) = \frac{pq}{N} \cdot \frac{p(q-1)}{N-1} \cdot \frac{p(q-2)}{N-2} \cdots \frac{p(q-n+1)}{N-n+1}.$$

This provides us a practical scheme for computing $P(X = 1)$ such that

$$\begin{cases} P_0 = \dfrac{pq}{N} = 1, \\ P_i = \dfrac{p(q-i)}{N-i} \cdot P_{i-1} \qquad (i = 1, 2, 3, \ldots, n-1). \end{cases}$$

It then results in

$$P(X=1) \approx 3.60565 \times 10^{-5}. \tag{3.6}$$

II. *Probability of* $X = 2$. Let us compute $P(X = 2)$. To this end, we introduce another random variable $X_2$ which denotes the number of sets including just two contaminated pellets for each permutation of $\Omega$. Let $x_2$ be a variable running from 1 to $\frac{n}{2}$. It is clear that

$$P(X = 2) = \sum_{x_2=1}^{\frac{n}{2}} P(X = 2,\, X_2 = x_2). \tag{3.7}$$

So it suffices to compute $P(X = 2,\, X_2 = x_2)$.

Then each $P(X = 2,\, X_2 = x_2)$ can be obtained by the following procedure:

1. First, compute the number of choice of $2x_2$ sites at which the double contaminated pellets are disposed. Of course, the choice of $x_2$ sets in which two contaminated pellets are disposed is ${}_qC_{x_2}$. For such a choice, the choice of two sites for contaminated pellets is ${}_pC_2$ per each set. Therefore, it counts ${}_qC_{x_2}[{}_pC_2]^{x_2}$.
2. Under (1), the permutations of $n$ pellets into the chosen $2x_2$ sites is ${}_nP_{2x_2}$.
3. Under (1) and (2), a collection of $N - 2x_2$ pellets (including $n - 2x_2$ contaminated ones) remains to be divided into $q$ sets. But any set other than those chosen in (1) must include at most one contaminated pellet. Then an analogous procedure to that explained above is available to compute the number of such permutations. Indeed, we have ${}_{q-x_2}C_{n-2x_2} \cdot [{}_pC_1]^{n-2x_2} \cdot (n - 2x_2)! \cdot (N - n)!$.

It then follows that

$$\begin{aligned} &P(X = 2,\, X_2 = x_2) \\ &\quad = \frac{{}_qC_{x_2} \cdot [{}_pC_2]^{x_2} \cdot {}_nP_{2x_2} \cdot {}_{q-x_2}C_{n-2x_2} \cdot [{}_pC_1]^{n-2x_2} \cdot (n - 2x_2)! \cdot (N - n)!}{N!} \\ &\quad = \frac{p^{n-x_2} \cdot (p-1)^{x_2} \cdot q! \cdot n! \cdot (N-n)!}{2^{x_2} \cdot x_2! \cdot (q - n + x_2)! \cdot (n - 2x_2)! \cdot N!}. \end{aligned}$$

It is easy to verify the following recurrence formula for $x_2$:

$$\begin{cases} P(X = 2,\, X_2 = 0) = P(X = 1), \\ P(X = 2,\, X_2 = x_2) = \dfrac{(p-1)(n - 2x_2 + 2)(n - 2x_2 + 1)}{2px_2(q - n + x_2)} \\ \qquad\qquad \times P(X = 2,\, X_2 = x_2 - 1) \qquad \left(x_2 = 1, 2, 3, \ldots, \frac{n}{2}\right). \end{cases}$$

Using this formula we can compute $P(X = 2,\, X_2 = x_2)$ for all $x_1 = 1, 2, 3, \ldots, \frac{n}{2}$. Then $P(X = 2)$ is obtained by the summation (3.7). Indeed,

$$P(X = 2) \approx 8.05853 \times 10^{-1}. \tag{3.8}$$

III. *Probability of* $X = 3$. We introduce a further random variable $X_3$ which denotes the number of sets including just three contaminated pellets for each permutation of $\Omega$. Let $x_3$ be a variable running from 1 to $\frac{n}{3}$. Then,

$$P(X = 3) = \sum_{\substack{1 \le x_3 \le \frac{n}{3} \\ 3 \le 2x_2 + 3x_3 \le n}} P(X = 3, X_3 = x_3,\, X_2 = x_2). \tag{3.9}$$

So let us compute $P(X = 3, X_3 = x_3, X_2 = x_2)$ for every pair $(x_3, x_2)$ such that $1 \leq x_3 \leq \frac{n}{3}$ and $3 \leq 2x_2 + 3x_3 \leq n$.

1. First, as before, compute the number of choice of $3x_3$ sites at which the triple contaminated pellets are disposed. The choice of $x_3$ sets in which three contaminated pellets are disposed is ${}_qC_{x_3}$. For such a choice, the choice of three sites for contaminated pellets is ${}_pC_3$ per each set. Therefore, it counts ${}_qC_{x_3}[{}_pC_3]^{x_3}$.
2. Under (1), the permutations of $n$ pellets into the chosen $3x_3$ sites is ${}_nP_{3x_3}$.
3. Under (1) and (2), a collection of $N - 3x_3$ pellets (including $n - 3x_3$ contaminated ones) remains to be divided into $q$ sets. But any set other than those chosen in (1) must include at most two contaminated pellets. Then an analogous procedure to that for the case where $X = 2$ is available to compute the number of such permutations. Indeed, we have

$$\begin{aligned}&{}_{q-x_3}C_{x_2} \cdot [{}_pC_2]^{x_2} \cdot {}_{n-3x_3}P_{2x_2} \cdot {}_{q-x_3-x_2}C_{n-3x_3-2x_2}[{}_pC_1]^{n-3x_3-2x_2}\\ &\qquad\qquad\qquad\qquad\qquad \times (n - 3x_3 - 2x_2)! \cdot (N - n)!.\end{aligned}$$

It then follows that

$$\begin{aligned}&P(X = 3, X_3 = x_3, X_2 = x_2)\\ &\quad = \big\{{}_qC_{x_3}[{}_pC_3]^{x_3} \cdot {}_nP_{3x_3} \cdot {}_{q-x_3}C_{x_2} \cdot [{}_pC_2]^{x_2} \cdot {}_{n-3x_3}P_{2x_2} \cdot {}_{q-x_3-x_2}C_{n-3x_3-2x_2}\\ &\qquad\quad \times [{}_pC_1]^{n-3x_3-2x_2} \cdot (n - 3x_3 - 2x_2)! \cdot (N - n)!\big\}\big/N!\\ &\quad = \frac{p^{n-2x_3-x_2} \cdot (p-1)^{x_3+x_2} \cdot (p-2)^{x_3} \cdot q! \cdot n! \cdot (N-n)!}{6^{x_3} \cdot 2^{x_2} \cdot x_3! \cdot x_2! \cdot (q - n + 2x_3 + x_2)! \cdot (n - 3x_3 - 2x_2)! \cdot N!}.\end{aligned}$$

To compute $P(X = 3)$ in an easy way, we rewrite (3.9) into

$$P(X = 3) = \sum_{x_2=0}^{\frac{n}{2}-2} \sum_{x_3=1}^{\left[\frac{n-2x_2}{3}\right]} P(X = 3, X_3 = x_3, X_2 = x_2), \tag{3.10}$$

where $\left[\frac{n-2x_2}{3}\right]$ denotes the integer part of $\frac{n-2x_2}{3}$, i.e., $0 \leq \frac{n-2x_2}{3} - \left[\frac{n-2x_2}{3}\right] < 1$. Then, for each fixed $x_2 = 0, 1, 2, \ldots, \frac{n}{2} - 2$, we verify the following recurrence formula for $x_3$:

$$\begin{cases} P(X = 3, X_3 = 0, X_2 = x_2) = P(X = 2, X_2 = x_2),\\ P(X = 3, X_3 = x_3, X_2 = x_2)\\ \quad = \dfrac{(p-1)(p-2)(n - 3x_3 - 2x_2 + 1)(n - 3x_3 - 2x_2 + 2)(n - 3x_3 - 2x_2 + 3)}{6p^2 x_3 (q - n + 2x_3 + x_2 - 1)(q - n + 2x_3 + x_2)}\\ \quad \times P(X = 3, X_3 = x_3 - 1, X_2 = x_2) \qquad \left(x_3 = 1, 2, 3, \ldots, \left[\frac{n-2x_2}{3}\right]\right). \end{cases}$$

For each fixed $0 \leq x_2 \leq \frac{n}{2} - 2$, we first compute the summation of the probabilities $P(X = 3, X_3 = x_3, X_2 = x_2)$ for $1 \leq x_3 \leq \left[\frac{n-2x_2}{3}\right]$. Then by the formula (3.10), we compute $P(X = 3)$. It then results in

$$P(X = 3) \approx 1.93364 \times 10^{-1}. \tag{3.11}$$

IV. *Probability of* $X = k$ *for* $k \geq 4$. By the similar procedures, we can develop our methods of computation for the cases where $k = 4, 5, 6, \dots, p$, and using those we can in fact compute $P(X = k)$ for all these $k$. For instance, we have

$$P(X = 4) \approx 1.07083 \times 10^{-3}. \tag{3.12}$$

By the way, in view of (3.6), (3.8), (3.11) and (3.12), we immediately verify that

$$P(X = 5) < 1 - \sum_{k=1}^{4} P(X = k) \approx 7.13 \times 10^{-4}. \tag{3.13}$$

Finally, let us consider the worst disposition that the $n$ contaminated pellets are divided into just $r = n/p = 60$ sets which therefore consist of entirely contaminated pellets. First, compute the number of choice of sites. Clearly, the number of choice of sets is ${}_qC_r$ which equals to that of choice of sites. The permutation of $n$ pellets to these chosen suites is $n!$. The permutation of non contaminated pellets is $(N - n)!$. Therefore,

$$P(X = p,\, X_p = r,\, X_{p-1} = \cdots = X_2 = 0) = \frac{{}_qC_r \cdot n! \cdot (N-n)!}{N!} = \frac{q! \cdot n! \cdot (N-n)!}{r! \cdot (q-r)! \cdot N!}.$$

In view of (3.5) we have

$$P(X = p,\, X_p = r,\, X_{p-1} = \cdots = X_2 = 0) = \frac{n! \cdot (q-n)!}{p^n \cdot r! \cdot (q-r)!}\, P(X = 1).$$

Here,

$$\frac{n! \cdot (q-n)!}{r! \cdot (q-r)!} = \frac{n(n-1)(n-2)\cdots[n-(n-r-1)]}{(q-r)(q-r-1)(q-r-2)\cdots[q-r-(n-r-1)]}$$

and

$$\frac{n}{q-r} > \frac{n-1}{q-r-1} > \frac{n-2}{q-r-2} > \cdots > \frac{n-(n-r-1)}{q-r-(n-r-1)}.$$

Since $\frac{n}{q-r} = \frac{600}{39994} < \frac{1}{60}$, we see that

$$P(X = p,\, X_p = r,\, X_{p-1} = \cdots = X_2 = 0) < \frac{1}{6^{n-r} \times 10^{3n-r}}\, P(X = 1), \tag{3.14}$$

which is an extremely small number.

**4 Conclusion** We have obtained the following results on diffusion rate of contaminant in the recycling line sketched by Figure 1.

Assume that one tray containing a unit amount of contaminant has entered the production line. Through the crashing, washing, melting and pelletizing processes, the contaminant diffuses into a certain number of pellets which is a unit grain of polystyrene of uniformed size to reproduce the new trays. By the experiment of pursuing some number of colored fragments of tray inserted in the line (Figure 2), we know that the temporal diffusion must be disregarded, although the contaminant spreads over a certain number, say $n$, of pellets. The $n$ contaminated pellets must be packed in a single packing box.

By the experiment of measuring magnitude of the X-ray in each part of the resultant sheet formed by a heating and sheeting machine (Figure 3), we know that it is reasonable to assume that $n$ is $6 \times 10^3$ and the $n$ contaminated pellets have a unified amount of contaminant, namely, $1/[6 \times 10^3]$ unit.

By the addition of three boxes of virgin pellets, we have a collection of $N = 4.0 \times 10^7$ pellets which includes the $n$ contaminant pellets. Through the blending and setting processes, these pellets are randomly divided into $q$ sets which consist uniformly of $p = 100$ pellets and yield just one new tray. Consequently, we have $q = 4.0 \times 10^5$, i.e., $N = pq$. Diffusion of the $n$ contaminated pellets over the $q$ sets can be known by the using the theory of combinatorial probability. Introduce a random variable $X$ which denotes the maximum number of contaminated pellets in a set through the $q$ sets in these divisions. Of course, $X$ takes a value $k$ from 1 to $p$. The probability of $X = k$ which is denoted by $P(X = k)$ can exactly be computed. For $k = 1, 2, 3, 4$ and 5, its approximate value or its estimate of value is given by (3.6), (3.8), (3.11), (3.12) and (3.13), respectively.

Consider a case of $X = k$ which takes place at probability $P(X = k)$. Then the sets containing $k$ contaminated pellets yield one recycling tray through the melting and sheeting processes. According to (3.3), the contaminant in a pellet diffuses in an area of sheet which corresponds to one tray at most with rate $\mathit{SDR} = 0.037264$. Therefore the recycling trays yielded by these sets are feared to contain at most contaminant of amount

$$\mathit{TDR} = \frac{1}{6.0 \times 10^3} \times 0.037264 \times k = \frac{k}{1.6101 \times 10^5}$$

unit. We then want to call this rate *the Total Diffusion Rate.*

The most favorable case is that $X = 1$. In this case, $\mathit{TDR}$ takes its minimum $1/[1.6101 \times 10^5]$, but as seen by (3.6) the probability is very small. The probability that either $X = 2$ or $X = 3$ takes place reaches to higher than 0.999. In these cases we have $\mathit{TDR} = 1/[8.0505 \times 10^4]$ or $1/[5.3670 \times 10^4]$, respectively. The worst case with realistic occurring probability might be, in view of (3.13), the case of $X = 5$. In this case, we have $\mathit{TDR} = 1/[3.2202 \times 10^4]$. To the contrary, the theoretically worst case is that $X = p\ (= 100)$. In such a case, $\mathit{TDR}$ attains its minimum $1/[1.6101 \times 10^3]$, but as seen by (3.14), its occurring probability is extremely small.

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# Submission to the SCMJ

In September 2012, the way of submission to Scientiae Mathematicae Japonicae (SCMJ) was changed. Submissions should be sent electronically (in PDF file) to the editorial office of International Society for Mathematical Sciences (ISMS).

(1) Preparation of files and Submission

a. Authors who would like to submit their papers to the SCMJ should make source files of their papers in LaTeX2e using the ISMS style file (scmjlt2e.sty) Submissions should be in PDF file compiled from the source files. Send the PDF file to s1bmt@jams.jp .

b. Prepare a Submission Form and send it to the ISMS. The required items to be contained in the form are:

1. Editor's name whom the author chooses from the Editorial Board (http://www.jams.or.jp/hp/submission_f.html )and would like to take in charge of the paper for refereeing.
2. Title of the paper.
3. Authors' names.
4. Corresponding author's name, e-mail address and postal address (affiliation).
5. Membership number in case the author is an ISMS member.

Japanese authors should write 3 and 4 both in English and in Japanese.

At http://www.jams.or.jp/hp/submission_f.html, the author can find the Submission Form. Fulfill the Form and sent it to the editorial office by pushing the button "transmission". Or, without using the Form, the author may send an e-mail containing the items 1-5 to s1bmt@jams.jp

(2) Registration of Papers

When the editorial office receives both a PDF file of a submitted paper and a Submission Form, we register the paper. We inform the author of the registration number and the received date. At the same time, we send the PDF file to the editor whom the author chooses in the Submission Form and request him/her to begin the process of refereeing. (Authors need not send their papers to the editor they choose.)

(3) Reviewing Process

a. The editor who receives, from the editorial office, the PDF file and the request of starting the reviewing process, he/she will find an appropriate referee for the paper.

b. The referee sends a report to the editor. When revision of the paper is necessary, the editor informs the author of the referee's opinion.

c. Based on the referee report, the editor sends his/her decision (acceptance of rejection) to the editorial office.

(4) a. Managing Editor of the SCMJ makes the final decision to the paper valuing the editor's decision, and informs it to the author.

b. When the paper is accepted, we ask the author to send us a source file and a PDF file of the final manuscript.

c. The publication charges for the ISMS members are free if the membership dues have been paid without delay. If the authors of the accepted papers are not the ISMS members, they should become ISMS members and pay ¥6,000 (US$75, Euro55) as the membership dues for a year, or should just pay the same amount without becoming the members.

## Items required in Submission Form

1. Editor's name who the authors wish will take in charge of the paper
2. Title of the paper
3. Authors' names

3'. 3. in Japanese for Japanese authors

4. Corresponding author's name and postal address (affiliation)

4'. 4. in Japanese for Japanese authors

5. ISMS membership number
6. E-mail address

# Call for ISMS Members

## Call for Academic and Institutional Members

**Discounted subscription price:** When organizations become the Academic and Institutional Members of the ISMS, they can subscribe our journal Scientiae Mathematicae Japonicae at the yearly price of US$225. At this price, they can add the subscription of the online version upon their request.

**Invitation of two associate members:** We would like to invite two persons from the organizations to the associate members with no membership fees. The two persons will enjoy almost the same privileges as the individual members. Although the associate members cannot have their own ID Name and Password to read the online version of SCMJ, they can read the online version of SCMJ at their organization.

To apply for the Academic and Institutional Member of the ISMS, please use the following application form.

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### Application for Academic and Institutional Member of ISMS

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| **Subscription of SCMJ**<br>Check one of the two. | ☐Print (US$225)<br>☐Print ＋ Online (US$225) |
| **University (Institution)** | |
| **Department** | |
| **Postal Address**<br>where SCMJ should be sent | |
| **E-mail address** | |
| **Person in charge** | Name:<br>Signature: |
| **Payment**<br>Check one of the two. | ☐Bank transfer ☐Credit Card (Visa, Master) |
| **Name of Associate Membership** | 1. |
| | 2. |

## Call for Individual Members

We call for individual members. The privileges to them and the membership dues are shown in “**Join ISMS !**” on the inside of the back cover.

### Items required in Membership Application Form

1. Name
2. Birth date
3. Academic background
4. Affiliation
5. 4’s address
6. Doctorate
7. Contact address
8. E-mail address
9. Special fields
10. Membership category (See **Table 1** in “**Join ISMS !**”)

### Individual Membership Application Form

| | |
|---|---|
| 1. **Name** | |
| 2. **Birth date** | |
| 3. **Academic background** | |
| 4. **Affiliation** | |
| 5. **4’s address** | |
| 6. **Doctorate** | |
| 7. **Contact address** | |
| 8. **E-mail address** | |
| 9. **Special fields** | |
| 10. **Membership category** | |

# Contributions (Gift to the ISMS)

We deeply appreciate your generous contributions to support the activities of our society.

The donation are used (1) to make medals for the new prizes (Kitagawa Prize, Kunugi Prize, and ISMS Prize), (2) to support the IVMS at Osaka University Nakanoshima Center, and (3) for a special fund designated by the contributors.

Your remittance to the following accounts of ours will be very much appreciated.

(1) Through a post office, remit to our giro account ( in Yen only ):
No. 00930-1-11872, Japanese Association of Mathematical Sciences (JAMS )
or send International Postal Money Order (in US Dollar or in Yen) to our address:
International Society for Mathematical Sciences
2-1-18 Minami Hanadaguchi, Sakai-ku, Sakai, Osaka 590-0075, Japan

(2) A/C 94103518, ISMS
CITIBANK, Japan Ltd., Shinsaibashi Branch
Midosuji Diamond Building
2-1-2 Nishi Shinsaibashi, Chuo-ku, Osaka 542-0086, Japan

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### Payment Instructions:

Payment can be made through a post office or a bank, or by credit card. Members may choose the most convenient way of remittance. Please note that we do not accept payment by bank drafts (checks). For more information, please refer to an invoice.

### Methods of Overseas Payment:

Payment can be made through (1) a post office, (2) a bank, (3) by credit card, or (4) UNESCO Coupons.

Authors or members may choose the most convenient way of remittance as are shown below. Please note that **we do not accept payment by bank drafts (checks)**.

(1) Remittance through a post office to our giro account No. 00930-1-11872 or send International Postal Money Order to our postal address (2) Remittance through a bank to our account No. 94103518 at Shinsaibashi Branch of CITIBANK (3) **Payment by credit cards** (AMEX, VISA, MASTER or NICOS), or (4) Payment by UNESCO Coupons.

### Methods of Domestic Payment:

Make remittance to:

(1) Post Office Transfer Account - 00930-3-73982 or

(2) Account No.7726251 at Sakai Branch, SUMITOMO MITSUI BANKING CORPORATION, Sakai, Osaka, Japan.

All of the correspondences concerning subscriptions, back numbers, individual and institutional memberships, should be addressed to the Publications Department, International Society for Mathematical Sciences.

# Join ISMS !

**ISMS Publications**: We published **Mathematica Japonica (M.J.)** in print, which was first published in 1948 and has gained an international reputation in about sixty years, and its offshoot **Scientiae Mathematicae (SCM)** both online and in print. In January 2001, the two publications were unified and changed to **Scientiae Mathematicae Japonicae (SCMJ)**, which is the "21st Century New Unified Series of Mathematica Japonica and Scientiae Mathematicae" and published both online and in print. Ahead of this, the online version of SCMJ was first published in September 2000. The whole number of SCMJ exceeds 270, which is the largest amount in the publications of mathematical sciences in Japan. The features of SCMJ are:

1) About 80 eminent professors and researchers of not only Japan but also 20 foreign countries join the Editorial Board. The accepted papers are published both online and in print. SCMJ is reviewed by Mathematical Review and Zentralblatt from cover to cover.
2) SCMJ is distributed to many libraries of the world. The papers in SCMJ are introduced to the relevant research groups for the positive exchanges between researchers.
3) **ISMS Annual Meeting:** Many researchers of ISMS members and non-members gather and take time to make presentations and discussions in their research groups every year.

**The privileges to the individual ISMS Members:**

(1) No publication charges
(2) Free access (**including printing out**) to the online version of SCMJ
(3) Free copy of each printed issue

**The privileges to the Institutional Members:**

Two associate members can be registered, free of charge, from an institution.

**Table 1:** Membership Dues for 2019

| **Categories** | **Domestic** | **Overseas** | **Developing countries** |
|---|---|---|---|
| 1-year Regular member | ￥8,000 | US$80， Euro75 | US$50, Euro47 |
| 1-year Students member | ￥4,000 | US$50， Euro47 | US$30， Euro28 |
| Life member* | Calculated as below* | US$750， Euro710 | US$440, Euro416 |
| Honorary member | Free | Free | Free |

(Regarding submitted papers,we apply above presented new fee after April 15 in 2015 on registoration date.) * Regular member between 63 - 73 years old can apply the category.
(73－age ) × ￥3,000
Regular member over 73 years old can maintain the qualification and the privileges of the ISMS members, if they wish.

Categories of 3-year members were abolished.

# CONTENTS

## Notices from the ISMS