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TOTAL FACILITY CONSTRUCTION PLANNING PROBLEM

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Abstract

This paper consider the following construction problem with various type facilities, i.e., type 1: an emergency facility, type 2:semi-obnoxious one, type 3: welcome one, type 4: not so far but not so near one and type 5: supply center of school lunch. There are finite possible construction sites $F_1, F_2, \cdots, F_n$ in a rectangular area $U = \{a \le x \le b, c \le y \le d\}$ and construction cost c_{ij} depends on the construction site F_j and facility type $T_i, i = 1, 2, 3, 4, 5, j = 1, 2, \ldots, n$. We use A-distance and except construction cost, for T_i,weighted maximum distance from the emergency facility to the hospital via accident site $D_{1,j}$ to be minimized among $j = 1, 2, ..., n$ For $T_\ell, \ell = 2, 3, 4$. the minimal satisfaction degree μ_{ij} with respect to the membership function about A-distance from the facility site to be maximized among $j = 1, 2, \ldots, n$ and for T_5, the latest lunch delivery time t_{5j} of schools among possible construction site F_j should be minimized. Main problem is as follows.

Each facility $T_i, i = 1, 2, 3, 4, 5$, is constructed at just one possible site so that total construction cost and weighted total sum $w_1 M_{1j_1} + w_2 M_{2j_2} + \cdots + w_5 M_{5j_5}$should be minimized where j_k is construction site of $T_k, k = 1, 2, 3, 4, 5.$. This problem becomes bi-criteria problem and we seek non-dominated solutions. Finally we discuss further research problems.

1 Introduction

There are huge amount of papers on facility location problems since Weber published his paper [?] and Hamacher et. al. [?] tried to classify them by using the similar codes to queuing and scheduling problems.

We considered many models on emergency facility location problem ([?], [?], [?], [?], [?]) and proposed an extended model of them. This paper is organized as follows. Section 2 formulates our model and defines non-dominated solutions. Section 3 proposes solution procedures to seek some non-dominated solutions. Section 4 summarizes results of our paper and discusses further research problems.

2 Problem formulation

We consider the following construction problem with various type facilities, i.e., type 1: an emergency facility, type 2:semi-obnoxious one (crematory, disposal center etc), type 3: welcome one (city hall), type 4: not so far but not so near one (shopping mall) and type 5: supply center of school lunch. There are finite possible construction sites $F_1, F_2, \cdots, F_n$ in a rectangular area $U = \{a \le x \le b, c \le y \le d\}$

and construction cost c_{ij} depends on the construction site F_j and facility type $T_i, i = 1,2,3,4,5, j = 1,2,\ldots,n$. We use A-distance and except construction cost for T_i,distance sum from the emergency facility to the hospital via accident site D_{1j} to be minimized among $j = 1,2,....,n$, For $T_\ell, \ell = 2,3,4$, the minimal satisfaction degree $\mu_{\ell j}$with respect to the membership function about A-distance from the facility site F_j to be maximized among $j = 1,2,...,n$,, and for T_5, latest lunch delivery time t_{5j} of schools among possible construction site F_j should be minimized. We denote D_{1j} as M_{1j}, $1 - \mu_{\ell j}, \ell = 2,3,,4$ and t_{5j} as M_{5j}. That is, corresponding to each facility type $k, k = 1,2,3,4,5$, we consider a sub-problem P_k and calculate $M_{kj}, j = 1,2,...,n$. First we review how to solve P_k and calculate $M_{kj}, j = 1,2,...,n$ in the next section. Main problem is as follows.

Each facility $T_i, i = 1,2,3,4,5$ is constructed at just one possible site so that total construction cost and weighted total sum $w_1M_{1j_1} + w_2M_{2j_2} + \cdots + w_5M_{5j_5}$should be minimized where j_k is construction site of $T_k, k = 1,2,3,4,5$. This problem is formulated as the following bi-criteria problem P.

$P : minimize\Sigma_{j=1}^n \Sigma_{i=1}^5 c_{ij}x_{ij}, : minimize\Sigma_{j=1}^n \Sigma_{i=1}^5, w_iM_{ij}x_{ij}$
subject to $\Sigma_{i=1}^5 x_{ij} = 1, j = 1,2,...,n, \Sigma_{j=1}^n x_{ij} = 1, i = 1,2,3,4,5,$
$x_{ij} = 0$ or $1, i = 1,2,3,4,5, j = 1,2,...,n$

3 Solution Procedure

(**A distance**) ([**?**])There exists a set of directions $A = \{\alpha_1, \alpha_2, \cdots, \alpha_a\}$ where $\alpha_i, i = 1,2,....,a$ is an angle from x axis in an orthogonal coordinate and let $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_a < 180^o$. Hereafter if no confusion occurs, directions $\alpha_i, i = 1,2,...,a$ and angles $\alpha_i, i = 1,2,...,a$ are used as the same meaning. Directions $\alpha_j.\alpha_{j+1}$ are called neighboring where $\alpha_a, \alpha_1$are also called neighboring, that is, α_{a+1} is interpreted as α_1.Further A line, a half line and a line segment are called A-directional (or A-oriented) if their directions are ones of $\alpha_i, i = 1,2,...,a$. Then A distance between two points$(p^1, p^2) \in R^2$ are defined as follows

$$d_A(p^1, p^2) = \begin{cases} d_2(p^1, p^2) & \text{if direction } \overline{p^1p^2} \text{ is A-oriented} \\ min\{d_A(p^1, p^3) + d_A(p^3, p^2) | p^3 \in R^2\} & \text{Otherwise} \end{cases} \tag{1}$$

where $d_2(p^1, p^2)$is the Euclidian distance between (p^1, p^2). That is, according to the results in [**?**], when $\alpha_j <$an angle of the line connecting demand point i with the facility site $(x, y) < \alpha_{j+1}$,

$d_i = M_1|m_2(p_i - x) - (q_i - y)| + M_2|m_1(p_i - x) - (q_i - y)|$ where $m_1 = max(tan\alpha_j, tan\alpha_{j+1}), m_2 = min(tan\alpha_j, tan\alpha_{j+1}), M_1 = \frac{\sqrt{1+m_1^2}}{m_1-m_2}, M_2 = \frac{\sqrt{1+m_2^2}}{m_1-m_2}$. If either α_j or α_{j+1} is 90^o, then we interpret $M_1 = \lim_{m_1\to\infty} \frac{\sqrt{1+m_1^2}}{m_1-m_2} = 1, M_2|m_1(p_i - x) - (q_i - y)| = \lim_{m_1\to\infty} \frac{\sqrt{1+m_2^2}}{m_1-m_2}|m_1(p_i - x) - (q_i - y)| = \sqrt{1+m_2^2}|p_i - x|$.

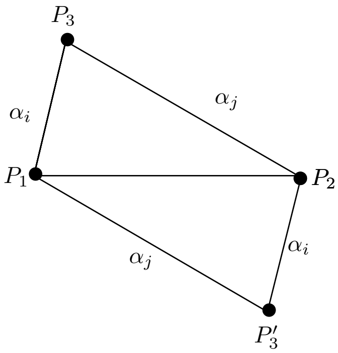

Fig.1 A-distance between P^1 and P^2

First we show the facility location of type 1. That is, an emergency facility problem P_1 as follows: There exist m hospitals $H_1, H_2, \cdots, H_m$. If an accident occurs, the ambulance cars in the facility site p rushes to the scene of accident and bring the injured persons to the nearest hospitals as soon as possible. Demand points (possible accident occurrence points) are distributed uniformly in U in a rectangular area. Let $S(Q)$ denotes the nearest hospital to the point $Q \in U$. Then the distance sum from p is $R(p,Q) = d_A(p,Q) + d_A(Q,S(Q))$ and $R(p) = max\{R(p,Q)|Q \in U\}$ should be minimized among $p \in F_1.F_2, \cdots, F_n$. $d_A(Q,S(Q))$ is calculated as below using Voronoi diagram with respect to hospitals $H_1, H_2, \cdots, H_m$.

(**Voronoi diagram**)

For a set of s points $V_1, V_2, \cdots, V_s$, Voronoi polygon $V_A(V_i)$ on point V_i with respect to A-distance on X is defined as follows:

$$V_A(V_i) = \cap_{j \neq i}\{Q|d_A(Q,V_i) \leq d_A(Q,V_j), Q \in X\}$$

The set of all Voronoi polygons for the points in V is a partition of some region on a plane X. Edge of Voronoi polygon is called Voronoi edge. Then we construct Voronoi diagram $VD_A(\mathbf{H})$ with respect to the set of hospital points $\mathbf{H} = H_1, H_2, \cdots, H_m$ and A-distance on the area X. To construct Voronoi diagram is done in $O(mlogm)$ computational time ([**?**]). Figure 2 illustrates Voronoi diagram in this case. According to the Theorem in [[**?**]], maximizer among Q with respect to $R(p)$ is one of the following points (a),(b) in Theorem 1.

Theorem 1.

(a) The intersection points of boundary of U and the Voronoi diagram

(b) Vertices of Voronoi diagram.

Let these points of Theorem 1 be $p_1^1, p_2^1, \cdots, p_{n_i}^1$

Then we can calculate $R(F_j), j = 1, 2, ..., n$ as follows.

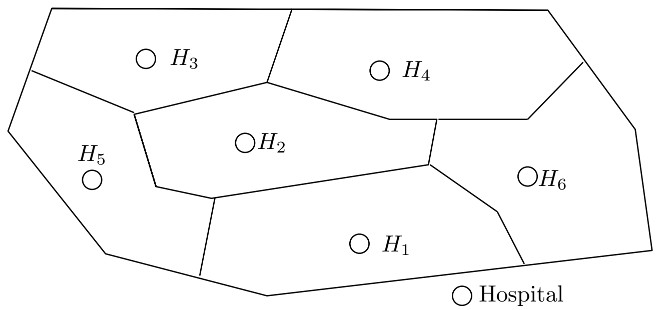

Fig.2 Voronoi diagram with respect to hospitals $H_1, H_2, H_3, H_4, H_5, H_6$

$R(F_j) = Max\{R(F_j, p_k^1)|k = 1, 2, ..., n_1\}, j = 1, 2, ..., n$

The optimal solution of P_1 is a minimizer of $min\{R(F_j), j = 1, 2, ..., n\}$. Next for type 2, 3, 4 facility, we consider A-distance $d_A(i, p)$ from demand point $p = (x, y) \in U$ to the facility site $F_i = (p_i, q_i)$ which is calculated as follows:

From the above results, when α <an angle of the line connecting demand point (x, y) with the facility site $F_i < \alpha_{i+1}$, $d_A(i, p) = M_1|m_2(p_i - x) - (q_i - y)| + M_2|m_1(p_i - x) - (q_i - y)|$ where $m_1 = max(tan\alpha_j, tan\alpha_{j+1}), m_2 = min(tan\alpha_j, tan\alpha_{j+1}), M_1 = \frac{\sqrt{1+m_1^2}}{m_1-m_2}, M_2 = \frac{\sqrt{1+m_2^2}}{m_1-m_2}$. Therefore A-distance $d_A(i, p)$ from demand points to facility site $F_i = min\{z_1^i, z_2^i, \cdots, z_a^i\}$ where $z_j^i, j = 1, 2, ..., a$ is the optimal value of following problem Z_j:

$Z_j : Minimizez$

subject to $M_1|m_2(p_i - x) - (q_i - y)| + M_2|m_1(p_i - x) - (q_i - y)| < z$

$m_2|x - p_i| \leq |y - q_i| \leq m_1|x - p_i|, (x, y) \in U$

This is a linear programming problem basically. For T_2, we consider the following membership function.

$$\mu_2(p) = \begin{cases} 0 & (d_A(i,p) \leq a_i) \\ \frac{d_A(i,p)-a_i}{b_i-a_i} & (a_i \leq d_A(i,p) \leq b_i) \\ 1 & (d_A(i,p) \geq b_i) \end{cases} \tag{2}$$

where $a_i < b_i$and subproblem is to maximize $\mu_{2i} = min\{\mu_{2i}(p)|p \in U\}$ with respect to $F_i, i = 1, 2, ..., n$. Optimal solution is a maximizer of μ_{2i}. For T_3, we consider the following membership function.

$$\mu_{3i}(p) = \begin{cases} 0 & (d_A(i,p) \geq e_i) \\ 1 - \frac{d_A(i,p)-c_i}{e_i-c_i} & (c_i \leq d_A(i,p) \leq e_i) \\ 1 & (d_A(i,p) \leq c_i) \end{cases} \tag{3}$$

where $c_i < e_i$ and subproblem is to maximize $\mu_{3i} = min\{\mu_{3i}(p)|p \in U\}$ with respect to $F_i, i = 1, 2, ..., n$. Optimal solution is a maxmizer of μ_{3i}. For T_4, we consider the following membership function.

$$\mu_{4i}(p) = \begin{cases} 0 & (d_A(i,p) \leq a_i) \\ \frac{d_A(i,p)-a_i}{b_i-a_i} & (a_i \leq d_A(i,p) \leq b_i) \\ 1 & (b_i \leq d_A(i,p) \leq c_i) \\ 1 - \frac{d_A(i,p)-c_i}{e_i-c_i} & (c_i \leq d_A(i,p) \leq e_i) \\ 0 & (d_A(i,p) \geq e_i) \end{cases} \tag{4}$$

where $a_i < b_i < c_i < e_i$ and subproblem is to maximize $\mu_{4i} = min\{\mu_{4i}(p)|p \in U\}$ with respect to $F_i, i = 1, 2, ..., n$. Optimal solution is a maximizer of μ_{4i}. Finally consider the supply center of school lunch T_5 given as follows. There are s schools $S_1.S_2, \cdots, S_s$ in urban area U. We consider the construction site of new supply center providing lunch for these schools among n possible sites $F_1, F_2, \cdots, F_n$ The trader delivers ingredients to the supply center every morning. After receiving these ingredients,the supply center starts to make lunch. Lunch for all schools should be ready on delivery time. The delivery cars must deliver lunch to be in lunch time of each school. For that purpose, we divide schools into groups corresponding to r delivery cars. We choose the best site of the center by minimizing the latest delivery time of lunch among all schools. First we calculate A-distances $d_A(i, j)$ from each possible site $F_j, i = 1, 2, ..., n$ to each school .$S_j, j = 1, 2, ..., s$. Sorting $d_A(i, j)$ for each F_i let the result be $d_A(i, i(1) \leq d_A(i, i(2)) \leq \cdots d_A(i, I(s))$. Then for eacf F_i, we divide schools into r trucks as follows.Choose r longest distances and assign school $S_{i(is-t+1)} todeliverycars$TR(t),t=1,2,...,r.$Thenletbe\tilde{d}_A(, k) = 2d_A(i, i(k)), k = 1, 2, ..., s - r$. Next we divide schools to r group by the following steps.

Step 1:Set $B_{i(t)} = d_A(i, i(s-t+1)) + \tilde{d}_A(i, t), t = 1, 2, ..., r-1, B_{i(r)} = d_A(i, i(s-r+1)) + \tilde{d}_A(i, i(s-r)), , k = s - r$ and $G(t) = S_{i(s-t+1)}, t = 1, 2, ..., r-1, G(r) = \{S_{i(s-r+1}, S_{i(s-r)}\}$

Step 2: Let $k = k+!$. if $k = 0$,terminates .Otherwise go to Step3.

Step 3: Let $B(s) \leftarrow min\{B_{i(u)}|u = 1, 2, ..., r\}$ and its minimizer be $t(k)$. Then $B_{i(t(k))} = B_{i(t(k))} + \tilde{d}_A(i, i(k)), G(t(k)) = G(t(k)) \cup S_{i(k)}$. Return to Step2.

Note that final $B_{i(u)}$ divided by the standard speed describes the total delivery time using TR_u to group of schools $G(u), u = 1, 2, \cdots, r$. Though heuristic, the above dividing method tries to make burden even, that is, minimizing the maximum burden among delivery cars. Let the maximum burden using the above dividing method for candidate site $F_i$be $BM(i)$. Here we assume the staring time of making lunch is and so finishing time preparing lunch is also fixed and so minimizer of $BM(i), i = 1, 2, ..., n$ is an optimal solution of P_5.Here $M_{5i} = \frac{BM(i)}{SP} + CT$ where CT is a starting time to making school lunch and SP is a standard speed of delivery truck. Based on the above discussion, we have the main problem P. $P : minimize\Sigma_{j=1}^{n}\Sigma_{i=1}^{5}c_{ij}x_{ij}, : minimize\Sigma_{j=1}^{n}\Sigma_{i=1}^{5}, w_i M_{ij}x_{ij}$

subject to $\Sigma_{i=1}^{5}x_{ij} = 1, j = 1, 2, ..., n, \Sigma_{j=1}^{n}x_{ij} = 1, i = 1, 2, 3, 4, 5,$

$x_{ij} = 0or1, i = 1, 2, 3, 4, 5, j = 1, 2, ..., n$ But this problem has bi-criteria and so we define non-dominated solution now,

(**Non-dominated solution**)

For two solutions, $\mathbf{F}^1 = (F_{j_1^1}, F_{j_2^1}, F_{j_3^1}, F_{j_4^1}, F_{j_5^1},)$, that is,$(x_{ij_i^1} = 1, i = 1, 2, ..., 5, otherx_{ij} = 0)$ and $\mathbf{F}^2 = (F_{j_1^2}, F_{j_2^2}, F_{j_3^2}, F_{j_4^2}, F_{j_5^2},)$, that is, $(x_{ij_i^2} = 1, i = 1, 2, ..., 5, otherx_{ij} = 0)$ if

$\Sigma_{j=1}^{n}\Sigma_{i=1}^{5}c_{ij}x_{j_j^1} \leq \Sigma_{j=1}^{n}\Sigma_{i=1}^{5}c_{ij}x_{j_j^2}, \Sigma_{j=1}^{n}\Sigma_{i=1}^{5}W_i M_{ij}x_{j_j^1} \leq \Sigma_{j=1}^{n}\Sigma_{i=1}^{5}W_i M_{ij}x_{j_j^2}$ and at least one inequality holds strictly inequality, then we call solution $\mathbf{F^1}$ dominates $\mathbf{F^2}$. If there exists no solution dominates $\mathbf{F}$, $\mathbf{F}$ is called non-dominated solution.

We seek some non-dominated solutions. First non-dominated solution is obtained from the solution minimizing the total construction cost, that is, optimal solution of the following assignment problem AP: minimize $\Sigma_{j=1}^{n}\Sigma_{i=1}^{5}c_{ij}x_{ij}$

subject to $\Sigma_{i=1}^{5}x_{ij} = 1, j = 1, 2, ..., n, \Sigma_{j=1}^{n}x_{ij} = 1, i = 1, 2, 3, 4, 5,$

$x_{ij} = 0or1, i = 1, 2, 3, 4, 5, j = 1, 2, ..., n$

This problem is a special transportation problem with 5 supply nodes and n demand nodes where upper supply quantity is 1 and each demand quantity is at most 1. Unit transportation cost is 1. Usually dummy n-5 supply nodes with big transportation cost to demand nodes and only one possible supply quantity, Or among $O(n^5)$ solutions, we find an optimal solution, that is minimizer of $\Sigma_{j=1}^{n}\Sigma_{i=1}^{5}c_{ij}x_{ij}$ is a non-dominated solution.Similarly an optimal solution of the following another assignment problem AM.

AM: minimize $\Sigma_{j=1}^{n}\Sigma_{i=1}^{5}, w_i M_{ij} x_{ij}$

subject to $\Sigma_{i=1}^{5}x_{ij} = 1, j = 1, 2, ..., n, \Sigma_{j=1}^{n}x_{ij} = 1, i = 1, 2, 3, 4, 5,$

$x_{ij} = 0 or 1, i = 1, 2, 3, 4, 5, j = 1, 2, ..., n$

Another one is given as follows:

Let $WM_{ij} = w_i M_{ij}, i = 1, 2, ..., 5, j = 1, 2, ..., n$ and sort them for each i=1,2,...,5. Then results be set $\mathbf{WM^i} = \{WM_i(i(1)), WM_i(i(2)), WM_i(i(3)), \cdots, WM_i(i(n_i))\}$ where $WM(i(i(1) < WM(i(i(2) < WM(i(i3) < \cdots < WM(i(i(n_i)), i = 1, 2, ..., 5, n$ is the number of different ones. We choose cheaper disjoint five construction sites from the list $\mathbf{WM^i} i = 1, 2, ..., 5$,and resulting one is a non-dominated solution. Of course, if there exists the same value $WM_{i_\ell}(i(k)) = WM_{i_\varsigma}(i(\tau))$, we prefer cheaper construction cost one.

4 Conclusion

We have a total construction planning model of various type facilities. Of course many other type facilities should be considered and also the case of given facilities are important to total planning including some case closing old ones. Further in a financial aspect, some facilities will be establish in a same place and in turn, in urban area, many barriers exist where inside we cannot pass and so make detours. Therefore more realistic situation should considered when we total planning. Further scenario analysis that considers near future situation including enviromental aspects.

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ON SUBWEAKLY b-CONTINUOUS FUNCTIONS

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Abstract. The purpose of this paper is to introduce a new class functions called, subweakly b-continuous functions. Also, we obtain its characterizations and its basic properties.

1 Introduction Functions and of course open functions stand among the most important notions in the whole of mathematical science. Many different forms of continuous functions have been introduced over the years. Various interesting problems arise when one considers continuity. Its importance is significant in various areas of mathematics and related sciences. In 1996, Andrijevic [2] introduced a weak form of open sets called b-open sets. In the same year, this notion was also called *sp*-open sets in the sense of Dontchev and Przemski [6] but one year later are called γ-open sets due to El-Atik [14]. Ekici [3, 7, 8, 9, 10, 11, 12, 13] studied some papers related with b-open sets. In this paper, we will continue the study of related functions by involving b-open sets. We introduce and characterize the concept of subweakly b-continuous functions in topological spaces.

2 Preliminaries Throughout this paper, spaces means topological spaces on which no separation axioms are assumed unless otherwise mentioned and $f:(X,\tau)\to(Y,\sigma)$ (or simply $f:X\to Y$) denotes a function f of a space (X,τ) into a space (Y,σ). Let A be a subset of a space (X,τ). The closure and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively. The θ-closure [24] of A, denoted by $Cl_\theta(A)$, is defined to be the set of all $x\in X$ such that $A\cap Cl(U)\neq\emptyset$ for every open neighbourhood U of x. If $A=Cl_\theta(A)$, then A is called θ-closed. The complement of θ-closed set is called θ-open. A subset A of (X,τ) is said to be regular open [23](resp. semi-open [15], preopen [16], α-open [20], b-open [2] or γ-open [14]) if $A=Int(Cl(A))$ (resp. $A\subset Cl(Int(A)), A\subset Int(Cl(A)), A\subset Int(Cl(Int(A))), A\subset Int(Cl(A))\cup Cl(Int(A))$). The complement of a semi-open (resp. preopen, b-open) set is called semi-closed [5] (resp.preclosed [16], b-closed [2]). The intersection of all semi-closed (resp. preclosed, b-closed) sets containing A is called the semiclosure [4] (resp. preclosure [16], b-closure [2]) of A and is denoted by $sCl(A)$ (resp. $pCl(A), bCl(A)$). For each $x\in X$, the family of all b-open sets containing x is denoted by $BO(X,\tau;x)$. The family of all α-open (resp. b-open) sets of a topological space (X,τ) is denoted by $\alpha O(X,\tau)$ (resp. $BO(X,\tau)$). A function $f:(X,\tau)\to(Y,\sigma)$ is said to be *α-continuous* [17] if for every $x\in X$ and every open set V of Y containing $f(x)$, there exists an α-open set U containing x such that $f(U)\subset V$. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be *weakly b-continuous* [22] if for every $x\in X$ and every open set V of Y containing $f(x)$, there exists $U\in BO(X,\tau;x)$ such that $f(U)\subset Cl(V)$.

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Lemma 2.1 [2] *Let A be a subset of a topological space (X, τ).*
(i) $x \in bCl(A)$ *if and only if* $A \cap U \neq \emptyset$ *for every* $U \in BO(X, \tau; x)$.
(ii) *Any union of b-open sets is b-open.*
(iii) $bCl(A)$ *is b-closed.*
(iv) A *is b-closed if and only if* $A = bCl(A)$.

3 Subweakly b-continuous functions

Definition 3.1 A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be *subweakly b-continuous* if there exists an open base $\mathcal{B}$ for the topology σ on Y for which $bCl(f^{-1}(V)) \subset f^{-1}(Cl(V))$ for every $V \in \mathcal{B}$.

Remark 3.2 (i) It is clear that weak b-continuity implies subweak b-continuity.
(ii) The converse of the implication of (i) above is not true in general as it can be seen from the following example: let (X, τ) and (Y, σ) be the following topological spaces, where $X := \{a, b, c, d\} = Y$, $\tau := \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, X\}$ and $\sigma := P(Y)$ and $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function defined by $f(a) = f(b) := a$, $f(c) := b$, $f(d) := c$. Then, there exists an open base $\mathcal{B}$ of the topology σ on Y such that $bCl(f^{-1}(V)) \subset f^{-1}(Cl(V))$ for every set $V \in \mathcal{B}$,i.e., $f : (X, \tau) \rightarrow (Y, \sigma)$ is subweakly b-continuous. Indeed, we take $\mathcal{B} := \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, Y\}$ and we have that: $BO(X, \tau) = P(X) \setminus \{\{d\}\}$ and $BC(X, \tau) = P(X) \setminus \{\{a, b, c\}\}$. And, the function $f : (X, \tau) \rightarrow (Y, \sigma)$ is not weakly b-continuous. Indeed. there exist a point $d \in X$ and a set $V := \{c, d\} \in \sigma$ such that $f(d) = c \in V$ and $f(U) \not\subset Cl(V)$ for every set $U \in BO(X, \tau; d)$, where $BO(X, \tau; d)$= $\{\{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$.

Theorem 3.3 *A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is subweakly b-continuous if and only if there is an open base $\mathcal{B}$ for the topology σ on Y for which $Cl(Int(f^{-1}(V))) \cap Int(Cl(f^{-1}(V))) \subset f^{-1}(Cl(V))$ for every $V \in \mathcal{B}$.*

Proof. The proof is clear, because it is well known that $bCl(A) = A \cup (Cl(Int(A)) \cap Int(Cl(A)))$ holds for every set A of (X, τ) (cf. Proposition 2.5 and its proof in [2]). **q.e.d**

Theorem 3.4 *If a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is subweakly b-continuous, then for every θ-open* (resp. *θ-closed*) *set V of (Y, σ), $f^{-1}(V)$ is a union of b-closed sets* (resp. *an intersection of b-open sets*).

Proof. Let $\mathcal{B}$ be an open base for the topology σ on Y for which $bCl(f^{-1}(V)) \subset f^{-1}(Cl(V))$ for every $V \in \mathcal{B}$. Let V be a θ-open subset of (Y, σ) with $x \in f^{-1}(V)$. Then there exist $W \in \mathcal{B}$ such that $f(x) \in W \subset Cl(W) \subset V$. Then $x \in bCl(f^{-1}(W)) \subset f^{-1}(Cl(W)) \subset f^{-1}(V)$. By Lemma 2.1, $bCl(f^{-1}(W))$ is b-closed; $f^{-1}(V)$ is a union of b-closed sets. The second case is proved by an argument similar to the first case above. **q.e.d.**

Recall that by the graph of a function $f : X \rightarrow Y$, we mean that $G(f) := \{(x, y)|\ x \in X, y = f(x)\}$ and by the graph function of f, say $g : X \rightarrow Y$, we mean that $g(x) := (x, f(x))$.

Theorem 3.5 *If $f : (X, \tau) \rightarrow (Y, \sigma)$ is subweakly b-continuous and (Y, σ) is Hausdorff, then $G(f)$ is b-closed in $(X \times Y, \tau \times \sigma)$.*

Proof. Let $(x, y) \in (X \times Y) \setminus G(f)$. Then $y \neq f(x)$. Let $\mathcal{B}$ be an open base for the topology σ on Y for which $bCl(f^{-1}(V)) \subset f^{-1}(Cl(V))$ for every $V \in \mathcal{B}$. Since (Y, σ)

is Hausdorff, there exist disjoint open sets V and W in Y with $y \in V$, $f(x) \in W$, and $V \in \mathcal{B}$. Then $f(x) \notin Cl(V)$ and hence $x \notin f^{-1}(Cl(V))$. Since f is subweakly b-continuous, $bCl(f^{-1}(V)) \subset f^{-1}(Cl(V))$ and hence $x \notin bCl(f^{-1}(V))$. Then we see that $(x, y) \in (X \setminus bCl(f^{-1}(V))) \times V \subset (X \times Y) \setminus G(f)$. Then by Lemmas 2.1, we have that $G(f)$ is b-closed. **q.e.d**

Theorem 3.6 *If $f : (X, \tau) \to (Y, \sigma)$ is subweakly b-continuous, then the graph function $g : (X, \tau) \to (X \times Y, \tau \times \sigma)$ is subweakly b-continuous.*

Proof. Let $\mathcal{B}$ be an open base for the topology σ on Y for which $bCl(f^{-1}(V)) \subset f^{-1}(Cl(V))$ for every $V \in \mathcal{B}$. Then $\mathcal{C} := \{U \times V | U \subset X \text{ is open and } V \in \mathcal{B}\}$ is an open base for the product topology $\tau \times \sigma$ on $X \times Y$. For $U \times V \in \mathcal{C}$, we have $bCl(g^{-1}(U \times V)) = bCl(U \cap f^{-1}(V)) \subset bCl(U) \cap bCl(f^{-1}(V)) \subset Cl(U) \cap f^{-1}(Cl(V)) = g^{-1}(Cl(U) \times Cl(V)) = g^{-1}(Cl(U \times V))$. Thus, the graph function $g : (X, \tau) \to (X \times Y, \tau \times \sigma)$ is subweakly b-continuous. **q.e.d.**

Theorem 3.7 *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $g : (X, \tau) \to (X \times Y, \tau \times \sigma)$ its graph function. Let $\mathcal{B}$ be an open base for the topology σ on Y. If $g : (X, \tau) \to (X \times Y, \tau \times \sigma)$ is subweakly b-continuous with respect to the open base $\mathcal{C} = \{U \times V | \ U \subset X$ is open and $V \in \mathcal{B}\}$ for the product topology $\tau \times \sigma$ on $X \times Y$, then f is subweakly b-continuous with respect to the open base $\mathcal{B}$.*

Proof. Let $V \in \mathcal{B}$. We have $bCl(f^{-1}(V)) = bCl(X \setminus f^{-1}(V)) = bCl(g^{-1}(X \times V)) \subset g^{-1}(Cl(X \times V)) = g^{-1}(X \times Cl(V))) = f^{-1}(Cl(V))$; hence f is subweakly b-continuous. **q.e.d.**

Definition 3.8 A topological space (X, τ) is said to be b-T_1 [10] if for each pair of distinct points x and y of X, there exists b-open sets U and V containing x and y, respectively such that $y \notin U$ and $x \notin V$.

Theorem 3.9 *If $f : (X, \tau) \to (Y, \sigma)$ is subweakly b-continuous injection and (Y, σ) is Hausdorff, then (X, τ) is b-T_1.*

Proof. Let x and y be distinct points in X. Since f is injective, $f(x) \neq f(y)$. Let $\mathcal{B}$ be an open base for the topology σ on Y for which $bCl(f^{-1}(V)) \subset f^{-1}(Cl(V))$ for every $V \in \mathcal{B}$. Since (Y, σ) is Hausdorff, there exist disjoint subsets V_1 and W_1 in (Y, σ) with $f(y) \in V_1$, $f(x) \in W_1$. There exists a subset $V \in \mathcal{B}$ such that $f(y) \in V, f(x) \notin V$, $V \cap W_1 = \emptyset$ and $V \subset V_1$. Then $f(x) \notin Cl(V)$; and hence $y \in f^{-1}(V) \subset bCl(f^{-1}(V))$ and $x \notin f^{-1}(Cl(V))$. Since f is subweakly b-continuous, $x \notin bCl(f^{-1}(V))$. Then, using Lemma 2.1(iii), we have $X \setminus bCl(f^{-1}(V))$ is a b-open set containing x but not y. By an argument similar to that of the above proof, it is shown that there exists a subset $W \in \mathcal{B}$ such that $X \setminus bCl(f^{-1}(W))$ is a b-open set containing y but not x. It follows that (X, τ) is b-T_1. **q.e.d**

Lemma 3.10 *Let (X, τ) be a topological space and A a subset of (X, τ). Then we have the following properties.*

(i) [14],[18, Proposition 3.9] (e.g. [1, Proof of Theorem 2.3(3)], [13, Lemma 2.2],[19, Lemma 3.2],[21, Lemma 5.2] , [25, Lemma 5.1]) *If $A \in \alpha O(X, \tau)$ and $U \in BO(X, \tau)$, then $U \cap A \in BO(A, \tau|A)$.*

(ii) [14] (e.g., [25, Lemma 5.2 (1)]) *If $A \in \alpha O(X, \tau)$ and $V \in BO(A, \tau|A)$ and then $V \in BO(X, \tau)$.*

(iii) [2, Proposition 2.4, Proposition 2.3(b)] *If $A \in \alpha O(X, \tau)$ and $U \in BO(X, \tau)$, then $U \cap A \in BO(X, \tau)$.*

(iv) [21, Lemma 5.3] *If $B \subset A \subset X$ and $A \in \alpha O(X,\tau)$, then $(bCl(B)) \cap A = bCl_A(B)$, where $bCl_A(B) := \bigcap\{F|\ F$ is b-closed in $(A, \tau|A)$ with $B \subset F \subset A\}$.*

Theorem 3.11 *If $f : (X,\tau) \to (Y,\sigma)$ is subweakly b-continuous and $A \in \alpha O(X,\tau)$, then $f|A : (A,\tau|A) \to (Y,\sigma)$ is subweakly b-continuous.*

Proof. Let $\mathcal{B}$ be an open base for the topology σ on Y for which $bCl(f^{-1}(V)) \subset f^{-1}(Cl(V))$ for every $V \in \mathcal{B}$. Then using Lemma 3.10 (iv) we have for $V \in \mathcal{B}$, $bCl_A((f|A)^{-1}$- $(Cl(V)) \subset A \cap bCl((f|A)^{-1}(V)) = A \cap bCl(A \cap f^{-1}(V)) \subset A \cap bCl(f^{-1}(V)) \subset A \cap f^{-1}(Cl(V)) = (f|A)^{-1}(Cl(V))$. Therefore, $f|A$ is subweakly b-continuous. **q.e.d.**

Theorem 3.12 *If $f : (X,\tau) \to (Y,\sigma)$ is subweakly b-continuous and E is an open subset of (Y,σ) with $f(X) \subset E$, then $f : (X,\tau) \to (E,\sigma|E)$ is subweakly b-continuous.*

Proof. Let $\mathcal{B}$ be an open base for the topology σ on Y for which $bCl(f^{-1}(V)) \subset f^{-1}(Cl(V))$ for every $V \in \mathcal{B}$. Then, the collection $\mathcal{C} := \{V \cap E|\ V \in \mathcal{B}\}$ is an open base for the relative topology $\sigma|E$ on E. Since E is open in (Y,σ), it is well known that $Cl(V) \cap E \subset Cl_E(V \cap E)$. Then, $bCl(f^{-1}(V \cap E)) \subset f^{-1}(Cl(V) \cap E) \subset f^{-1}(Cl_E(V \cap E))$; hence $f : (X,\tau) \to (E,\sigma|E)$ is subweakly b-continuous. **q.e.d.**

Theorem 3.13 *Let $f : (X,\tau) \to (X,\tau)$ be subweakly b-continuous and let $A \subset X$ such that $f(X) \subset A$ and $f|A$ is the identity function on A. Then, if (X,τ) is Hausdorff, then A is b-closed.*

Proof. Assume A is not b-closed. Let $x \in bCl(A) \setminus A$. Let $\mathcal{B}$ be an open base for the topology τ on X for which $bCl(f^{-1}(V)) \subset f^{-1}(Cl(V))$ for every $V \in \mathcal{B}$. Since $x \notin A$, $f(x) \neq x$. Since (X,τ) is Hausdorff, there exist disjoint open sets V and W with $x \in V$, $f(x) \in W$ and $V \in \mathcal{B}$. Let $U \in BO(X,\tau;x)$. Then $x \in U \cap V$ which is b-open in (X,τ) by Lemma 3.10(iii). Since $x \in bCl(A)$, $(U \cap V) \cap A \neq \emptyset$. Let $y \in (U \cap V) \cap A$. Since $y \in A$, $f(y) = y \in V$ and hence $y \in f^{-1}(V)$. Therefore, $y \in U \cap f^{-1}(V)$ and hence $U \cap f^{-1}(V) \neq \emptyset$ and, using Lemma 2.1(i), we have $x \in bCl(f^{-1}(V))$. However, $f(x) \in W$ which is open and disjoint from V. So $f(x) \notin Cl(V)$ or, that is, $x \notin f^{-1}(Cl(V))$, which contradicts the assumption that f is also subweakly b-continuous. Therefore, A is b-closed. **q.e.d.**

Theorem 3.14 *If $f_1 : (X,\tau) \to (Y,\sigma)$ is α-continuous, $f_2 : (X,\tau) \to (Y,\sigma)$ is subweakly b-continuous, and (Y,σ) is Hausdorff, then the set $A := \{x \in X|\ f_1(x) = f_2(x)\}$ is b-closed in (X,τ).*

Proof. Let $x \in X \setminus A$. Then $f_1(x) \neq f_2(x)$. Let $\mathcal{B}$ be an open base for the topology σ on Y for which $bCl(f^{-1}(V)) \subset f^{-1}(Cl(V))$ for every $V \in \mathcal{B}$. Since (Y,σ) is Hausdorff, there exist disjoint open sets V and W with $f_1(x) \in V, f_2(x) \in W$ and $V \in \mathcal{B}$. Then $f_2(x) \notin Cl(V)$ and hence $x \notin f_2^{-1}(Cl(V))$. Then, since f_2 is subweakly b-continuous, $x \in X \setminus bCl(f_2^{-1}(V))$. Thus, $x \in f_1^{-1}(V) \cap (X \setminus bCl(f_2^{-1}(V))) \subset X \setminus A$. By Lemma 2.1(iii), $X \setminus bcl(f_2^{-1}(V))$ is b-open in (X,τ). Since $f_1^{-1}(V)$ is α-open in (X,τ), it follows from Lemma 3.10(iii) that the intersection of these sets is b-open in (X,τ). It follows from Lemma 2.1(ii) that $X \setminus A$ is b-open; and hence A is b-closed in (X,τ). **q.e.d.**

Recall that a subset A of a topological space (X,τ) is said to be b-*dense* [18] if $bcl(A) = X$.

Corollary 3.15 *Assume that $f_1 : (X,\tau) \to (Y,\sigma)$ is α-continuous, $f_2 : (X,\tau) \to (Y,\sigma)$ is subweakly b-continuous, and (Y,σ) is Hausdorff. If f_1 and f_2 agree on a b-dense set, then $f_1 = f_2$.*

Proof. Let $A := \{x \in X |\ f_1(x) = f_2(x)\}$ and let U be a b-dense set in (X, τ) on which f_1 and f_2 agree. Then, since $U \subset A$, we have $X = bCl(U) \subset bCl(A) = A$ (cf. Theorem 3.14) and hence $f_1 = f_2$. **q.e.d.**

Theorem 3.16 *If $f_j : (X, \tau) \to (Y_j, \sigma_j)$ is subweakly b-continuous for each $j \in \Lambda_m$ where $\Lambda_m := \{1, 2, ..., m\}(m > 1)$, then $f : (X, \tau) \to (\prod_{j=1}^m Y_j, \prod_{j=1}^m \sigma_j)$ given by $f(x) := (f_1(x), f_2(x), ..., f_m(x))$ is subweakly b-continuous.*

Proof. For each $j \in \Lambda$, let $\mathcal{B}_j$ be an open base for the topology on Y_j for which $bCl(f_j^{-1}(V_j)) \subset f_j^{-1}(Cl(V_j))$ for every $V_j \in \mathcal{B}_j$. Then, let $\mathcal{B} := \{\prod_{j=1}^m V_j | V_j \in \mathcal{B}_j (j \in \Lambda_m)\}$ be an open base for the topology $\prod_{j=1}^m \sigma_j$ on $\prod_{j=1}^m Y_j$. For every set $V := \prod_{j=1}^m V_j \in \mathcal{B}$, $bCl(f^{-1}(V))$=$bCl(\bigcap\{f_j^{-1}(V_j) | j \in \Lambda\}) \subset \bigcap\{bCl(f_j^{-1}(V_j)) |\ j \in \Lambda_m\} \subset \bigcap\{f_j^{-1}(Cl(V_j)) | j \in \Lambda_m\}$=$f^{-1}(\prod_{j=1}^m (Cl(V_j)))$=$f^{-1}(Cl(V))$. Thus, f is subweakly b-continuous. **q.e.d.**

Theorem 3.17 *If $f_j : (X_j, \tau_j) \to (Y_j, \sigma_j)$ is subweakly b-continuous for each $j \in \Lambda_m$ where $\Lambda_m := \{1, 2, ..., m\}(m > 1)$, then a function $\prod_{j=1}^m f_j : (\prod_{j=1}^m X_j, \prod_{j=1}^m \tau_j) \to (\prod_{j=1}^m Y_j, \prod_{j=1}^m \sigma_j)$ defined by $(\prod_{j=1}^m f_j)(x) := (f_1(x), f_2(x), ..., f_m(x))$ is subweakly b-continuous.*

Proof. For each $j \in A$, let $\mathcal{B}_j$ be an open base for the topology on Y_j for which $bCl(f_j^{-1}(V_j)) \subset f_j^{-1}(Cl(V_j))$ for every $V_j \in \mathcal{B}_j$. Let $\mathcal{B} := \{\prod_{j=1}^m V_j | V_j \in \mathcal{B}_j (j \in \Lambda_m)\}$ be an open base for the topology $\prod_{j=1}^m \sigma_j$ on $\prod_{j=1}^m Y_j$. For every set $V := \prod_{j=1}^m V_j \in \mathcal{B}$, we have that: $bCl((\prod_{j=1}^m f_j)^{-1}(V)) \subset \prod_{j=1}^m bCl((f_j)^{-1}(V_j)) \subset \prod_{j=1}^m (f_j)^{-1}(Cl(V_j))$ $=(\prod_{j=1}^m f_j)^{-1}(\prod_{j=1}^m Cl(V_j))$ $=(\prod_{j=1}^m f_j)^{-1}(Cl(\prod_{j=1}^m V_j))$ $=(\prod_{j=1}^m f_j)^{-1}(Cl(V))$. Thus, $\prod_{j=1}^m f_j$ is subweakly b-continuous. **q.e.d.**

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O-UNION AND O-DECOMPOSITION ON HYPER K-ALGEBRAS

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ABSTRACT. In this paper, we define a O-union of two hyper K-algebras and O-decomposition of a hyper K-algebra. In general, the O-union of two hyper K-algebra is not a hyper K-algebra. But, if a hyper K-algebra $(H, \circ, 0)$, be the O-union of two hyper K-algebras $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$, we investigate which properties of $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$ is transferred to $(H, \circ, 0)$ and conversely. Also we show that a hyper K-algebra $(H, \circ, 0)$ where $x \in x \circ y$ can be decomposed into a positive implicative hyper BCK-algebra and a hyper K-algebra.

1. INTRODUCTION

The concept of BCK-algebra that is a generalization of set difference and propositional calculi was established by Imai and Iséki [3] in 1966. In Ref. [4], Jun et al. applied the hyper structures BCK-algebra. In 1934, Marty [5] introduced for the first time the hyper structure theory in the 8th congress of Scandinavian Mathematicians. In Ref. [2], Borzooei et al. introduced the generalization of BCK-algebra and hyper BCK-algebra, called hyper K-algebra. They studied properties of hyper K-algebra. In this article, the aim is to define the O-union and O-decomposition on hyper K-algebras. Section 2, concerns definitions and theorems that are needed in the sequel. Section 3, we give O-union's definition of two hyper K-algebras and O-decomposition of a hyper K-algebra into two hyper K-algebras and finally in Section 4, we study transferable properties on O-Union (decomposition) hyper K-algebras.

2. PRELIMINARIES

In this section we give some definitions and theorems that are needed in the sequel.

Definition 2.1. [2] Let H be a set containing 0 and the function $\circ : H \times H \to P^*(H)(:= P(H) \setminus \emptyset)$ is called a hyper operation on H. Then $(H, \circ, 0)$ is called a hyper K-algebra (hyper BCK-algebra) if it satisfies HK1-HK5 (BHK1-BHK4).

$$\begin{array}{ll}
\mathrm{H}K1 : (x \circ z) \circ (y \circ z) < x \circ y, & \mathrm{B}HK1 : (x \circ z) \circ (y \circ z) \ll x \circ y, \\
\mathrm{H}K2 : (x \circ y) \circ z = (x \circ z) \circ y, & \mathrm{B}HK2 : (x \circ y) \circ z = (x \circ z) \circ y, \\
\mathrm{H}K3 : x < x, & \mathrm{B}HK3 : x \circ H \ll x, \\
\mathrm{H}K4 : x < y, y < x \Rightarrow x = y, & \mathrm{B}HK4 : x \ll y, y \ll x \Rightarrow x = y. \\
\mathrm{H}K5 : 0 < x. &
\end{array}$$

for all $x, y, z \in H$, where $x < y (x \ll y) \Leftrightarrow 0 \in x \circ y$. For any $A, B \subseteq H, A < B$ if there exist $a \in A$ and $b \in B$ such that $a < b$. Moreover, $A \ll B$ if for all $a \in A$ there exist $b \in B$ such that $a \ll b$. A hyper K-algebra $(H, \circ, 0)$ is bounded if there exist an element $e \in H$ such that $x < e$ for all $x \in H$.

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Definition 2.2. [2] Let S be a nonempty set of a hyper K-algebra $(H, \circ, 0)$ containing 0. If S is a hyper K-algebra with respect to the hyper operation $\circ$ on H, we say that S is a hyper K-subalgebra of H.

Theorem 2.3. [2] *Let S be a nonempty set of a hyper K-algebra $(H, \circ, 0)$. Then S is a hyper K-subalgebra of H iff $x \circ y \subseteq S$ for all $x, y \in S$.*

Theorem 2.4. [7] *Let H be a set containing 0, $P_0(H) := \{A \subseteq H : 0 \in A\}$ and $S = \{f | f : H \to P_0(H)$ is a function$\}$. Then $\circ_f : H \times H \to P^*(H)$ where*

$$x \circ_f y := \begin{cases} f(x), & if x = y, \\ \{x\}, & if x \neq y. \end{cases}$$

is a hyperoperation. Moreover, the following statements are equivalent:
(1) $(H, o_f, 0)$ *is a hyper K-algebra,*
(2) $f(x)\, o_f\, y = f(x)$ *for all* $y \neq x, y \in H$,
(3) $x \neq y$ *and* $y \in f(x)$ *imply* $y \in f(y)$ *and* $f(y) \subseteq f(x)$.
This hyper K-algebra is called a quasi union hyper K-algebra.

Theorem 2.5. [7] *Let $(H, \circ, 0)$ be a quasi union hyper K-algebra. Then the following statements are equivalent:*
(1) *H is a positive implicative hyper K-algebra,*
(2) $f(x) = \{0\}$ *or* $f(x) = \{0, x\}$ *for all* $x \in H$,
(3) *H is a hyper BCK-algebra.*

Definition 2.6. [2, 9] Let I be a subset of a hyper K-algebra containing 0. Then I is said to be a hyper K-ideal (weak hyper K-ideal) of H if $x \circ y < I$ ($x \circ y \subseteq I$) and $y \in I$ imply $x \in I$ for all $x, y \in H$.

Notation: Let A and I be nonempty subsets of a hyper K-algebra H. We set $AR_1I := A \subseteq I$, $AR_2I := A \cap I \neq \emptyset$, and $AR_3I := A < I$.

Definition 2.7. [1] A nonempty subset of a hyper K-algebra H such that $0 \in I$, for all $x, y, z \in H$, and $i, j, k \in \{1, 2, 3\}$ is said to be
(1) implicative hyper K-ideal of H if $((x \circ z) \circ (y \circ x)) < I$, $z \in I \Rightarrow x \in I$,
(2) positive implicative hyper K-ideal of type (i, j, k) if $(x \circ y) \circ zR_iI$ and $y \circ zR_jI$ imply that $x \circ zR_kI$,
(3) commutative hyper K-ideal of type (i, j) if $(x \circ y) \circ zR_iI$, $z \in I$ imply that $x \circ (y \circ (y \circ x))R_jI$.

Theorem 2.8. [1] *Let I be a hyper K-ideal of hyper K-algebra H. Then I is an implicative hyper K-ideal iff $x \circ (y \circ x) < I$ implies that $x \in I$, for any $x, y \in H$.*

3. O-union and O-decomposition on the hyper K-algebras

In this section, at first we define O-union of two hyper K-algebras and O-decomposition of a hyper K-algebra into two hyper K-algebras, and then we study transferable properties on O-Union (decomposition) hyper K-algebras.

Definition 3.1. Let $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$ are two hyper K-algebras and $\circ := \circ_1 \cup \circ_2$ i.e. $x \circ y = (x \circ_1 y) \cup (x \circ_2 y)$. If $(H, \circ, 0)$ be a hyper K-algebra then we say $(H, \circ, 0)$ is O-union of two hyper K-algebras $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$. Moreover, a hyper K-algebra

$(H, \circ, 0)$ is called O-decomposition into two hyper K-algebras $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$ if $\circ = \circ_1 \cup \circ_2$, for all $x, y \in H$. If $\circ$ be different from $\circ_1$ and $\circ_2$, we say that $(H, \circ, 0)$ is a proper O-decomposition.

Example 3.2. The hyper K-algebra $(H, \circ, 0)$

$\circ$	0	1	2
0	{0}	{0,1,2}	{ 0,1,2}
1	{1}	{0,1,2}	{0,1,2}
2	{2}	{1,2}	{0,1,2}

can be O-decomposed into two hyper K-algebras $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$ as follows:

$\circ_1$	0	1	2
0	{0}	{0,1,2}	{ 0,1,2}
1	{1}	{0,2}	{0,1,2}
2	{2}	{2}	{0,1,2}

$\circ_2$	0	1	2
0	{0}	{0,1,2}	{ 0,1,2}
1	{1}	{0,1,2}	{0,2}
2	{2}	{1,2}	{0,1,2}

The O-decomposition of a hyper K-algebra $(H, \circ, 0)$ is not unique, since the hyper K-algebra $(H, \circ, 0)$ in example 3.2 is O-decomposed as follows:

$\circ_3$	0	1	2
0	{0}	{0,1}	{ 0,1}
1	{1}	{0,1}	{0}
2	{2}	{2}	{0,1}

$\circ_4$	0	1	2
0	{0}	{0,1,2}	{ 0,1,2}
1	{1}	{0,1,2}	{1,2}
2	{2}	{1,2}	{0,2}

The following example shows that a hyper K-algebra $(H, \circ, 0)$ can not be O-decomposed into two proper hyper K-algebras.

Example 3.3.

$\circ$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0,1}	{1}
2	{2}	{2}	{0}

$\circ_1$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0}	{1}
2	{2}	{2}	{0}

$\circ_2$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0,1}	{1}
2	{2}	{2}	{0}

The following example shows that O-union of two hyper K-algebras $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$ is not a hyper K-algebra.

Example 3.4. Let $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$ are hyper K-algebras as follows. Then $(H, \circ, 0)$, the O-union of them is not hyper K-algebra, because $1 < 2, 2 < 1$ but $1 \neq 2$.

$\circ_1$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0}	{1}
2	{2}	{0,1}	{0,1,2}

$\circ_2$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0,1}	{0}
2	{2}	{2}	{0,2}

$\circ$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0,1}	{0,1}
2	{2}	{0,1,2}	{0,1,2}

Theorem 3.5. *Any O-union of two quasi union hyper K-algebras is a quasi union hyper K-algebra.*

Proof. Suppose $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$ are two quasi union hyper K-algebras, therefore there are two functions $f, g : H \to P_0(H)$ and $\circ : H \times H \to P^*(H)$ such that

$$x \circ_f y := \begin{cases} (f \cup g)(x) & , ifx = y; \\ \{x\} & , ifx \neq y. \end{cases}$$

It is clear that $\circ$ is a hyperopration, we show that H is a quasi union hyper K-algebra. Let $y \in (f \cup g)(x) = (x \circ_1 x) \cup (x \circ_2 x)$ for any $x, y \in H$. So $y \in x \circ_1 x$ or $y \in x \circ_2 x$. Since

$(H, \circ_1, 0)$ and $(H, \circ_2, 0)$ are two quasi union hyper K-algebras, we get $y \in y \circ_1 y \subseteq x \circ_1 x$ or $y \in y \circ_2 y \subseteq x \circ_2 x$. Therefore $y \in (f \cup g)(y) \subseteq (f \cup g)(x)$, and the proof is completed. $\square$

Theorem 3.6. *Let $(H, \circ, 0)$ be a hyper K-algebra such that $x \in x \circ y$ for all $x, y \in H$. Then H is O-decomposition into a positive implicative hyper BCK-algebra $(H, \circ_1, 0)$ and a hyper K-algebra $(H, \circ_2, 0)$.*

Proof. Let $(H, \circ, 0)$ be a hyper K-algebra, since $x \in x \circ y$ we can define $\circ_1 : H \times H \to H$ as follows:

$$x \circ_1 y := \begin{cases} \{x\} & , if x \neq y; \\ \{0\} & , if x = y. \end{cases}$$

It is clear that $(H, \circ_1, 0)$ is a quasi union hyper K-algebra. By Theorem 2.5(1) and (2), $(H, \circ_1, 0)$ is a positive implicative hyper BCK-algebra. So $(H, \circ, 0)$ is written as O-decomposition into a hyper BCK-algebra $(H, \circ_1, 0)$ and at least a hyper K-algebra $(H, \circ_2, 0)$ where $\circ_2 = \circ$. $\square$

Example 3.7. The hyper K-algebra $(H, \circ, 0)$ with following cayley table is O-decomposition into a hyper BCK-algebra $(H, \circ_1, 0)$ and a proper hyper K-algebra $(H, \circ_2, 0)$.

$\circ$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0,1}	{0,1}
2	{2}	{2}	{0,2}

$\circ_1$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0}	{1}
2	{2}	{2}	{0}

$\circ_2$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0,1}	{0}
2	{2}	{2}	{0,2}

The following example shows that the condition $x \in x \circ y$ in the theorem 3.6 is necessary.

Example 3.8. By the following cayley table, $(H, \circ, 0)$ is a hyper K-algebra,

$\circ$	0	1	2	3
0	{0}	{0}	{0}	{0}
1	{1}	{0,1}	{1,3}	{2}
2	{2}	{0,2}	{0,2}	{2}
3	{3}	{3}	{3}	{0}

If $\circ = \circ_1 \cup \circ_2$ then there are 36 hyper oprations on H for $\circ_1$ as follows:

$\circ_1$	0	1	2	3
0	{0}	{0}	{0}	{0}
1	{1}	{0} or {0,1}	{1} or {3}or {1,3}	{2}
2	{2}	{0} or {2} or {0,2}	{0} or {0,2}	{2}
3	{3}	{3}	{3}	{0}

by checking all these cases, we see that $(H, \circ_1, 0)$ is not a hyper BCK-algebra. So $(H, \circ, 0)$ is not written as O-decomposition into a BCK-algebra and a hyper K-algebra.

4. Transferable properties

In this section we study transferable properties on O-Union (decomposition) hyper K-algebras.

Theorem 4.1. *Let $(H, \circ, 0)$ be O-decomposition into two hyper K-algebras $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$. Then S is subalgebra of $(H, \circ, 0)$ if and only if S is subalgebra of $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$.*

Proof. It is clear. $\square$

Theorem 4.2. *Let $(H, \circ, 0)$ be O-decomposition into two hyper K-algebras $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$. Then I is a weak hyper K-ideal of $(H, \circ, 0)$ if and only if I is a weak hyper K-ideal of $(H, \circ_1, 0)$ or $(H, \circ_2, 0)$.*

Proof. Suppose I be a weak hyper K-ideal of $(H, \circ_1, 0)$ or $(H, \circ_2, 0)$, $x \circ y \subseteq I$ and $y \in I$. Then $x \circ_1 y \subseteq I$ and $x \circ_2 y \subseteq I$ for all $x, y \in H$. Since I is a weak hyper K-ideal of $(H, \circ_1, 0)$ or $(H, \circ_2, 0)$ then $x \in I$.
Conversely, suppose I be a weak hyper K-ideal of $(H, \circ, 0)$ and $x \circ_1 y \subseteq I$ or $x \circ_2 y \subseteq I$ and $y \in I$. If $x \circ_i y \nsubseteq I$ for some $i \in \{1, 2\}$, then I is a weak hyper K-ideal of $(H, \circ_1, 0)$ or $(H, \circ_2, 0)$, otherwise $x \circ_i y \subseteq I$ for any $i \in \{1, 2\}$ and we have $x \circ y = x \circ_1 y \cup x \circ_2 y \subseteq I$, therefore $x \in I$. □

Theorem 4.3. *Let $(H, \circ, 0)$ is O-decomposition into two hyper K-algebras $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$. Then*

(1) *If e be a upper bound of $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$, then e is a upper bound of H.*
(2) *If I be a hyper K-ideal of $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$, then I is a hyper K-ideal of H.*
(3) *If I be an implicative hyper K-ideal of $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$, then I is an implicative hyper K-ideal of H.*

Proof. (1): By hypothesis we have $0 \in x \circ_1 e$ and $0 \in x \circ_2 e$ for all $x \in H$. So $0 \in x \circ e$ and e is a upper bound of H.
(2): Let $x \circ y < I$ and $y \in I$, so $x \circ_1 y < I$ or $x \circ_2 y < I$, since I is hyper K-ideal of $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$ we get $x \in I$.
(3): Let $x \circ (y \circ x) < I$, so $x \circ_1 (y \circ_1 x) < I$ or $x \circ_2 (y \circ_2 x) < I$, by assumption we have $x \in I$. □

The following example shows that the converse of theorem 4.3 (1) is not true in general.

Example 4.4. Let $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$ are hyper K-algebras as follows and $(H, \circ, 0)$ be O-union of them. Then the two hyper K-algebras are not bounded but 1 is a upper bound of $(H, \circ, 0)$.

$\circ_1$	0	1	2	3
0	{0}	{0}	{0}	{0}
1	{1}	{0,1}	{1}	{1}
2	{2}	{0}	{0,2}	{2}
3	{3}	{3}	{0,1,3}	{0,1,3}

$\circ_2$	0	1	2	3
0	{0}	{0}	{0}	{0}
1	{1}	{0,1}	{1}	{1}
2	{2}	{2}	{0,2}	{2}
3	{3}	{0,1,2}	{0,1,3}	{0,1,2,3}

$\circ$	0	1	2	3
0	{0}	{0}	{0}	{0}
1	{1}	{0,1}	{1}	{1}
2	{2}	{0,2}	{0,2}	{2}
3	{3}	{0,1,2,3}	{0,1,3}	{0,1,2,3}

The following example shows that, in the theorem 4.3 (2) and (3) we can not use "or" instead of "and".

Example 4.5. Let $(H, \circ, 0)$, $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$ be as follows. Then $I = \{0, 1\}$ is a hyper K-ideal of $(H, \circ_1, 0)$, but I is not a hyper K-ideal of $(H, \circ_2, 0)$ and $(H, \circ, 0)$. Also I is an implicative hyper K-ideal of $(H, \circ_1, 0)$ and it is not implicative hyper K-ideal of $(H, \circ_2, 0)$ and $(H, \circ, 0)$. Because $2 \circ_2 (2 \circ_2 2) < I$ but $2 \notin I$.

$\circ_1$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0,1}	{1}
2	{2}	{2}	{0}

$\circ_2$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0,1}	{1}
2	{2}	{0}	{0,2}

$\circ$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0,1}	{1}
2	{2}	{0,2}	{0,2}

Theorem 4.6. *Let $(H, \circ, 0)$ be O-decomposition into two hyper K-algebras $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$. Then*

(1) *If I be a hyper K-ideal of $(H, \circ, 0)$, then I is a hyper K-ideal of $(H, \circ_1, 0)$ or $(H, \circ_2, 0)$.*
(2) *If I be an implicative hyper K-ideal of $(H, \circ, 0)$, then I is an implicative hyper K-ideal of $(H, \circ_1, 0)$ or $(H, \circ_2, 0)$.*
(3) *If I be a positive implicative hyper K-ideal of type (i, j, k) in $(H, \circ, 0)$, where $i, j, k \in \{1, 2, 3\}$. Then I is a positive implicative hyper K-ideal of the same type in $(H, \circ_1, 0)$ or $(H, \circ_2, 0)$.*

Proof. (1): Suppose $x \circ_1 y < I$ or $x \circ_2 y < I$ and $y \in I$ for all $x, y \in H$, then $x \circ y < I$. Since I is hyper K-ideal of H, we have $x \in I$, i.e. I is a hyper K-ideal of $(H, \circ_1, 0)$ or $(H, \circ_2, 0)$.
(2): Suppose $x \circ_1 (y \circ_1 x) < I$ for all $x, y \in H$, then $x \circ (y \circ x) < I$. Since I is an implicative hyper K-ideal of H, by Theorem 2.8, $x \in I$ and I is an implicative hyper K-ideal of $(H, \circ_1, 0)$ or $(H, \circ_2, 0)$.
(3): It is sufficient to prove for type $(1, 1, 1)$, the proof for other types is similar. If $(x \circ_i y) \circ_i z \nsubseteq I$ or $y \circ_i z \nsubseteq I$ for some $i \in \{1, 2\}$, then I is positive implicative hyper K-ideal of type $(1, 1, 1)$ in $(H, \circ_1, 0)$ or $(H, \circ_2, 0)$. Otherwise if $(x \circ_1 y) \circ_1 z \subseteq I$, $(x \circ_2 y) \circ_2 z \subseteq I$, $y \circ_1 z \subseteq I$ and $y \circ_2 z \subseteq I$, then $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$. Since I is a positive implicative hyper K-ideal of type $(1, 1, 1)$ in $(H, \circ, o)$, then $x \circ z \subseteq I$ and we get $x \circ_1 z \subseteq I$ and $x \circ_2 z \subseteq I$. Therefore in general I is positive implicative hyper K-ideal of type $(1, 1, 1)$ in $(H, \circ_1, o)$ or $(H, \circ_2, o)$ and the proof is completed. □

The following example shows that the converse of theorem 4.6 (3) is not true in general.

Example 4.7. Consider the following hyper K-algebras $(H, \circ, 0)$, $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$. Then $(H, \circ, 0)$ is O-decomposition into $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$, and $I = \{0, 1\}$ is positive implicative hyper K-ideal of type $(2, 1, 2)$ in $(H, \circ_1, 0)$, but I is not positive implicative hyper K-ideal of type $(2, 1, 2)$ in $(H, \circ_2, 0)$, since $(2 \circ_2 1) \circ_2 0 \cap I \neq \emptyset$ and $1 \circ_2 0 \subseteq I$ but $2 \circ_2 0 \cap I = \emptyset$. Hence I is not positive implicative hyper K-ideal of type $(2, 1, 2)$ in H.

$\circ$	0	1	2
0	{0}	{0,1,2}	{ 0,1,2}
1	{1}	{0,2}	{1,2}
2	{2}	{0,1,2}	{0,1,2}

$\circ_1$	0	1	2
0	{0}	{0}	{ 0}
1	{1}	{0}	{1}
2	{2}	{2}	{0,2}

$\circ_2$	0	1	2
0	{0}	{0,1,2}	{ 0,1,2}
1	{1}	{0,2}	{1,2}
2	{2}	{0,1}	{0,1,2}

Theorem 4.8. *Let $(H, \circ, 0)$ be O-decomposition into two hyper K-algebras $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$. If I be a positive implicative hyper K-ideal of types $(1, 1, 2)$ and $(1, 1, 3)$ in $(H, \circ_1, 0)$ or $(H, \circ_2, 0)$. Then I is a positive implicative hyper K-ideal of the same type in H.*

Proof. Let I is a positive implicative hyper K-ideal of type $(1, 1, 2)$ in $(H, \circ_1, 0)$ or $(H, \circ_2, 0)$, $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$, then $(x \circ_1 y) \circ_1 z \subseteq I, (x \circ_2 y) \circ_2 z \subseteq I$, $y \circ_1 z \subseteq I$ and $y \circ_2 z \subseteq I$. By hypothesis we get that $x \circ_1 z \cap I \neq \emptyset$ or $x \circ_2 z \cap I \neq \emptyset$, hence $x \circ z \cap I \neq \emptyset$ and I is a positive implicative hyper K-ideal of type $(1, 1, 2)$ in $(H, \circ, 0)$. The proof for type $(1, 1, 3)$ is similar. □

Theorem 4.9. *Let $(H, \circ, 0)$ be O-decomposition into two hyper K-algebras $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$. If I be a positive implicative hyper K-ideal of type $(1, 1, 1)$, $(1, j, k)$ or $(i, 1, k)$*

where $i, j \in \{1, 2, 3\}$ *and* $k \neq 1$ *in* $(H, \circ_1, 0)$ *and* $(H, \circ_2, 0)$*. Then* I *is a positive implicative hyper K-ideal of the same type in* H*.*

Proof. Let I is a positive implicative hyper K-ideal of type $(1, 1, 1)$ in $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$, $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$, so $(x \circ_1 y) \circ_1 z \subseteq I, (x \circ_2 y) \circ_2 z \subseteq I$, $y \circ_1 z \subseteq I$ and $y \circ_2 z \subseteq I$. Since I is a positive implicative hyper K-ideal of type $(1, 1, 1)$ in $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$, we have $x \circ_1 z \subseteq I$ and $x \circ_2 z \subseteq I$, so $x \circ z \subseteq I$. The proof for the others is similar. □

Theorem 4.9 is not true for other cases, the following example shows this for type $(2, 2, 3)$.

Example 4.10. The hyper K-algebra $(H, \circ, 0)$ is O-decomposition into $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$ as follows and $I = \{0, 1\}$ is a positive implicative of type $(2, 2, 3)$ in $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$, but I is not a positive implicative of type $(2, 2, 3)$ in H. Since $(2 \circ 3) \circ 1 \cap I \neq \emptyset$, $3 \circ 1 \cap I \neq \emptyset$ and $2 \circ 1 \not\subseteq I$.

$\circ$	0	1	2	3
0	{0}	{0}	{0}	{0}
1	{1}	{0,1}	{1}	{1}
2	{2}	{2}	{0,2}	{0,1,2}
3	{3}	{0,3}	{3}	{0,3}

$\circ_1$	0	1	2	3
0	{0}	{0}	{0}	{0}
1	{1}	{0,1}	{1}	{1}
2	{2}	{2}	{0,2}	{2}
3	{3}	{0}	{3}	{0,3}

$\circ_2$	0	1	2	3
0	{0}	{0}	{0}	{0}
1	{1}	{0,1}	{1}	{1}
2	{2}	{2}	{0,2}	{0,1}
3	{3}	{3}	{3}	{0,3}

Theorem 4.11. *Let* $(H, \circ, 0)$ *be O-decomposition into two hyper K-algebras* $(H, \circ_1, 0)$ *and* $(H, \circ_2, 0)$ *and the nonempty subset* I *of* H *be a commutative hyper K-ideal of type* $(i, j); i, j \in \{1, 2, 3\}$ *in* H*. Then* I *is a commutative hyper K-ideal of type* (i, j) *in* $(H, \circ_1, 0)$ *or* $(H, \circ_2, 0)$*.*

Proof. We prove theorem for type $(2, 2)$ and the proof for the other types is similar. Let $(x \circ_1 y) \circ_1 z \cap I \neq \emptyset$ or $(x \circ_1 y) \circ_1 z \cap I \neq \emptyset$ and $z \in I$, so $(x \circ y) \circ z \cap I \neq \emptyset$. Since I is a commutative hyper K-ideal of type $(2, 2)$ in H, we have $x \circ (y \circ (y \circ x)) \cap I \neq \emptyset$. Thus $x \circ_1 (y \circ_1 (y \circ_1 x)) \cap I \neq \emptyset$ or $x \circ_2 (y \circ_2 (y \circ_2 x)) \cap I \neq \emptyset$. □

Theorem 4.12. *Let* $(H, \circ, 0)$ *be O-decomposition into two hyper K-algebras* $(H, \circ_1, 0)$ *and* $(H, \circ_2, 0)$ *and the nonempty subset* I *of* H *be a commutative hyper K-ideal of type* $(1, 1)$ *or* $(i, j); i \in \{1, 2, 3\}, j \in \{2, 3\}$ *in* $(H, \circ_1, 0)$ *and* $(H, \circ_2, 0)$*, then* I *is a commutative hyper K-ideal of the same type in* H*.*

Proof. We prove theorem for type $(1, 1)$ and the proof for the other types is similar. Let $(x \circ y) \circ z \subseteq I$ and $z \in I$. So $(x \circ_1 y) \circ_1 z \subseteq I$ and $(x \circ_2 y) \circ_2 z \subseteq I$. Since I is a commutative hyper K-ideal of type $(1, 1)$ in $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$, we have $x \circ_1 (y \circ_1 (y \circ_1 x)) \subseteq I$ and $x \circ_2 (y \circ_2 (y \circ_2 x)) \subseteq I$. Finally $x \circ (y \circ (y \circ x)) \subseteq I$ and I is a commutative hyper K-ideal of type $(1, 1)$ in H. □

The following example shows that, in the theorem 4.12 we can not use "or" instead of "and".

Example 4.13. Consider the following hyper K-algebras $(H, \circ, 0)$, $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$. Then $(H, \circ, 0)$ is O-decomposition into $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$, and $I = \{0, 1\}$ is a commutative hyper K-ideal of type $(1, 1)$ in $(H, \circ_1, 0)$, but I is not commutative hyper K-ideal of type $(1,1)$ in $(H, \circ_2, 0)$. Since $(1 \circ_2 0) \circ_2 0 \subseteq I$ and $1 \circ_2 (0 \circ_2 (0 \circ_2 1)) = \{0,1,2\} \nsubseteq I$. Finally $(1 \circ 0) \circ 0 \subseteq I$ but $1 \circ (0 \circ (0 \circ 1)) = \{0,1,2\} \nsubseteq I$, so I is not a commutative hyper K-ideal of type $(1,1)$ in H.

$\circ$	0	1	2
0	{0}	{0,1,2}	{0,1,2}
1	{1}	{0,1,2}	{0,1,2}
2	{2}	{1,2}	{0,1,2}

$\circ_1$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0}	{1}
2	{2}	{2}	{0,2}

$\circ_2$	0	1	2
0	{0}	{0,1,2}	{0,1,2}
1	{1}	{0,1,2}	{0,2}
2	{2}	{1,2}	{0,1,2}

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CONVERGENCE OF NETS IN POSETS VIA AN IDEAL

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Abstract. It is well known that the meaning of the convergence in posets stings the interest of many investigators such as R. F. Anderson, J. C. Mathews and V. Olejček (see, for example [13, 14]). Among others, the notions of the order-convergence and of the o_2-convergence in posets were studied in details, presenting necessary and sufficient conditions under of which these convergences are topological. Many researchers give a special attention to the study of these convergences in different posets, inserting new knowledge in the classical theory of posets's convergence. In this paper, we introduce the ideal-order-convergence in posets, proving results which are based on this notion. We insert topologies in posets and we study their properties. We also give a sufficient and necessary condition for the ideal-order-convergence in a poset to be topological. The introduction of a weaker form of the ideal-order-convergence in posets, called ideal-o_2-convergence, completes our study.

Introduction

The order-convergence in posets was introduced by G. Birkhoff [1]. In general, the order-convergence is not topological, that is a poset X may not have a topology τ so that nets order-converge if and only if they converge with respect to the topology τ on X [14,22]. Then, much attention was paid to those posets in which the order-convergence is topological [15–17, 23]. Also, modifications of the order-convergence was studied in [13, 18, 20, 22, 23].

Meanwhile with the study of the order-convergence in posets, the notion of the o_2-convergence was communicated by the authors in [13, 18]. In fact, the o_2-convergence is a generalization of the order-convergence and, as the order-convergence, the o_2-convergence is also, not topological in general. Also in [20], many sufficient and necessary conditions were given so that this kind of convergence be topological.

On the other hand, in recent years, a lot of papers have been written on statistical convergence and ideal convergence in metric and topological spaces (see, for instance, [2, 3, 7–9, 12]).

In the present paper we introduce and study the notion of convergence of nets in posets via an ideal. We proceed with the following enumeration: In Section 1, we recall some definitions which will be used in the rest of the paper. In Section 2, we define the notion of the ideal-order-convergence in posets proving classical results for the notion of convergence. In Section 3, we introduce topologies in posets and we give a sufficient and necessary condition for the ideal-order-convergence in a poset to be topological. In Section 4, we study the ideal-order-convergence in Cartesian products of posets. Finally, in Section 5, the concepts of the ideal-o_2-convergence and the topological ideal-o_2-convergence in posets are developed.

1 Preliminaries

In this section we recall some definitions that are needed in the sequel and we refer to [1] for more details. We shall frequently denote posets by their underlying sets, and we

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write X for $(X, \leqslant)$. We will also use the following symbols $(a,b) = \{x \in X : a < x < b\}$, $[a,b] = \{x \in X : a \leqslant x \leqslant b\}$, $(a,b] = \{x \in X : a < x \leqslant b\}$, and $[a,b) = \{x \in X : a \leqslant x < b\}$. In addition, by writing $A \subseteq_{fin} B$ we mean that the set A is a finite subset of the set B.

(1) A subset A of a poset X is said to be directed if $A \neq \emptyset$, and for any $a_1, a_2 \in A$ there exists $a \in A$ such that $a_1 \leqslant a$ and $a_2 \leqslant a$.

(2) A subset A of a poset X is said to be filtered if $A \neq \emptyset$, and for any $a_1, a_2 \in A$ there exists $a \in A$ such that $a \leqslant a_1$ and $a \leqslant a_2$.

If $(D_1, \leqslant_1)$ and $(D_2, \leqslant_2)$ are directed sets, then the Cartesian product $D_1 \times D_2$ is directed by $\leqslant$, where $(d_1, d_2) \leqslant (d'_1, d'_2)$ if and only if $d_1 \leqslant_1 d'_1$ and $d_2 \leqslant_2 d'_2$.

A net in a set X is an arbitrary function x from a non-empty directed preordered set D to X. If $x(d) = x_d$, for all $d \in D$, then the net x will be denoted by the symbol $(x_d)_{d \in D}$.

Let X be a topological space. A net $(x_d)_{d \in D}$ in X is said to *topology-converge* to a point $x \in X$, if for every open neighborhood U of x, $x_d \in U$ eventually. In this case we write $(x_d)_{d \in D} \xrightarrow{t} x$.

A net $(y_\lambda)_{\lambda \in \Lambda}$ in X is said to be a *semi-subnet* of the net $(x_d)_{d \in D}$ in X if there exists a function $\varphi : \Lambda \to D$ such that $y = x \circ \varphi$, or equivalently, $y_\lambda = x_{\varphi(\lambda)}$ for every $\lambda \in \Lambda$. We write $(y_\lambda)^{\varphi}_{\lambda \in \Lambda}$ to indicate the fact that φ is the function mentioned above.

A family $\mathcal{I}$ of subsets of a non-empty set D is called an *ideal* if $\mathcal{I}$ has the following properties:

(1) $\emptyset \in \mathcal{I}$.

(2) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.

(3) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

The ideal $\mathcal{I}$ is called *non-trivial* if $D \notin \mathcal{I}$.

Suppose that $(y_\lambda)^{\varphi}_{\lambda \in \Lambda}$ is a semi-subnet of the net $(x_d)_{d \in D}$ in X. For every ideal $\mathcal{I}$ of the directed set D, we consider the family $\{A \subseteq \Lambda : \varphi(A) \in \mathcal{I}\}$. This family is an ideal on Λ which will be denoted by $\mathcal{I}_\Lambda(\varphi)$.

A *filter* $\mathcal{F}$ in a non-empty set X is a family of subsets of X that has the following properties:

(1) $X \in \mathcal{F}$.

(2) If $A \in \mathcal{F}$ and $B \supseteq A$, then $B \in \mathcal{F}$.

(3) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

If $\emptyset \notin \mathcal{F}$, we say that $\mathcal{F}$ is a *proper* filter.

Given a filter $\mathcal{F}$ on a set X, let $M = \{(x, F) \in X \times \mathcal{F} : x \in F\}$ and for $(x, F), (y, G) \in M$ define $(x, F) \geqslant (y, G)$ if and only if $F \subseteq G$. It is easily seen that $\geqslant$ directs M. The map $s_{\mathcal{F}} : M \to X$ with $s_{\mathcal{F}}(x, F) = x$, is a net in X, which is called the *net associated with* $\mathcal{F}$. If (X, τ) is a topological space, then $\mathcal{F} \xrightarrow{t} x \in X$ with respect to τ if and only if $s_{\mathcal{F}} \xrightarrow{t} x$ with respect to τ.

Dually, given a net $s : M \to X$ on a set X, define

$$\mathcal{F}_s = \{F \subseteq X : \{s(m) : m \geqslant m_0\} \subseteq F \text{ for some } m_0 \in M\}.$$

Then $\mathcal{F}_s$ is a filter on X, which is called the *filter associated with* s. If (X, τ) is a topological space, then $s \xrightarrow{t} x$ with respect to τ if and only if $\mathcal{F}_s \xrightarrow{t} x$ with respect to τ.

Definition 1.1 [9] Let X be a topological space. A net $(x_d)_{d \in D}$ in X is said to *$\mathcal{I}$-topology-converge* to a point $x \in X$, where $\mathcal{I}$ is an ideal on D, if for every open neighborhood U of x, $\{d \in D : x_d \notin U\} \in \mathcal{I}$. In this case we write $(x_d)_{d \in D} \xrightarrow{\mathcal{I}-t} x$.

Definition 1.2 [1] Let X be a poset. A net $(x_d)_{d\in D}$ in X is said to *order-converge* to a point $x \in X$ if there exist subsets A and B of X such that:

(1) A is directed and B is filtered.

(2) $x = \bigvee A = \bigwedge B$.

(3) For every $a \in A$ and $b \in B$, there exists $d_0 \in D$ such that $a \leqslant x_d \leqslant b$ hold for all $d \geqslant d_0$.

In this case we write $(x_d)_{d\in D} \xrightarrow{o} x$.

Given a poset X, by $\mathcal{T}_X^o$ we denote the set consisting of all subsets U of X satisfying the following property: If $(x_d)_{d\in D} \xrightarrow{o} x \in U$, then there exists $d_0 \in D$ such that $x_d \in U$ for every $d \geqslant d_0$. The set $\mathcal{T}_X^o$ forms a topology on X, which is called *the order topology on* X (see [21, 23]).

Definition 1.3 [19] Let X be a poset and $x, y, z \in X$. We define:

(1) $x \ll y$, if for any directed subset $A \subseteq X$, for which $\bigvee A$ exists and $y \leqslant \bigvee A$, there is $a \in A$ such that $x \leqslant a$.

(2) $z \rhd y$, if for any filtered subset $B \subseteq X$, for which $\bigwedge B$ exists and $\bigwedge B \leqslant y$, there is $b \in B$ such that $b \leqslant z$.

Clearly, if $x, y, z \in X$, then the following implications hold: $x \ll y \Rightarrow x \leqslant y$, and $z \rhd x \Rightarrow z \geqslant x$.

Definition 1.4 [19] A poset X is called *doubly continuous* if for each element $x \in X$, the set $\{a \in X : a \ll x\}$ is directed, the set $\{b \in X : b \rhd x\}$ is filtered and

$$x = \bigvee\{a \in X : a \ll x\} = \bigwedge\{b \in X : b \rhd x\}.$$

Definition 1.5 [23] The order-convergence in a poset X is called *topological*, if there exists a topology τ on X such that for every net $(x_d)_{d\in D}$ in X and $x \in X$ we have $(x_d)_{d\in D} \xrightarrow{o} x$ if and only if $(x_d)_{d\in D} \xrightarrow{t} x$ with respect to τ.

Proposition 1.6 [23] Let X be a complete lattice. If X satisfies the two infinite distributivity (the meet-infinite distributivity and the join-infinite distributivity) laws, then the following are equivalent:

(1) The order-convergence on X is topological.

(2) X is doubly continuous.

(3) X is a completely distributive lattice.

In the next we recall some definitions and results from [16].

Definition 1.7 Let X be a poset and $x, y, z \in X$. We define:

(1) $x \ll_S y$, if for every directed subset D of X with $\bigvee D = y$, there exists $d \in D$ such that $x \leqslant d$.

(2) $z \rhd_S y$, if for every filtered subset G of X with $\bigwedge G = y$, there exists $g \in G$ such that $z \geqslant g$.

Clearly, if $x, y, z \in X$, then the following implications hold: $x \ll y \Rightarrow x \ll_S y \Rightarrow x \leqslant y$, and $z \rhd x \Rightarrow z \rhd_S x \Rightarrow z \geqslant x$. Also for a poset X and $x \in X$ we use the following symbols: $\Downarrow_S x = \{a \in X : a \ll_S x\}$, $\Uparrow_S x = \{b \in X : x \ll_S b\}$, $\downarrow\downarrow_S x = \{c \in X : x \rhd_S c\}$ and $\uparrow\uparrow_S x = \{d \in X : d \rhd_S x\}$.

Definition 1.8 A poset X is called

(1) *S-doubly continuous* if for each element $x \in X$, the sets $\Downarrow_S x$ and $\uparrow\uparrow_S x$ are directed and filtered, respectively and $\bigvee \Downarrow_S x = \bigwedge \uparrow\uparrow_S x = x$, and

(2) *S^*-doubly continuous* if it is S-doubly continuous, and for every $x \in X$, $y \in \Downarrow_S x$ and $z \in \uparrow\uparrow_S x$, there exist $y_0 \in \Downarrow_S x$ and $z_0 \in \uparrow\uparrow_S x$ such that $[y_0, z_0] \subseteq \Uparrow_S y \cap \downarrow\downarrow_S z$.

Proposition 1.9 If X is a doubly continuous poset, then X is S^*-doubly continuous.

Definition 1.10 Let X be a poset.

(1) A filter $\mathcal{F}$ in X *order-converges to* x in the sense of Birkhoff if there exist a directed set D and a filtered set G such that $\bigvee D = x = \bigwedge G$ and $[a, b] \in \mathcal{F}$ for all $a \in D$ and $b \in G$. In this case, we write $\mathcal{F} \xrightarrow{O} x$.

(2) A subset U of X is called a *B-open* set if for any filter $\mathcal{F}$ that order converges to $x \in U$, there exists $F \in \mathcal{F}$ such that $F \subseteq U$. The set $\mathcal{T}_X$ of all B-open subsets of X forms a topology on X, which is called the *B-topology* on X.

Proposition 1.11 Let X be a poset and $U \subseteq X$. Then, $U \in \mathcal{T}_X$ if and only if for any directed subset D of X and any filtered subset G of X with $\bigvee D = \bigwedge G = x \in U$, there exist $d_0 \in D$ and $g_0 \in G$ such that $[d_0, g_0] \subseteq U$.

Theorem 1.12 For a poset X, the order-convergence in X is topological if and only if X is an S^*-doubly continuous poset.

2 Ideal-oder convergence

In this section we introduce the ideal-order-convergence in posets and prove some of its properties.

Definition 2.1 Let X be a poset. A net $(x_d)_{d\in D}$ in X is said to *$\mathcal{I}$-order-converge* to a point $x \in X$, where $\mathcal{I}$ is an ideal on D, if there exist subsets A and B of X such that:

(1) A is directed and B is filtered.

(2) $x = \bigvee A = \bigwedge B$.

(3) For every $a \in A$ and $b \in B$, $\{d \in D : x_d \notin [a, b]\} \in \mathcal{I}$.

Notation 2.2 Let $(x_d)_{d\in D}$ be a net in a poset X and let $\mathcal{I}$ be a non-trivial ideal on D. If $(x_d)_{d\in D}$ $\mathcal{I}$-order-converges to $x \in X$, then the point x is called the *$\mathcal{I}$-o-limit* of the net $(x_d)_{d\in D}$. In this case we write $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o} x$.

The ideal-convergences with respect to non-trivial ideals can be reduced to convergences of semi-subnets. More precisely, the following fact holds:

Proposition 2.3 *Let $(x_d)_{d\in D}$ be a net in a poset X and $\mathcal{I}$ a non-trivial ideal on D. Then there exists a semi-subnet $(y_\lambda)^{\varphi_\mathcal{I}}_{\lambda\in\Lambda_\mathcal{I}}$ of $(x_d)_{d\in D}$ such that for every $A \subseteq X$,*

$\{d \in D : x_d \notin A\} \in \mathcal{I}$ *if and only if there exists $\lambda_0 \in \Lambda_\mathcal{I}$ such that $y_\lambda \in A$ for all $\lambda \geqslant \lambda_0$.*

In particular, for $x \in X$ and a topology τ on X,

(1) $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-t} x$ *with respect to τ if and only if* $(y_\lambda)^{\varphi_\mathcal{I}}_{\lambda\in\Lambda_\mathcal{I}} \xrightarrow{t} x$ *with respect to τ.*

(2) $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o} x$ *if and only if* $(y_\lambda)^{\varphi_\mathcal{I}}_{\lambda\in\Lambda_\mathcal{I}} \xrightarrow{o} x$.

Proof. Set $\Lambda_{\mathcal{I}} = \{(d,I) \in D \times \mathcal{I} : d \notin I\}$ and define a preorder $\leqslant$ on $\Lambda_{\mathcal{I}}$ by letting $(d,I) \leqslant (d',I')$ if and only if $I \subseteq I'$ for $(d,I),(d',I') \in \Lambda_{\mathcal{I}}$. Since $\mathcal{I}$ is non-trivial, $(\Lambda_{\mathcal{I}}, \leqslant)$ is directed. Let $\varphi_{\mathcal{I}} : \Lambda_{\mathcal{I}} \to D$ such that $(d,I) \mapsto d$ be the projection. Then the semi-subnet $(y_\lambda)^{\varphi_{\mathcal{I}}}_{\lambda\in\Lambda_{\mathcal{I}}}$ of $(x_d)_{d\in D}$ is as required. Indeed, let $\{d \in D : x_d \notin A\} \in \mathcal{I}$ for some $A \subseteq X$. If we set $I_0 = \{d \in D : x_d \notin A\}$ and $\lambda_0 = (d_0, I_0)$, then for each $\lambda = (d,I) \geqslant \lambda_0$ (i.e. $I \supseteq I_0$) we have $y_\lambda = x_d \in A$.

Conversely, let that for some $A \subseteq X$ there exists $\lambda_0 = (d_0, I_0) \in \Lambda_{\mathcal{I}}$ such that $y_\lambda = x_d \in A$ for all $\lambda = (d,I) \geqslant \lambda_0$. Then $\{d \in D : x_d \notin A\} \subseteq I_0 \in \mathcal{I}$.
(1) Take $A = U$ an arbitrary τ-open neighborhood of x.
(2) Take $A = [a,b]$ an arbitrary interval. □

Proposition 2.4 *Suppose that the net $(x_d)_{d\in D}$ in X $\mathcal{I}$-order-converges to $x, y \in X$, where $\mathcal{I}$ is a non-trivial ideal on D. Then, $x = y$.*

Proof. It follows directly from Proposition 2.3 and the fact that a limit of order-convergence is uniquely determined (see Remark 1 in p.15 of [11]). □

Example 2.5 Let $(x_d)_{d\in D}$ be a net in a poset X and $x \in X$. We consider the family

$$\{A \subseteq D : A \subseteq \{d \in D : d \ngeqslant d_0\} \text{ for some } d_0 \in D\}.$$

This family is a non-trivial ideal on D which will be denoted by $\mathcal{I}_D$. The net $(x_d)_{d\in D}$ order-converges to x if and only if $(x_d)_{d\in D} \xrightarrow{\mathcal{I}_D - o} x$.

Example 2.6 Let $X = \{x\} \cup \{a_i : i \in \mathbb{N}\}$, where $\mathbb{N}$ denotes the set of all natural numbers. The order $\leqslant$ on X is defined as follows:

(O1) $a_i < x$, for every $i \in \mathbb{N}$.

(O2) For all $i, j \in \mathbb{N}$, if $i < j$, then $a_i < a_j$.

Then, $(a_i)_{i\in\mathbb{N}} \xrightarrow{o} x$. Indeed, for the subsets $A = \{a_i : i \in \mathbb{N}\}$ and $B = \{x\}$ of X we have:

(1) A is directed and B is filtered.

(2) $x = \bigvee A = \bigwedge B$.

(3) For every $i \in \mathbb{N}$, there exists $j_0 \in \mathbb{N}$ ($j_0 = i$) such that $a_i \leqslant a_j \leqslant x$ hold for all $j \geqslant j_0$.

Generally, for every admissible ideal $\mathcal{I}$ on $\mathbb{N}$, namely, $\mathcal{I}$ contains all finite subsets of $\mathbb{N}$, we have $(a_i)_{i\in\mathbb{N}} \xrightarrow{\mathcal{I}-o} x$. Let $\mathcal{I}_e$ be the ideal of even numbers on $\mathbb{N}$. Then, the net $(a_i)_{i\in\mathbb{N}}$ does not $\mathcal{I}_e$-order-converge to x.

Proposition 2.7 *If $(x_d)_{d\in D}$ is a net with $x_d = x$ for every $d \in D$, then $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o} x$ holds for every ideal $\mathcal{I}$ of D.*

Proof. The sets $A = B = \{x\}$ satisfy the conditions of Definition 2.1. Particularly, for the condition (3) we have $\{d \in D : x_d = x \notin \{x\}\} = \emptyset \in \mathcal{I}$. □

Proposition 2.8 *If $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o} x$, then for every semi-subnet $(y_\lambda)^{\varphi}_{\lambda\in\Lambda}$ of the net $(x_d)_{d\in D}$ we have $(y_\lambda)^{\varphi}_{\lambda\in\Lambda} \xrightarrow{\mathcal{I}_\Lambda(\varphi)-o} x$.*

Proof. Let $(y_\lambda)^{\varphi}_{\lambda\in\Lambda}$ be a semi-subnet of the net $(x_d)_{d\in D}$. Suppose that A and B are subsets of X such that:

(1) A is directed and B is filtered.

(2) $x = \bigvee A = \bigwedge B$.

(3) For every $a \in A$ and $b \in B$, $\{d \in D : x_d \notin [a,b]\} \in \mathcal{I}$.

It suffices to prove that for every $a \in A$ and $b \in B$, $\{\lambda \in \Lambda : y_\lambda \notin [a,b]\} \in \mathcal{I}_\Lambda(\varphi)$. Let $C = \{\lambda \in \Lambda : y_\lambda \notin [a,b]\}$. If $C = \emptyset$, then we are done. Suppose that $C \neq \emptyset$. We prove that $\varphi(C) \in \mathcal{I}$. Let $\varphi(\lambda) \in \varphi(C)$, where $\lambda \in C$. Since $y_\lambda = x_{\varphi(\lambda)} \notin [a,b]$, we have $\varphi(\lambda) \in \{d \in D : x_d \notin [a,b]\}$ which means that $\varphi(C) \subseteq \{d \in D : x_d \notin [a,b]\}$. Since $\{d \in D : x_d \notin [a,b]\} \in \mathcal{I}$, $\varphi(C) \in \mathcal{I}$. □

Proposition 2.9 *Let X be a poset and $x, y, z \in X$. If $y \ll_S x$ and $z \rhd_S x$, then for every net $(x_d)_{d\in D}$ in X, which $\mathcal{I}$-order-converges to x, where $\mathcal{I}$ is a non-trivial ideal on D, $\{d \in D : x_d \notin [y,z]\} \in \mathcal{I}$.*

Proof. Let $y \ll_S x$, $z \rhd_S x$ and $(x_d)_{d\in D}$ be a net in X which $\mathcal{I}$-order-converges to x, where $\mathcal{I}$ is a non-trivial ideal on D. Then, there exist subsets A and B of X such that:

(1) A is directed and B is filtered.

(2) $x = \bigvee A = \bigwedge B$.

(3) For each $a \in A$ and $b \in B$, $\{d \in D : x_d \notin [a,b]\} \in \mathcal{I}$.

Since $y \ll_S x$, there exists $a_0 \in A$ such that $y \leqslant a_0$ and since $z \rhd_S x$, there exists $b_0 \in B$ such that $b_0 \leqslant z$. By assumption, for $a_0 \in A$ and $b_0 \in B$ we have that $\{d \in D : x_d \notin [a_0,b_0]\} \in \mathcal{I}$. Since $\{d \in D : x_d \notin [y,z]\} \subseteq \{d \in D : x_d \notin [a_0,b_0]\}$, we have that $\{d \in D : x_d \notin [y,z]\} \in \mathcal{I}$. □

Corollary 2.10 *Let X be a poset and $x, y, z \in X$. If $y \ll x$ and $z \rhd x$, then for every net $(x_d)_{d\in D}$ in X, which $\mathcal{I}$-order-converges to x, where $\mathcal{I}$ is a non-trivial ideal on D, $\{d \in D : x_d \notin [y,z]\} \in \mathcal{I}$.*

Proposition 2.11 *Let X be a S-doubly continuous poset, $(x_d)_{d\in D}$ be a net in X, $x \in X$, and $\mathcal{I}$ be a non-trivial ideal on D. If for every $y, z \in X$ with $y \ll_S x$ and $z \rhd_S x$ we have $\{d \in D : x_d \notin [y,z]\} \in \mathcal{I}$, then $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o} x$.*

Proof. Is a direct consequence of the Definitions 2.1 and 1.8. □

Proposition 2.12 *Let X be a doubly continuous poset, $(x_d)_{d\in D}$ be a net in X, $x \in X$, and $\mathcal{I}$ be a non-trivial ideal on D. If for every $y, z \in X$ with $y \ll x$ and $z \rhd x$ we have $\{d \in D : x_d \notin [y,z]\} \in \mathcal{I}$, then $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o} x$.*

Proof. Is a direct consequence of the Definitions 2.1 and 1.4. □

3 Topologies in posets

In this section we introduce topologies in posets and we give a sufficient and necessary condition for the ideal-order-convergence in a poset to be topological.

Proposition 3.1 *Let X be a set and let $\mathcal{C}_X$ be a class consisting of triads $((x_d)_{d\in D}, x, \mathcal{I})$, where $(x_d)_{d\in D}$ is a net in X, $x \in X$, and $\mathcal{I}$ is a non-trivial ideal on D. The family*

$$\{U \subseteq X : \{d \in D : x_d \notin U\} \in \mathcal{I} \text{ for every } ((x_d)_{d\in D}, x, \mathcal{I}) \in \mathcal{C}_X, x \in U\}$$

is a topology $\tau(\mathcal{C}_X)$ on X.

Proof. Obviously $\emptyset \in \tau(\mathcal{C}_X)$. Moreover, since $\{d \in D : x_d \notin X\} = \emptyset \in \mathcal{I}$, $X \in \tau(\mathcal{C}_X)$. Let $U, V \in \tau(\mathcal{C}_X)$ and $((x_d)_{d\in D}, x, \mathcal{I}) \in \mathcal{C}_X$, $x \in U \cap V$. Then, $\{d \in D : x_d \notin U\} \in \mathcal{I}$ and $\{d \in D : x_d \notin V\} \in \mathcal{I}$. Therefore,

$$\{d \in D : x_d \notin U \cap V\} = \{d \in D : x_d \notin U\} \cup \{d \in D : x_d \notin V\} \in \mathcal{I}$$

which means that the intersection $U \cap V \in \tau(\mathcal{C}_X)$. Now, let $U_i \in \tau(\mathcal{C}_X)$, $i \in I$ and $((x_d)_{d\in D}, x, \mathcal{I}) \in \mathcal{C}_X$, $x \in \cup_{i\in I} U_i$. Then, $\{d \in D : x_d \notin U_{i_0}\} \in \mathcal{I}$ for some $i_0 \in I$. Since

$$\{d \in D : x_d \notin \cup_{i\in I} U_i\} \subseteq \{d \in D : x_d \notin U_{i_0}\} \in \mathcal{I},$$

we have $\{d \in D : x_d \notin \cup_{i\in I} U_i\} \in \mathcal{I}$. Hence, $\cup_{i\in I} U_i \in \tau(\mathcal{C}_X)$. $\square$

Proposition 3.2 *If $((x_d)_{d\in D}, x, \mathcal{I}) \in \mathcal{C}_X$, then $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-t} x$ with respect to $\tau(\mathcal{C}_X)$.*

Proof. Let $((x_d)_{d\in D}, x, \mathcal{I}) \in \mathcal{C}_X$ and U be an open neighborhood of x. Since $x \in U \in \tau(\mathcal{C}_X)$, by the definition of the topology $\tau(\mathcal{C}_X)$, we have $\{d \in D : x_d \notin U\} \in \mathcal{I}$. Therefore, $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-t} x$ with respect to $\tau(\mathcal{C}_X)$. $\square$

Notation 3.3 For an arbitrary poset X, we denote by $\mathcal{C}_X^o$ the class consisting of triads $((x_d)_{d\in D}, x, \mathcal{I})$, where $(x_d)_{d\in D}$ is a net in X, $x \in X$, and $\mathcal{I}$ is a non-trivial ideal on D such that $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o} x$.

Proposition 3.4 *Let X be a poset. Then, $\tau(\mathcal{C}_X^o) = \mathcal{T}_X^o$.*

Proof. Firstly, we prove that $\tau(\mathcal{C}_X^o) \subseteq \mathcal{T}_X^o$. Let $U \in \tau(\mathcal{C}_X^o)$ and a net $(x_d)_{d\in D} \xrightarrow{o} x \in U$. Then by Example 2.5 $(x_d)_{d\in D} \xrightarrow{\mathcal{I}_D-o} x$. By the definition of $\mathcal{I}_D$ it follows that $(x_d)_{d\in D}$ is eventually in U. Thus $U \in \mathcal{T}_X^o$.

We prove the opposite direction $\mathcal{T}_X^o \subseteq \tau(\mathcal{C}_X^o)$. Let $U \in \mathcal{T}_X^o$ and a net $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o} x \in U$, where $\mathcal{I}$ is a non-trivial ideal on D. Then by Proposition 2.3 the net $(y_\lambda)^{\varphi_\mathcal{I}}_{\lambda\in\Lambda_\mathcal{I}} \xrightarrow{o} x$. That is, there exists $\lambda_0 \in \Lambda_\mathcal{I}$ such that $y_\lambda \in U$ for all $\lambda \geqslant \lambda_0$. Thus $\{d \in D : x_d \notin U\} \in \mathcal{I}$, which means that $U \in \tau(\mathcal{C}_X^o)$. $\square$

The following result is a characterization of open sets in $\mathcal{T}_X^o$.

Lemma 3.5 Let X be a poset and $U \subseteq X$. Then, $U \in \mathcal{T}_X^o$ if and only if for any directed subset D of X and any filtered subset F of X with $\bigvee D = \bigwedge F = x \in U$, there exist $d_0 \in D$ and $f_0 \in F$ such that $[d_0, f_0] \subseteq U$.

Proof. Let $U \in \mathcal{T}_X^o$, D be a directed subset of X, F be a filtered subset of X and $\bigvee D = \bigwedge F = x \in U$. Suppose that for each $d \in D$ and $f \in F$ there exist $g_{(d,f)} \in X$ with $d \leqslant g_{(d,f)} \leqslant f$ and $g_{(d,f)} \notin U$. The Cartesian product $D \times F$ is directed if we define $(d', f') \geqslant (d, f)$ to mean that $d' \geqslant d$ and $f' \leqslant f$. Then, $(g_{(d,f)})_{(d,f)\in D\times F} \xrightarrow{o} x$, and, therefore, the net $(g_{(d,f)})_{(d,f)\in D\times F}$ converges to x, with respect to $\mathcal{T}_X^o$, contradiction. Thus, for some $d_0 \in D$ and $f_0 \in F$ we get $[d_0, f_0] \subseteq U$.

Now, let $U \subseteq X$ and suppose that for any directed subset D of X and any filtered subset F of X with $\bigvee D = \bigwedge F = x \in U$, there exist $d_0 \in D$ and $f_0 \in F$ such that $[d_0, f_0] \subseteq U$. Consider a net $(x_\lambda)_{\lambda\in\Lambda} \xrightarrow{o} x \in U$. Then, by Definition 1.2 there exist a directed subset E of X and a filtered subset G of X with $\bigvee E = \bigwedge G = x$ and for every $e \in E$ and $g \in G$, there exists $\lambda_{e,g} \in \Lambda$ such that $x_\lambda \in [e, g]$ for every $\lambda \geqslant \lambda_{e,g}$. By hypothesis there exist $e_0 \in E$ and $g_0 \in G$ such that $[e_0, g_0] \subseteq U$. Consequently, there exists $\lambda_0 \in \Lambda$ such that $x_\lambda \in [e_0, g_0] \subseteq U$ for every $\lambda \geqslant \lambda_0$. Hence, by the definition of the topology $\mathcal{T}_X^o$ we have $U \in \mathcal{T}_X^o$. $\square$

Lemma 3.6 *Let X be a poset and $U \subseteq X$. Then, $U \in \tau(\mathcal{C}^o_X)$ if and only if for any directed subset D of X and any filtered subset F of X with $\bigvee D = \bigwedge F = x \in U$, there exist $d_0 \in D$ and $f_0 \in F$ such that $[d_0, f_0] \subseteq U$.*

Proof. The proof is similar to the proof of Lemma 3.5. □

Remark 3.7 *We observe that Proposition 3.4 it follows, alternatively, as a direct consequence of the Lemmas 3.5 and 3.6. Also, given a poset X, in view of Lemma 3.5 and Proposition 1.11 we have that the topology $\mathcal{T}^o_X$ on X is equal to the B-topology on X (see Definition 1.10).*

Corollary 3.8 *Let X be a poset. If $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o} x$, then $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-t} x$ with respect to $\mathcal{T}^o_X$.*

Proof. Is similar to Proposition 3.2. □

Proposition 3.9 *Let X be a poset. The topology $\mathcal{T}^o_X$ is the finest topology τ on X such that ideal-order-convergence implies ideal-topology-convergence with respect to τ.*

Proof. Let τ be a topology on X such that ideal-order-convergence implies ideal-topology-convergence with respect to τ. We prove that $\tau \subseteq \mathcal{T}^o_X$. Let $U \in \tau$. It suffices to prove that for every $((x_d)_{d\in D}, x, \mathcal{I}) \in \mathcal{C}^o_X$, $x \in U$ we have that $\{d \in D : x_d \notin U\} \in \mathcal{I}$ (see Proposition 3.4). Let $((x_d)_{d\in D}, x, \mathcal{I}) \in \mathcal{C}^o_X$. Then, $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o} x$ and, by assumption, $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-t} x$ with respect to τ. Therefore, $\{d \in D : x_d \notin U\} \in \mathcal{I}$. □

Definition 3.10 The ideal-order-convergence in a poset X is called *topological*, if there exists a topology τ on X such that for every net $(x_d)_{d\in D}$ in X, $x \in X$ and for every non-trivial ideal $\mathcal{I}$ of D, $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o} x$ if and only if $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-t} x$ with respect to τ.

Proposition 3.11 *Let X be a poset such that the ideal-order-convergence is topological and let τ be the corresponding topology on X. Then, $\tau \subseteq \mathcal{T}^o_X$.*

Proof. Is a direct consequence of the Proposition 3.9. □

Proposition 3.12 *The ideal-order-convergence in a poset X is topological if and only if the order-convergence in X is topological.*

Proof. Consider a poset X and suppose that the ideal-order-convergence in X is topological. Let $(x_d)_{d\in D}$ be a net in X and $x \in X$. For the non-trivial ideal $\mathcal{I}_D$ of D (see Example 2.5) we have that $(x_d)_{d\in D} \xrightarrow{\mathcal{I}_D-o} x$ if and only if $(x_d)_{d\in D} \xrightarrow{\mathcal{I}_D-t} x$ with respect to some topology τ on X. Therefore, $(x_d)_{d\in D} \xrightarrow{o} x$ if and only if $(x_d)_{d\in D}$ converges to x with respect to τ. Thus, the order-convergence in X is topological.

Conversely, suppose that the order-convergence in X is topological. Let $(x_d)_{d\in D}$ be a net in X, $\mathcal{I}$ a non-trivial ideal on D and $x \in X$. Then by Proposition 2.3 and hypothesis we have the following equivalences: $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o} x$ if and only if $(y_\lambda)^{\varphi_{\mathcal{I}}}_{\lambda\in\Lambda_{\mathcal{I}}} \xrightarrow{o} x$ if and only if $(y_\lambda)^{\varphi_{\mathcal{I}}}_{\lambda\in\Lambda_{\mathcal{I}}} \xrightarrow{t} x$ with respect to some topology τ on X if and only if $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-t} x$ with respect to τ. Thus, the ideal-order-convergence in X is topological. □

As the study of the notion of the ideal-order-convergence is extended, it raises the necessity to clarify, in which posets, is the ideal-order-convergence topological. Following [16] we prove that for a poset X the ideal-order-convergence is topological if and only if X is an S^*-doubly continuous poset.

Proposition 3.13 *Let X be a poset.*

(1) *If $\mathcal{F}$ is a filter on X and $s_{\mathcal{F}}$ is its associated net, then $\mathcal{F} \xrightarrow{O} x \in X$ (in the sense of Definition 1.10) if and only if $s_{\mathcal{F}} \xrightarrow{o} x$ (in the sense of Definition 1.2).*

(2) *If $s : M \to X$ is a net in X and $\mathcal{F}_s$ is its associated filter, then $s \xrightarrow{o} x \in X$ (in the sense of Definition 1.2) if and only if $\mathcal{F}_s \xrightarrow{O} x$ (in the sense of Definition 1.10).*

Proof. (1) Suppose that $\mathcal{F} \xrightarrow{O} x \in X$. Then, there exist a directed set $D \subseteq X$ and a filtered set $G \subseteq X$ such that $\vee D = \wedge G = x$ and $[a, b] = E \in \mathcal{F}$ for all $a \in D$ and $b \in G$. It follows that for every $(f, F) \geqslant (e, E)$ equivalently $F \subseteq E$, we have $s_{\mathcal{F}}(f, F) = f \in F \subseteq E \Rightarrow a \leqslant s_{\mathcal{F}}(f, F) \leqslant b$. Thus $s_{\mathcal{F}} \xrightarrow{o} x$.

Conversely, let $s_{\mathcal{F}} \xrightarrow{o} x \in X$. Then, there exist a directed set $D \subseteq X$ and a filtered set $G \subseteq X$ such that $\vee D = \wedge G = x$ and for every $a \in D$ and $b \in G$ there exists $m_0 = (f_0, F_0) \in M$ such that $a \leqslant s_{\mathcal{F}}(m) \leqslant b$ for all $m \geqslant m_0$. Then, for all $f \in F_0$ we have $a \leqslant s_{\mathcal{F}}(f, F_0) = f \leqslant b$, since $(f, F_0) \geqslant (f_0, F_0)$. Thus, $F_0 \subseteq [a, b]$. So $[a, b] \in \mathcal{F}$ and $\mathcal{F} \xrightarrow{O} x$.

(2) Suppose that $s \xrightarrow{o} x \in X$. Then, there exist a directed set $D \subseteq X$ and a filtered set $G \subseteq X$ such that $\vee D = \wedge G = x$ and for every $a \in D$ and $b \in G$ there exists $m_0 \in M$ such that $a \leqslant s(m) \leqslant b$ for all $m \geqslant m_0$, which means that $[a, b] \supseteq \{s(m) : m \geqslant m_0\} \in \mathcal{F}_s$ and thus $\mathcal{F}_s \xrightarrow{O} x$.

Conversely, let $\mathcal{F}_s \xrightarrow{O} x \in X$. Then, there exist a directed set $D \subseteq X$ and a filtered set $G \subseteq X$ such that $\vee D = \wedge G = x$ and $[a, b] \in \mathcal{F}_s$ for all $a \in D$ and $b \in G$. This means that for some $m_0 \in M$ we have $\{s(m) : m \geqslant m_0\} \subseteq [a, b]$ and thus $s \xrightarrow{o} x$. □

We observe that the coincidence of $\mathcal{T}_X^o$ and B-topology on X is, also, immediate from Proposition 3.13.

Proposition 3.14 *The order-convergence in a poset X (in the sense of Definition 1.2) is topological if and only if the order-convergence in X (in the sense of Definition 1.10) is topological.*

Proof. Is a direct consequence of Proposition 3.13. □

Proposition 3.15 *For a poset X, the ideal-oder convergence is topological for the $\mathcal{T}_X^o$ topology if and only if X is an S^*-doubly continuous poset.*

Proof. Is a direct consequence of Theorem 1.12, Remark 3.7, Proposition 3.12 and Proposition 3.14. □

4 Ideal-order-convergence in Cartesian products of posets

In this section we study ideal-order-convergence in the Cartesian product of two posets X and Y.

For an ideal (resp., filter) $\mathcal{I}$ on a set X, let $\mathcal{I}^*$ denote the dual filter (resp., ideal) on $\mathcal{I}$, that is, $\mathcal{I}^* = \{A \subseteq X : X \setminus A \in \mathcal{I}\}$. For filters $\mathcal{F}_1$ and $\mathcal{F}_2$ on sets D_1 and D_2, respectively, let $\mathcal{F}_1 \times \mathcal{F}_2$ denote the product filter, that is,

$$\mathcal{F}_1 \times \mathcal{F}_2 = \{A \subseteq D_1 \times D_2 : F_1 \times F_2 \subseteq A \text{ for some } F_1 \in \mathcal{F}_1 \text{ and some } F_2 \in \mathcal{F}_2\}.$$

Then the following trivial facts hold:

(1) An ideal (resp., filter) $\mathcal{I}$ on a set X is non-trivial if and only if so is the dual filter (resp., ideal) $\mathcal{I}^*$.

(2) If filters $\mathcal{F}_1$ and $\mathcal{F}_2$ on sets D_1 and D_2, respectively, are non-trivial, so is the product filter $\mathcal{F}_1 \times \mathcal{F}_2$.

Proposition 4.1 *Let D_1, D_2 be two directed sets and let $\mathcal{I}_1$, $\mathcal{I}_2$ be two non-trivial ideals on D_1 and D_2, respectively. The family $(\mathcal{I}_1^* \times \mathcal{I}_2^*)^*$ is a non-trivial ideal on $D_1 \times D_2$, which will denote by $\mathcal{I}_1 \times \mathcal{I}_2$.*

Proof. Is an easy consequence of the above discussion. □

Proposition 4.2 *Let X and Y be two posets. Then, we have $(x_{d_1})_{d_1 \in D_1} \xrightarrow{\mathcal{I}_1 - o} x$ and $(y_{d_2})_{d_2 \in D_2} \xrightarrow{\mathcal{I}_2 - o} y$, where $\mathcal{I}_1$ and $\mathcal{I}_2$ are two non-trivial ideals of D_1 and D_2, respectively if and only if $((x_{d_1}, y_{d_2}))_{(d_1,d_2) \in D_1 \times D_2} \xrightarrow{\mathcal{I}_1 \times \mathcal{I}_2 - o} (x, y)$.*

Proof. Let $(x_{d_1})_{d_1 \in D_1} \xrightarrow{\mathcal{I}_1 - o} x$ and $(y_{d_2})_{d_2 \in D_2} \xrightarrow{\mathcal{I}_2 - o} y$. We prove that

$$((x_{d_1}, y_{d_2}))_{(d_1,d_2) \in D_1 \times D_2} \xrightarrow{\mathcal{I}_1 \times \mathcal{I}_2 - o} (x, y).$$

There exist subsets A_1, B_1 and A_2, B_2 of X and Y, respectively such that:

(1) A_1, A_2 are directed and B_1, B_2 are filtered.

(2) $x = \bigvee A_1 = \bigwedge B_1$ and $y = \bigvee A_2 = \bigwedge B_2$.

(3) For every $a_1 \in A_1$ and $b_1 \in B_1$, $\{d_1 \in D_1 : x_{d_1} \notin [a_1, b_1]\} \in \mathcal{I}_1$.

(4) For every $a_2 \in A_2$ and $b_2 \in B_2$, $\{d_2 \in D_2 : y_{d_2} \notin [a_2, b_2]\} \in \mathcal{I}_2$.

We set $A = A_1 \times A_2$ and $B = B_1 \times B_2$. Then:

(5) A is directed and B is filtered.

(6) $(x, y) = \bigvee A = \bigwedge B$.

Let $(a_1, a_2) \in A$ and $(b_1, b_2) \in B$. We prove that

$$\{(d_1, d_2) \in D_1 \times D_2 : (x_{d_1}, y_{d_2}) \notin [(a_1, a_2), (b_1, b_2)]\} \in \mathcal{I}_1 \times \mathcal{I}_2.$$

It suffices to prove that

$$W = \{(d_1, d_2) \in D_1 \times D_2 : (x_{d_1}, y_{d_2}) \in [(a_1, a_2), (b_1, b_2)]\} \in \mathcal{I}_1^* \times \mathcal{I}_2^*.$$

We set $I_1 = \{d_1 \in D_1 : x_{d_1} \notin [a_1, b_1]\}$ and $I_2 = \{d_2 \in D_2 : y_{d_2} \notin [a_2, b_2]\}$.
Then $D_1 \setminus I_1 = \{d_1 \in D_1 : x_{d_1} \in [a_1, b_1]\} \in \mathcal{I}_1^*$ and $D_2 \setminus I_2 = \{d_2 \in D_2 : y_{d_2} \in [a_2, b_2]\} \in \mathcal{I}_2^*$.
We see that

$$(D_1 \setminus I_1) \times (D_2 \setminus I_2) \subseteq W.$$

Therefore, $W \in \mathcal{I}_1^* \times \mathcal{I}_2^*$.

Conversely, let $((x_{d_1}, y_{d_2}))_{(d_1,d_2) \in D_1 \times D_2} \xrightarrow{\mathcal{I}_1 \times \mathcal{I}_2 - o} (x, y)$. We prove that

$$(x_{d_1})_{d_1 \in D_1} \xrightarrow{\mathcal{I}_1 - o} x.$$

There exist subsets A and B of $X \times Y$ such that:

(7) A is directed and B is filtered.

(8) $(x, y) = \bigvee A = \bigwedge B$.

(9) For every $(a_1, a_2) \in A$ and $(b_1, b_2) \in B$,
$\{(d_1, d_2) \in D_1 \times D_2 : (x_{d_1}, y_{d_2}) \notin [(a_1, a_2), (b_1, b_2)]\} \in \mathcal{I}_1 \times \mathcal{I}_2$.

We set:
$A_1 = \{x_1 \in X : (x_1, y_1) \in A \text{ for some } y_1 \in Y\}$,
$B_1 = \{x_1 \in X : (x_1, y_1) \in B \text{ for some } y_1 \in Y\}$.
Then A_1 is directed, B_1 is filtered and $x = \bigvee A_1 = \bigwedge B_1$.
We prove that:

(10) For every $a_1 \in A_1$ and $b_1 \in B_1$, $\{d_1 \in D_1 : x_{d_1} \notin [a_1, b_1]\} \in \mathcal{I}_1$.

Let $a_1 \in A_1$ and $b_1 \in B_1$. Then, there exist $a_2, b_2 \in Y$ such that $(a_1, a_2) \in A$ and $(b_1, b_2) \in B$. Hence,

$$\{(d_1, d_2) \in D_1 \times D_2 : (x_{d_1}, y_{d_2}) \notin [(a_1, a_2), (b_1, b_2)]\} \in \mathcal{I}_1 \times \mathcal{I}_2,$$

or equivalently

$$W = \{(d_1, d_2) \in D_1 \times D_2 : (x_{d_1}, y_{d_2}) \in [(a_1, a_2), (b_1, b_2)]\} \in \mathcal{I}_1^* \times \mathcal{I}_2^*.$$

Therefore, there exist $I_1 \in \mathcal{I}_1$ and $I_2 \in \mathcal{I}_2$ such that $(D_1 \setminus I_1) \times (D_2 \setminus I_2) \subseteq W$. Since

$$\{d_1 \in D_1 : x_{d_1} \in [a_1, b_1]\} \supseteq D_1 \setminus I_1 \in \mathcal{I}_1^*,$$

we have $\{d_1 \in D_1 : x_{d_1} \notin [a_1, b_1]\} \in \mathcal{I}_1$. Similarly, we get $(y_{d_2})_{d_2 \in D_2} \xrightarrow{\mathcal{I}_2 - o} y$. □

Proposition 4.3 *Let X and Y be two posets. Then, $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o} x$ and $(y_d)_{d\in D} \xrightarrow{\mathcal{I}-o} y$ if and only if $((x_d, y_d))_{d\in D} \xrightarrow{\mathcal{I}-o} (x, y)$.*

Proof. Is similar to the proof of Proposition 4.2. □

Based on the ideas of papers [4–6], we will use Proposition 3.4 to prove the following two propositions.

Proposition 4.4 *Let X and Y be two posets. Then, $\mathcal{T}_X^o \times \mathcal{T}_Y^o \subseteq \mathcal{T}_{X\times Y}^o$.*

Proof. Suppose that $U_X \in \mathcal{T}_X^o$ and $U_Y \in \mathcal{T}_Y^o$. It suffices to prove that $U_X \times U_Y \in \mathcal{T}_{X\times Y}^o$. Let $(((x_d, y_d))_{d\in D}, (x, y), \mathcal{I}) \in \mathcal{C}_{X\times Y}^o$, $(x, y) \in U_X \times U_Y$. From Proposition 4.3, it follows that $((x_d)_{d\in D}, x, \mathcal{I}) \in \mathcal{C}_X^o$ and $((y_d)_{d\in D}, y, \mathcal{I}) \in \mathcal{C}_Y^o$, where $x \in U_X$ and $y \in U_Y$. Therefore,

$$\{d \in D : x_d \notin U_X\} \in \mathcal{I} \text{ and } \{d \in D : y_d \notin U_Y\} \in \mathcal{I}.$$

Since $\{d \in D : (x_d, y_d) \notin U_X \times U_Y\} = \{d \in D : x_d \notin U_X\} \cup \{d \in D : y_d \notin U_Y\} \in \mathcal{I}$, we conclude that $\{d \in D : (x_d, y_d) \notin U_X \times U_Y\} \in \mathcal{I}$ and, consequently, the product $U_X \times U_Y \in \mathcal{T}_{X\times Y}^o$. □

Proposition 4.5 *Let X and Y be two posets. The Cartesian product topology $\mathcal{T}_X^o \times \mathcal{T}_Y^o$ coincides with the topology $\mathcal{T}_{X\times Y}^o$ if the latter has a base of Cartesian product sets.*

Proof. By Proposition 4.4 it suffices to prove that $\mathcal{T}_{X\times Y}^o \subseteq \mathcal{T}_X^o \times \mathcal{T}_Y^o$. Consider any product $U_X \times U_Y$ which is open in the topology $\mathcal{T}_{X\times Y}^o$. We prove that $U_X \times U_Y \in \mathcal{T}_X^o \times \mathcal{T}_Y^o$. For this purpose we show that $U_X \in \mathcal{T}_X^o$ and $U_Y \in \mathcal{T}_Y^o$. Let $((x_d)_{d\in D}, x, \mathcal{I}) \in \mathcal{C}_X^o$, $x \in U_X$. Let $y \in U_Y$ and consider the net $(y_d)_{d\in D}$, where $y_d = y$ for every $d \in D$. By Propositions 2.7 and 4.3 we have $(((x_d, y_d))_{d\in D}, (x, y), \mathcal{I}) \in \mathcal{C}_{X\times Y}^o$, $(x, y) \in U_X \times U_Y$. Since $U_X \times U_Y \in \mathcal{T}_{X\times Y}^o$, we have $\{d \in D : (x_d, y_d) \notin U_X \times U_Y\} \in \mathcal{I}$. Now, since

$$\{d \in D : x_d \notin U_X\} \subseteq \{d \in D : (x_d, y_d) \notin U_X \times U_Y\},$$

we have $\{d \in D : x_d \notin U_X\} \in \mathcal{I}$. Therefore, $U_X \in \mathcal{T}_X^o$. Similarly, we can see that $U_Y \in \mathcal{T}_Y^o$. □

5 Ideal-o_2-convergence and ideal-o_2-topology

A generalization of the ideal-order-convergence in posets, the so-called ideal-o_2-convergence, is discussed in this section. Moreover, an investigation of the topological ideal-o_2-convergence in posets completes this section.

We will need the following notions.

Definition 5.1 [13, 18] Let X be a poset. A net $(x_d)_{d\in D}$ in X is said to *o_2-converge* to a point $x \in X$ if there exist subsets M and N of X such that:

(1) $x = \bigvee M = \bigwedge N$.

(2) For each $m \in M$ and $n \in N$, there exists $d_0 \in D$ such that $m \leqslant x_d \leqslant n$ hold for all $d \geqslant d_0$.

In this case we write $(x_d)_{d\in D} \xrightarrow{o_2} x$.

Definition 5.2 [20] Let X be a poset and $x, y, z \in X$. We define:

(1) $x \ll_\alpha y$, if for every net $(x_d)_{d\in D}$ in X with $(x_d)_{d\in D} \xrightarrow{o_2} y$ there exists $d_0 \in D$ such that $x_d \geqslant x$ for every $d \geqslant d_0$.

(2) $z \rhd_\alpha y$, if for every net $(x_d)_{d\in D}$ in X with $(x_d)_{d\in D} \xrightarrow{o_2} y$ there exists $d_0 \in D$ such that $x_d \leqslant z$ for every $d \geqslant d_0$.

Definition 5.3 [20] A poset X is called *α-doubly continuous* if for each element $x \in X$, $x = \bigvee\{a \in X : a \ll_\alpha x\} = \bigwedge\{b \in X : b \rhd_\alpha x\}$.

Definition 5.4 [10] A poset X is called *O_2-doubly continuous* if it satisfies the following conditions:

(1) X is α-doubly continuous and

(2) if $y \ll_\alpha x$ and $z \rhd_\alpha x$, then there exist $A \subseteq_{fin} \{a \in X : a \ll_\alpha x\}$ and $B \subseteq_{fin} \{b \in X : b \rhd_\alpha x\}$ such that $y \ll_\alpha c$ and $z \rhd_\alpha c$ for each $c \in \bigcap\limits_{m\in A} \bigcap\limits_{n\in B} [m, n]$.

Definition 5.5 Let X be a poset. A net $(x_d)_{d\in D}$ in X is said to *$\mathcal{I}$-o_2-converge* to a point $x \in X$, where $\mathcal{I}$ is a non-trivial ideal on D, if there exist subsets M and N of X such that:

(1) $x = \bigvee M = \bigwedge N$.

(2) For each $m \in M$ and $n \in N$, $\{d \in D : x_d \notin [m, n]\} \in \mathcal{I}$.

Notation 5.6 Let $(x_d)_{d\in D}$ be a net in a poset X and let $\mathcal{I}$ be a non-trivial ideal on D. If $(x_d)_{d\in D}$ $\mathcal{I}$-o_2-converges to $x \in X$, then the point x is called the *$\mathcal{I}$-o_2-limit* of the net $(x_d)_{d\in D}$. In this case we write $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o_2} x$.

Proposition 5.7 *Let X be a poset, $(x_d)_{d\in D}$ be a net in X and $\mathcal{I}$ a non-trivial ideal on D. Then $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o_2} x$ if and only if $(y_\lambda)^{\varphi_\mathcal{I}}_{\lambda\in\Lambda_\mathcal{I}} \xrightarrow{o_2} x$.*

Proof. Is similar to Proposition 2.3 (2). □

Proposition 5.8 *If a net $(x_d)_{d\in D}$ in X $\mathcal{I}$-o_2-converges to $x, y \in X$, where $\mathcal{I}$ is a non-trivial ideal on D, then $x = y$.*

Proof. It follows directly from Proposition 5.7 and the fact that a limit of o_2-convergence is uniquely determined (see Remark 3 (2) of [20]). □

Proposition 5.9 *Let $(x_d)_{d\in D}$ be a net in a poset X and let $\mathcal{I}$ be a non-trivial ideal on D. If $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o} x$, where $x \in X$, then $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o_2} x$. Therefore, the $\mathcal{I}$-order-convergence implies the $\mathcal{I}$-o_2-convergence.*

Proof. Is a direct consequence of the Definitions 2.1 and 5.5. □

The converse of Proposition 5.9 is not necessarily true as the following example verifies.

Example 5.10 Let $(\mathbb{Z}, \leqslant)$ be the poset represented by the following diagram:

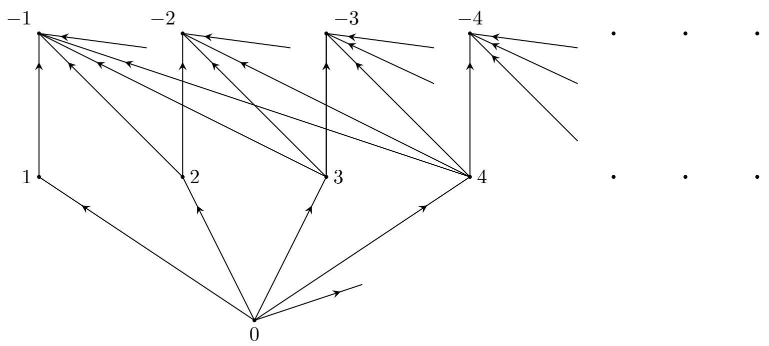

Figure 1: The poset $(\mathbb{Z}, \leqslant)$

Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$. For the net $(a_n)_{n\in\mathbb{N}}$, where $a_n = n$, $n \in \mathbb{N}$, we have $(a_n)_{n\in\mathbb{N}} \xrightarrow{\mathcal{I}-o_2} 0$. Indeed, for the subsets $M = \{0\}$ and $N = \{-n : n \in \mathbb{N}\}$ of $\mathbb{Z}$ we have:

(1) $0 = \bigvee M = \bigwedge N$.

(2) For every $n \in \mathbb{N}$, $\{m \in \mathbb{N} : a_m \notin [0, -n]\} \in \mathcal{I}$.

But the net $(a_n)_{n\in\mathbb{N}}$ does not $\mathcal{I}$-order-converge to 0, because the subset N of $\mathbb{Z}$ is not filtered.

Remark 5.11 From Proposition 5.9 we can, easily, see that Propositions 2.7, 2.8, Corollary 2.10, and Propositions 4.2, 4.3 are satisfied, also, for the notion of $\mathcal{I}$-o_2-convergence.

Notation 5.12 For an arbitrary poset X, we denote by $\mathcal{C}_X^{o_2}$ the class consisting of triads $((x_d)_{d\in D}, x, \mathcal{I})$, where $(x_d)_{d\in D}$ is a net in X, $x \in X$, and $\mathcal{I}$ is a non-trivial ideal on D such that $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o_2} x$. The corresponding topology $\tau(\mathcal{C}_X^{o_2})$ on X (see Proposition 3.1) is called the *ideal-o_2-topology* on X.

Proposition 5.13 *For any poset X, $\tau(\mathcal{C}_X^{o_2}) = \mathcal{T}_X^{o_2} \subseteq \mathcal{T}_X^{o}$.*

Proof. The equality is similar to the proof of Proposition 3.4 taking into account Proposition 5.7. The inclusion it follows immediately from the definitions. □

Proposition 5.14 *If $((x_d)_{d\in D}, x, \mathcal{I}) \in C_X^{o_2}$, then $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-t} x$ with respect to $\tau(C_X^{o_2})$.*

Proof. It is similar to the proof of Proposition 3.2. □

Remark 5.15 The Corollary 3.8 and the Propositions 3.9, 4.4, 4.5 are satisfied for the ideal-o_2-convergence, replacing the correspondent notions.

Definition 5.16 The ideal-o_2-convergence in a poset X is called *topological*, if there exists a topology τ on X such that for every net $(x_d)_{d\in D}$ in X, $x \in X$ and for every non-trivial ideal $\mathcal{I}$ of D, $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o_2} x$ if and only if $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-t} x$ with respect to τ.

Proposition 5.17 *The ideal-o_2-convergence in a poset X is topological if and only if the o_2-convergence in X is topological.*

Proof. Is similar to the proof of Proposition 3.12 taking into account Propositions 2.3 and 5.7. □

Proposition 5.18 *Let X be a chain and $x_1, x_2 \in X$. Then, $(x_1, x_2) \in \tau(\mathcal{C}_X^{o_2})$.*

Proof. It suffices to prove that for every $((x_d)_{d\in D}, x, \mathcal{I}) \in \mathcal{C}_X^{o_2}$, $x \in (x_1, x_2)$ we have that $\{d \in D : x_d \notin (x_1, x_2)\} \in \mathcal{I}$. Let $((x_d)_{d\in D}, x, \mathcal{I}) \in \mathcal{C}_X^{o_2}$. Then, $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o_2} x$. Therefore, there exist subsets M and N of X such that:

(1) $x = \bigvee M = \bigwedge N$.

(2) For each $m \in M$ and $n \in N$, $\{d \in D : x_d \notin [m, n]\} \in \mathcal{I}$.

Let $m_0 \in M$ and $n_0 \in N$ such that $x_1 \leqslant m_0 < x < n_0 \leqslant x_2$. Then,

$$\{d \in D : x_d \notin [m_0, n_0]\} \in \mathcal{I}.$$

Since

$$\{d \in D : x_d \notin (x_1, x_2)\} \subseteq \{d \in D : x_d \notin [m_0, n_0]\},$$

we have $\{d \in D : x_d \notin (x_1, x_2)\} \in \mathcal{I}$. □

Proposition 5.19 Let X be a poset and $x, y, z \in X$. Then, the following statements hold:

(1) $x \ll_\alpha y$ if and only if for every net $(x_d)_{d\in D}$ in X and every non-trivial ideal $\mathcal{I}$ on D such that $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o_2} y$ we have $\{d \in D : x_d \ngeqslant x\} \in \mathcal{I}$.

(2) $z \rhd_\alpha y$ if and only if for every net $(x_d)_{d\in D}$ in X and every non-trivial ideal $\mathcal{I}$ on D such that $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o_2} y$ we have $\{d \in D : x_d \nleqslant z\} \in \mathcal{I}$.

Proof. (1) ($\Leftarrow$) Let $(x_d)_{d\in D}$ be net in X such that $(x_d)_{d\in D} \xrightarrow{o_2} y$. Consider the ideal $\mathcal{I}_D$. Then, $(x_d)_{d\in D} \xrightarrow{\mathcal{I}_D-o_2} y$ and therefore, $\{d \in D : x_d \ngeqslant x\} \in \mathcal{I}_D$. By the definition of $\mathcal{I}_D$ there exists $d_0 \in D$ such that $\{d \in D : x_d \ngeqslant x\} \subseteq \{d \in D : d \ngeqslant d_0\}$. Therefore, $x_d \geqslant x$ for every $d \geqslant d_0$.

($\Rightarrow$) Let $(x_d)_{d\in D}$ be a net in X and $\mathcal{I}$ a non-trivial ideal on D such that $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o_2} y$. Then, by Proposition 5.7, $(y_\lambda)_{\lambda\in\Lambda_{\mathcal{I}}}^{\varphi_{\mathcal{I}}} \xrightarrow{o_2} y$. Thus, there exists $\lambda_0 \in \Lambda_{\mathcal{I}}$ such that $y_\lambda \geqslant x$ for all $\lambda \geqslant \lambda_0$. By Proposition 2.3 $\{d \in D : x_d \ngeqslant x\} \in \mathcal{I}$.

(2) Is similar to the proof of (1). □

Proposition 5.20 *The ideal-o_2-convergence in a poset X is topological if and only if X is an O_2-doubly continuous poset.*

Proof. According to Theorem 4.11 in [10] and Proposition 5.17 we have the result. □

Corollary 5.21 *The ideal-o_2-convergence in every finite lattice, every chain or antichain is topological.*

Proof. Is a direct consequence of Remark 3.3 in [10] and Proposition 5.20. □

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HIGHER ORDER APPROXIMATION OF THE DISTRIBUTION OF ANOVA TESTS FOR HIGH-DIMENSIONAL TIME SERIES

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Abstract. Analysis of variance (ANOVA) is tailored for independent observations. Recently, there has been considerable demand for the ANOVA of high-dimensional and dependent observations in many fields. Thus, it is important to analyze the differences among big data's averages of areas from all over the world, such as the financial and manufacturing industries. However, the numerical accuracy of ANOVA for such observations has been inadequately developed. Thus, herein, we study the Edgeworth expansion of distribution of ANOVA tests for high-dimensional and dependent observations. Specifically, we present the second-order approximation of classical test statistics proposed for independent observations. We also provide numerical examples for simulated high-dimensional time-series data.

1 Introduction Analysis of variance (ANOVA) is a type of hypothesis testing method for the null hypothesis of "no treatment effect". It is generally used to test the null hypothesis that the means of three or more populations of within-group means are all equal. Moreover, this method shows whether the within-group means are equal.

ANOVA has a long history in statistics. Gauss founded it in the late 1800s, and Markoff developed it in the early 1900s. Many test statistics for ANOVA and multivariate analysis of variance (MANOVA) have been proposed, primarily under independent disturbances of a MANOVA model. The early applications can be found in [10] and [14]. In addition, [3] and [4] obtained general theoretical results. They derived asymptotic expansions of the null and non-null distributions of the likelihood ratio test-statistics. [2] discussed higher-order approximations (Edgeworth expansions) and their validity. Furthermore, [8] developed higher-order asymptotic expansions of the null and non-null distributions of the likelihood ratio test statistic, Lawley-Hotelling test statistic, and Bartlett-Nanda-Pillai test statistic under high-dimensional and i.i.d. settings. Moreover, in a time-series analysis, [13] discussed the Edgeworth expansions for various statistics. Recently, under a high-dimensional time-series setting, [12] discussed the first-order asymptotics of Lawley-Hotelling test statistic, likelihood ratio test statistic, and Bartlett-Nanda-Pillai test statistic.

In the current era of big data, an analysis of high-dimensional time-series data is required in practical problems, such as those in economics, finance, and bioinformatics. Especially, the accuracy of statistical decisions for high-dimensional time-series data has become increasingly important. Many data analysts need accurate methods for the equivalence of the within-group means of big data, because this analysis is very basic. MANOVA will be useful for these needs. However, from the viewpoint of the numerical accuracy of approximations, higher-order asymptotics of ANOVA test statistics for high-dimensional data are not adequately developed. In the present study, we focus on Edgeworth expansions of distributions of Lawley-Hotelling test statistic, likelihood ratio test statistic, and Bartlett-Nanda-Pillai test statistic.

In this paper, we consider a one-way MANOVA model whose disturbance process is generated by a high-dimensional stationary process.

Herein, let δ_{ij} be Kronecker's delta, $\boldsymbol{I}_p$ be the p-dimensional identity matrix, $\boldsymbol{O}_{\mathrm{P}}(a_n)$ be an order of the probability that is, for a sequence of random variables $\{X_n\}$ and $\{a_n\}$, $0 < a_n \in \mathbb{R}$, $\{a_n^{-1} X_n\}$ is bounded in probability, and let $\boldsymbol{O}_{\mathrm{P}}^U(\cdot)$ be a $p \times p$ matrix whose elements are probability order $\boldsymbol{O}_{\mathrm{P}}(\cdot)$ with respect to all elements uniformly. In addition, let $|\cdot|$ be the determinant of $\cdot$, $\|\cdot\|$ be the Euclidean norm of $\cdot$, and $\mathbb{1}$ be the indicator function.

2 Problems and Preliminaries Throughout this paper, we consider the MANOVA model under which a q-tuple of p-dimensional time series $\boldsymbol{X}_{i1}, \cdots, \boldsymbol{X}_{in_i}$, $i = 1, \ldots, q$ satisfies

$$\boldsymbol{X}_{it} = \boldsymbol{\mu} + \boldsymbol{\alpha}_i + \boldsymbol{\epsilon}_{it}, \quad t = 1, \cdots, n_i, \quad i = 1, \cdots, q, \tag{1}$$

where $\mu \in \mathbb{R}^p$ is the global mean of the model (1), the disturbances $\boldsymbol{\epsilon}_i \equiv \{\boldsymbol{\epsilon}_{i1}, \cdots \boldsymbol{\epsilon}_{in_i}\}$ are kth-order stationary with mean $\mathbf{0}$, lag u autocovariance matrix $\boldsymbol{\Gamma}(u) = (\Gamma_{jk}(u))_{1 \le j,k \le p}$, $u \in \mathbb{Z}$, and n_i is the observation length of the ith group. Furthermore, the total observation length of all groups $n = \sum_{i=1}^q n_i$ and $\{\boldsymbol{\epsilon}_i\}$, $i = 1, \cdots, q$ are mutually independent. We impose a further standard assumption, which is called homoscedasticity (e.g., Ch. 8.9 of [1]). Now $\boldsymbol{\alpha}_i$ denotes the effect of the ith treatment, which measures the deviation from $\boldsymbol{\mu}$ satisfying $\sum_{i=1}^q \boldsymbol{\alpha}_i = \mathbf{0}$. Because the treatment effects sum to zero, we discuss the problem of testing:

$$H: \ \boldsymbol{\alpha}_1 = \cdots = \boldsymbol{\alpha}_q = \mathbf{0} \text{ vs. } A: \ \boldsymbol{\alpha}_i \neq \mathbf{0} \text{ for some } i. \tag{2}$$

The null hypothesis H implies that all effects are zero.

For our high-dimensional dependent observations, we use the Lawley-Hotelling test statistic $\tilde{T}_1$, likelihood ratio test statistic $\tilde{T}_2$, and Bartlett-Nanda-Pillai test statistic $\tilde{T}_3$:

$$\begin{aligned} \tilde{T}_1 &\equiv n\mathrm{tr}\hat{\mathcal{S}}_H \hat{\mathcal{S}}_E^{-1}, \\ \tilde{T}_2 &\equiv -n \log |\hat{\mathcal{S}}_E| / |\hat{\mathcal{S}}_E + \hat{\mathcal{S}}_H|, \\ \tilde{T}_3 &\equiv n\mathrm{tr}\hat{\mathcal{S}}_H (\hat{\mathcal{S}}_E + \hat{\mathcal{S}}_H)^{-1}, \end{aligned}$$

where

$$\hat{\mathcal{S}}_H \equiv \sum_{i=1}^q n_i (\hat{\boldsymbol{X}}_{i\cdot} - \hat{\boldsymbol{X}}_{\cdot\cdot})(\hat{\boldsymbol{X}}_{i\cdot} - \hat{\boldsymbol{X}}_{\cdot\cdot})' \text{ and } \hat{\mathcal{S}}_E \equiv \sum_{i=1}^q \sum_{t=1}^{n_i} (\boldsymbol{X}_{it} - \hat{\boldsymbol{X}}_{i\cdot})(\boldsymbol{X}_{it} - \hat{\boldsymbol{X}}_{i\cdot})' \ \text{ with}$$

$$\hat{\boldsymbol{X}}_{i\cdot} = \frac{1}{n_i} \sum_{t=1}^{n_i} \boldsymbol{X}_{it} \text{ and } \hat{\boldsymbol{X}}_{\cdot\cdot} = \frac{1}{n} \sum_{i=1}^q \sum_{t=1}^{n_i} \boldsymbol{X}_{it}.$$

Now, we call $\hat{\mathcal{S}}_H$ and $\hat{\mathcal{S}}_E$ the between-group sums of squares and products (SSP) and the within-group SSP, respectively. To derive the stochastic expansion of $n^{-1}\hat{\mathcal{S}}_E$ in Section 4, we introduce

$$\hat{\mathcal{S}}_i \equiv (n_i - 1)^{-1} \sum_{t=1}^{n_i} (\boldsymbol{X}_{it} - \hat{\boldsymbol{X}}_{i\cdot})(\boldsymbol{X}_{it} - \hat{\boldsymbol{X}}_{i\cdot})', \tag{3}$$

$$\boldsymbol{V} = \sum_{i=1}^q \sqrt{\frac{n_i}{n}} \boldsymbol{V}_i, \ \boldsymbol{V}_i = \sqrt{n_i}(\hat{\mathcal{S}}_i - \boldsymbol{I}_p). \tag{4}$$

In addition, to derive the Edgeworth expansion of distributions of the three test statistics under H, we impose the following assumptions:

Assumption 1

$$\frac{p^{3/2}}{\sqrt{n}} \to 0 \ \ as\ n, p \to \infty, \tag{5}$$

$$\frac{n_i}{n} \to \rho_i > 0 \ \ as\ n \to \infty, \tag{6}$$

where ρ_i is a positive constant which is independent of n and p for every i.

Here, the condition (6) implies the orders of n_i and n are asymptotically the same.

Assumption 2 *For the p-vectors $\boldsymbol{\epsilon}_{it} = (\epsilon_{it}^{(1)}, \cdots, \epsilon_{it}^{(p)})'$ given in (1), there exists an $\ell \geq 0$ with*

$$\sum_{t_1,\ldots,t_{k-1}=-\infty}^{\infty} \{1 + |t_j|\}^{\ell} |c^i_{a_1,\cdots,a_k}(t_1, \cdots, t_{k-1})| < \infty,$$

for $j = 1, \cdots, k-1$ and any k-tuple $a_1, \cdots, a_k \in \{1, \cdots, p\}$ and $i = 1, \cdots, q$, when $k = 2, 3, \cdots$. Here $c^i_{a_1,\cdots,a_k}(t_1, \cdots, t_{k-1}) = \mathrm{cum}\{\epsilon_{it_1}^{(a_1)}, \cdots, \epsilon_{it_k}^{(a_k)}\}$.

If $\epsilon_{it}^{(a_{m_1})}, \cdots, \epsilon_{it}^{(a_{m_h})}$ for any h-tuple $m_1, \cdots, m_h \in \{1, \cdots, k\}$ are independent of $\epsilon_{it}^{(a_{m_{h+1}})}, \cdots, \epsilon_{it}^{(a_{m_k})}$ for the remaining $(k-h)$-tuple $m_{h+1}, \cdots, m_k \in \{1, \cdots, k\}$, then $c^i_{a_{m_1},\cdots,a_{m_k}}(t_{m_1}, \cdots, t_{m_k-1}) = 0$ ([5], p. 19). Assumption 2 implies that if the time points of a group of $\epsilon_{it_l}^{(a_*)}$'s are well separated from the remaining time points of $\epsilon_{it_s}^{(a_*)}$'s, the values of $c^i_{a_1,\cdots,a_k}(t_1, \cdots, t_{k-1})$ become small (and hence summable) (see [5, p.19]).This property is natural for stochastic processes with short memory. We introduce a concrete example of the high-dimensional process ϵ_i's which satisfy Assumption 2. That is $DCC\text{-}GARCH(p, q)$ model (9). [9] expressed a typical component of this model as

$$\sum_{l=0}^{\infty} \sum_{j_l < j_{l-1} < \cdots < j_1 < t} b_{t-j_1} \cdots b_{j_{l-1}-j_l} \eta_{j_1} \cdots \eta_{j_l} \tag{7}$$

where η_j's are i.i.d. with $E\eta_j^2 < \infty$. By (7), we can easily check this model satisfies Assumption 2.

Assumption 3

$$\boldsymbol{\Gamma}(j) = \mathbf{0} \ \ for\ all\ j \neq 0. \tag{8}$$

Assumption 3 means that the disturbance process $\{\boldsymbol{\epsilon}_i\}$ is an uncorrelated process. Now, note that the condition (8) is not very severe because of the very practical nonlinear time-series model

DCC-GARCH(q, r)

$$\boldsymbol{\epsilon}_{it} = \boldsymbol{H}_{it}{}^{1/2}\boldsymbol{\eta}_{it}, \quad \boldsymbol{\eta}_{it} \overset{i.i.d.}{\sim} (\boldsymbol{0}, \boldsymbol{I}_p),$$

$$\boldsymbol{H}_{it} = \boldsymbol{D}_{it}\boldsymbol{R}_{it}\boldsymbol{D}_{it}, \quad \boldsymbol{D}_{it} = diag\left[\sqrt{\sigma_{it}^{(1)}}, \cdots, \sqrt{\sigma_{it}^{(p)}}\right],$$

$$\boldsymbol{\epsilon}_{it} = \begin{pmatrix} \epsilon_{it}^{(1)} \\ \vdots \\ \epsilon_{it}^{(p)} \end{pmatrix}, \quad \sigma_{it}^{(j)} = c_j + a_j \sum_{l=1}^{r} \left\{\epsilon_{i,t-l}^{(j)}\right\}^2 + b_j \sum_{l=1}^{q} \sigma_{i,t-l}^{(j)}, \tag{9}$$

$$\boldsymbol{R}_{it} = (diag\,[\boldsymbol{Q}_{it}])^{-1/2}\,\boldsymbol{Q}_{it}\,(diag\,[\boldsymbol{Q}_{it}])^{-1/2},$$

$$\tilde{\boldsymbol{\epsilon}}_{it} = \begin{pmatrix} \tilde{\epsilon}_{it}^{(1)} \\ \vdots \\ \tilde{\epsilon}_{it}^{(p)} \end{pmatrix}, \quad \tilde{\epsilon}_{it}^{(j)} = \frac{\epsilon_{it}^{(j)}}{\sqrt{\sigma_{it}^{(j)}}}, \quad \boldsymbol{Q}_{it} = (1-\alpha-\beta)\tilde{\boldsymbol{Q}} + \alpha\tilde{\boldsymbol{\epsilon}}_{i,t-1}\tilde{\boldsymbol{\epsilon}}'_{i,t-1} + \beta\boldsymbol{Q}_{i,t-1},$$

(see [7]) satisfies (8). Here, $\tilde{\boldsymbol{Q}}$, the unconditional correlation matrix, is a constant positive semidefinite matrix, and $\boldsymbol{H}_{it}$'s are measurable with respect to $\boldsymbol{\eta}_{i,t-1}, \boldsymbol{\eta}_{i,t-2}, \cdots$.

3 Main Results

In what follows, without loss of generality, we assume $\boldsymbol{\Gamma}(0) = \boldsymbol{I}_p$, and $\boldsymbol{\mu} = \boldsymbol{0}$ because the three test statistics $\tilde{T}_1$, $\tilde{T}_2$, and $\tilde{T}_3$ are invariant under linear transformation, our discussion for $\boldsymbol{X}_{it}$ remains valid for the case where we apply a linear transformation $\{\boldsymbol{\Gamma}(0)\}^{-1/2}$ to $\boldsymbol{X}_{it}$. We derive the stochastic expansion of the standardized versions T_1, T_2, and T_3 of the three test statistics $\tilde{T}_1$ (Lawley-Hotelling test statistic), $\tilde{T}_2$ (likelihood ratio test statistic), and $\tilde{T}_3$ (Bartlett-Nanda-Pillai test statistic), respectively:

$$T_1 \equiv \frac{1}{\sqrt{2(q-1)}}\left\{\frac{n}{\sqrt{p}}\mathrm{tr}\hat{\mathcal{S}}_H\hat{\mathcal{S}}_E^{-1} - \sqrt{p}(q-1)\right\}, \tag{10}$$

$$T_2 \equiv -\frac{1}{\sqrt{2(q-1)}}\left\{\frac{n}{\sqrt{p}}\log|\hat{\mathcal{S}}_E|/|\hat{\mathcal{S}}_E + \hat{\mathcal{S}}_H| + \sqrt{p}(q-1)\right\}, \tag{11}$$

$$T_3 \equiv \frac{1}{\sqrt{2(q-1)}}\left\{\frac{n}{\sqrt{p}}\mathrm{tr}\hat{\mathcal{S}}_H(\hat{\mathcal{S}}_E + \hat{\mathcal{S}}_H)^{-1} - \sqrt{p}(q-1)\right\}. \tag{12}$$

This section provides their Edgeworth expansions. Lemmas and all proofs are provided in Section 4.

Theorem 1 *Suppose Assumptions 1-3. Then, under the null hypothesis H, we have the following Edgeworth expansions:*

$$\begin{aligned} P(T_i < z) \;=\; & \Phi(z) - \phi(z)\left\{p^{-1/2}\cdot\frac{c_3}{6}(z^2-1) + p^{-1}\cdot\frac{c_4}{24}(z^3-3z)\right\} \\ & + \boldsymbol{o}\left(p^{-1}\right), \quad (i = 1, 2, 3) \end{aligned} \tag{13}$$

where

$$\Phi(z) = \int_{-\infty}^{z}\phi(y)dy, \quad \phi(y) = (2\pi)^{-1/2}exp\left(-\frac{y^2}{2}\right),$$

and

$$
\begin{aligned}
c_3 &= \left(\frac{2}{q-1}\right)^{3/2}\left\{q-3+3\sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^2-\sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^3\right\},\\
c_4 &= \left(\frac{2}{q-1}\right)^{2}\left\{q-4+6\sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^2-4\sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^3-\sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^4\right\}.
\end{aligned}
$$

Remark 1 *This asymptotic result is an extended version of [8] and [12]. Our setting in Section 2 shows we can apply this result to not only high-dimensional i.i.d. data (that was discussed in [8]) but also high-dimensional time series data. Also, an approximation of the three test statistics T_i, $i = 1, 2, 3$ in Theorem 1 is more accurate than one of them in [12] because we investigated the higher order asymptotic structure of T_i, $i = 1, 2, 3$ by using Edgeworth expansion method.*

4 Asymptotic theory for main results In this section, we provide the lemmas and their proofs. In what follows, we use the same linear transformation as in Section 3. First, the stochastic expansion of $n^{-1}\hat{\mathcal{S}}_E$ and $\hat{\mathcal{S}}_H$ is given.

Lemma 1 *Suppose Assumptions 1-3. Then, under null hypothesis H, the following (14)-(16) hold true;*

$$\frac{1}{n}\hat{\mathcal{S}}_E = \boldsymbol{I}_p + \frac{1}{\sqrt{n}}\boldsymbol{V} - \frac{q}{n}\boldsymbol{I}_p + \boldsymbol{O}_{\mathrm{P}}^{U}\left(n^{-3/2}\right), \tag{14}$$

$$\left\{\frac{1}{n}\hat{\mathcal{S}}_E\right\}^{-1} = \boldsymbol{I}_p - \frac{1}{\sqrt{n}}\boldsymbol{V} + \frac{1}{n}(\boldsymbol{V}^2 + q\boldsymbol{I}_p) + \boldsymbol{O}_{\mathrm{P}}^{U}\left(n^{-3/2}\right), \tag{15}$$

$$\hat{\mathcal{S}}_H = \boldsymbol{O}_{\mathrm{P}}^{U}(1). \tag{16}$$

Proof (Lemma 1) *By (4), write $n^{-1}\hat{\mathcal{S}}_E$ as*

$$
\begin{aligned}
\frac{1}{n}\hat{\mathcal{S}}_E &= \frac{1}{n}\sum_{i=1}^{q}(n_i-1)\hat{\mathcal{S}}_i\\
&= \frac{1}{n}\sum_{i=1}^{q}(n_i-1)\left(\boldsymbol{I}_p + \frac{1}{\sqrt{n_i}}\boldsymbol{V}_i\right)\\
&= \boldsymbol{I}_p + \frac{1}{\sqrt{n}}\boldsymbol{V} - \frac{q}{n}\boldsymbol{I}_p - \frac{1}{n}\sum_{i=1}^{q}\frac{1}{\sqrt{n_i}}\boldsymbol{V}_i. \qquad (17)
\end{aligned}
$$

In what follows, for each i, we will show $\boldsymbol{V}_i = \boldsymbol{O}_{\mathrm{P}}^{U}(1)$. By the null hypothesis H and $\boldsymbol{\mu} = \boldsymbol{0}$, we rewrite $\hat{\mathcal{S}}_i$ as follows:

$$
\begin{aligned}
\hat{\mathcal{S}}_i &= n_i(n_i-1)^{-1}\left(\frac{1}{n_i}\sum_{t=1}^{n_i}\boldsymbol{X}_{it}\boldsymbol{X}_{it}' - \hat{\boldsymbol{X}}_{i\cdot}\hat{\boldsymbol{X}}_{i\cdot}'\right)\\
&= n_i(n_i-1)^{-1}\left(\boldsymbol{A}-\boldsymbol{B}\right)\ (say), \qquad (18)
\end{aligned}
$$

where $\boldsymbol{A} = 1/n_i \sum_{t=1}^{n_i} \boldsymbol{X}_{it}\boldsymbol{X}_{it}'$ *and* $\boldsymbol{B} = \hat{\boldsymbol{X}}_{i\cdot}\hat{\boldsymbol{X}}_{i\cdot}'$. *We observe*

$$\begin{aligned}
&E\{\boldsymbol{A}\} = \boldsymbol{I}_p \quad \text{and} \\
&Cov\{A_{jk}, A_{lm}\} \\
&\quad = \frac{1}{n_i} \sum_{s=-n_i+1}^{n_i-1} \left(1 - \frac{|s|}{n_i}\right) \{c_{jl}(s)c_{km}(s) + c_{jm}(s)c_{kl}(s) + c_{jklm}^{i}(0, s, s)\} \\
&\quad = \boldsymbol{O}\left(\frac{1}{n_i}\right) = \boldsymbol{O}\left(\frac{1}{n}\right) \quad \text{uniformly in } j, k, l, m \text{ by Assumption 2.}
\end{aligned} \tag{19}$$

Hence, $\boldsymbol{A} = \boldsymbol{I}_p + \boldsymbol{O}_{\mathrm{P}}^{U}(1/\sqrt{n})$. *Next, we observe*

$$\begin{aligned}
&E(\hat{\boldsymbol{X}}_{i\cdot}) = \boldsymbol{\alpha}_i \quad \text{and} \\
&Cov\{\hat{\boldsymbol{X}}_{i\cdot}, \hat{\boldsymbol{X}}_{i\cdot}\} \\
&\quad = \left\{\frac{1}{n_i} \sum_{s=-n_i+1}^{n_i-1} \left(1 - \frac{|s|}{n_i}\right) c_{jk}(s)\right\} \\
&\quad = \boldsymbol{O}^{U}\left(\frac{1}{n_i}\right).
\end{aligned} \tag{20}$$

Thus,

$$\boldsymbol{B} = \boldsymbol{O}_{\mathrm{P}}^{U}\left(\frac{1}{n}\right). \tag{21}$$

Therefore,

$$\hat{\mathcal{S}}_i = \boldsymbol{I}_p + \boldsymbol{O}_{\mathrm{P}}^{U}\left(\frac{1}{\sqrt{n}}\right),$$

and

$$\boldsymbol{V}_i = \boldsymbol{O}_{\mathrm{P}}^{U}(1). \tag{22}$$

By using (17) and (22), we can get

$$\frac{1}{n}\hat{\mathcal{S}}_E = \boldsymbol{I}_p + \frac{1}{\sqrt{n}}\boldsymbol{V} - \frac{q}{n}\boldsymbol{I}_p + \boldsymbol{O}_{\mathrm{P}}^{U}\left(n^{-3/2}\right), \tag{14}$$

and

$$\left\{\frac{1}{n}\hat{\mathcal{S}}_E\right\}^{-1} = \left\{\boldsymbol{I}_p + \frac{1}{\sqrt{n}}\boldsymbol{V} - \frac{q}{n}\boldsymbol{I}_p + \boldsymbol{O}_{\mathrm{P}}^{U}\left(n^{-3/2}\right)\right\}^{-1} = \{\boldsymbol{I}_p - \boldsymbol{M}_n\}^{-1} \text{ (say)}.$$

It is known that

$$\{\boldsymbol{I}_p - \boldsymbol{M}_n\}^{-1} = \sum_{k=0}^{\infty} \boldsymbol{M}_n^k \tag{23}$$

(see p. 169 of [11]). From Assumption 1, it follows that

$$\begin{aligned}
\boldsymbol{M}_n^0 &= \boldsymbol{I}_p, \\
\boldsymbol{M}_n &= -\frac{1}{\sqrt{n}}\boldsymbol{V} + \frac{q}{n}\boldsymbol{I}_p + \boldsymbol{O}_{\mathrm{P}}^{U}\left(n^{-3/2}\right), \\
\boldsymbol{M}_n^2 &= \frac{1}{n}\boldsymbol{V}^2 + \boldsymbol{O}_{\mathrm{P}}^{U}\left(n^{-3/2}\right), \\
\boldsymbol{M}_n^k &= \boldsymbol{O}_{\mathrm{P}}\left(n^{-\frac{k}{2}}\right)\boldsymbol{H}, \quad k \geq 3,
\end{aligned}$$

where $\boldsymbol{H}$ *is a* $p \times p$*-matrix and* $\boldsymbol{H} = \boldsymbol{O}_{\mathrm{P}}^{U}(1)$*. Then, we obtain*

$$\left\{\frac{1}{n}\hat{\mathcal{S}}_E\right\}^{-1} = \boldsymbol{I}_p - \frac{1}{\sqrt{n}}\boldsymbol{V} + \frac{1}{n}(\boldsymbol{V}^2 + q\boldsymbol{I}_p) + \boldsymbol{O}_{\mathrm{P}}^{U}\left(n^{-3/2}\right). \tag{15}$$

Next, we show $\hat{\mathcal{S}}_H = \boldsymbol{O}_{\mathrm{P}}^{U}(1)$*. To this end, we recall*

$$\hat{\mathcal{S}}_H = \sum_{i=1}^{q} n_i(\hat{\boldsymbol{X}}_{i\cdot} - \hat{\boldsymbol{X}}_{\cdot\cdot})(\hat{\boldsymbol{X}}_{i\cdot} - \hat{\boldsymbol{X}}_{\cdot\cdot})'. \tag{24}$$

From (20), we observe that $\hat{\boldsymbol{X}}_{i\cdot} = \boldsymbol{\alpha}_i + \boldsymbol{O}_{\mathrm{P}}^{U}\left(1/\sqrt{n_i}\right)$, $\sum_{i=1}^{q}\boldsymbol{\alpha}_i = \boldsymbol{0}$*, and similarly,* $\hat{\boldsymbol{X}}_{\cdot\cdot} = \boldsymbol{O}_{\mathrm{P}}^{U}(1/\sqrt{n})$*. Thus, we have*

$$\hat{\mathcal{S}}_H = \boldsymbol{O}_{\mathrm{P}}^{U}(1). \tag{16}$$

Note that (14), (15), and (16) are derived for the multivariate i.i.d. case, e.g., [8, p.164].

Lemma 2 *Suppose Assumputions 1-3. Then, under null hypothesis* H*, it holds that*

$$\tilde{T}_i = U^{(0)} + \frac{1}{\sqrt{n}}U^{(1)} + \frac{1}{n}\left(U^{(2)} + \beta_i R^{(2)}\right) + \boldsymbol{O}_{\mathrm{P}}\left(\frac{p^{3/2}}{n}\right), \; i = 1, 2, 3, \tag{25}$$

where

$$\begin{aligned}
U^{(0)} &= \mathrm{tr}\hat{\mathcal{S}}_H, \\
U^{(1)} &= -\mathrm{tr}\{\hat{\mathcal{S}}_H\boldsymbol{V}\}, \\
U^{(2)} &= \mathrm{tr}\{\hat{\mathcal{S}}_H(\boldsymbol{V}^2 + q\boldsymbol{I}_p)\}, \\
R^{(2)} &= \mathrm{tr}\{\hat{\mathcal{S}}_H^2\}, \; \textit{and} \\
(\beta_1, \beta_2, \beta_3) &= \left(0, -\frac{1}{2}, -1\right).
\end{aligned}$$

Proof (Lemma 2) *From Lemma 1, it follows that*

$$\begin{aligned}
\tilde{T}_1 &= \mathrm{tr}\left[\hat{\mathcal{S}}_H\left\{\frac{1}{n}\hat{\mathcal{S}}_E\right\}^{-1}\right] \\
&= \mathrm{tr}\left[\hat{\mathcal{S}}_H\left\{\boldsymbol{I}_p - \frac{1}{\sqrt{n}}\boldsymbol{V} + \frac{1}{n}(\boldsymbol{V}^2 + q\boldsymbol{I}_p) + \boldsymbol{O}_{\mathrm{P}}^{U}\left(n^{-3/2}\right)\right\}\right] \\
&= \mathrm{tr}\hat{\mathcal{S}}_H - \frac{1}{\sqrt{n}}\mathrm{tr}\{\hat{\mathcal{S}}_H\boldsymbol{V}\} + \frac{1}{n}\mathrm{tr}\{\hat{\mathcal{S}}_H(\boldsymbol{V}^2 + q\boldsymbol{I}_p)\} + \mathrm{tr}\left\{\hat{\mathcal{S}}_H \cdot \boldsymbol{O}_{\mathrm{P}}^{U}\left(n^{-3/2}\right)\right\}.
\end{aligned}$$

From (16),

$$\tilde{T}_1 = \mathrm{tr}\hat{\mathcal{S}}_H - \frac{1}{\sqrt{n}}\mathrm{tr}\{\hat{\mathcal{S}}_H \boldsymbol{V}\} + \frac{1}{n}\mathrm{tr}\{\hat{\mathcal{S}}_H(\boldsymbol{V}^2 + q\boldsymbol{I}_p)\} + \boldsymbol{O}_{\mathrm{P}}\left(\frac{p^{3/2}}{n}\right). \tag{25}$$

Next, to derive (25), first, note that for every matrix $\boldsymbol{F}$ and the matrix differential operator d

$$\begin{aligned} d\log|\boldsymbol{F}| &= \mathrm{tr}(\boldsymbol{F}^{-1}d\boldsymbol{F}), \\ d\boldsymbol{F}^{-1} &= -\boldsymbol{F}^{-1}(d\boldsymbol{F})\boldsymbol{F}^{-1}, \end{aligned}$$

and (23) (e.g., [11]). Then, a modification of Proposition 6.1.5 of [6] and Lemma 1 shows that for

$$f := n\log\left|\boldsymbol{I}_p + \frac{1}{n}\hat{\mathcal{S}}_H\left\{\frac{1}{n}\hat{\mathcal{S}}_E^{-1}\right\}\right|,$$

we have that

$$f = \sum_{m=0}^{\infty}\frac{1}{m!}d^m f.$$

where d^m's are m-th differentials of f which are calculated by

$$\begin{aligned} d^0 f &= \mathrm{tr}\{\hat{\mathcal{S}}_H\} - \frac{1}{2n}\mathrm{tr}\{\hat{\mathcal{S}}_H^2\} + \boldsymbol{O}_{\mathrm{P}}\left(p\cdot n^{-2}\right), \\ d^1 f &= -\frac{1}{\sqrt{n}}\mathrm{tr}\{\hat{\mathcal{S}}_H\boldsymbol{V}\} + \frac{1}{n}\mathrm{tr}\{\hat{\mathcal{S}}_H(\boldsymbol{V}^2 + q\boldsymbol{I}_p)\} + \boldsymbol{O}_{\mathrm{P}}\left(p^2\cdot n^{-3/2}\right), \\ d^m f &= \boldsymbol{O}_{\mathrm{P}}\left(p\cdot n^{-2}\right), \quad m \geq 2. \end{aligned}$$

Thus, we obtain

$$\tilde{T}_2 = \mathrm{tr}\{\hat{\mathcal{S}}_H\} - \frac{1}{\sqrt{n}}\mathrm{tr}\{\hat{\mathcal{S}}_H\boldsymbol{V}\} + \frac{1}{n}\left[\mathrm{tr}\{\hat{\mathcal{S}}_H(\boldsymbol{V}^2 + q\boldsymbol{I}_p)\} - \frac{1}{2}\mathrm{tr}\{\hat{\mathcal{S}}_H^2\}\right] + \boldsymbol{O}_{\mathrm{P}}\left(p^2\cdot n^{-3/2}\right). \tag{25}$$

From Lemma 1 and (23), it follows that

$$\begin{aligned} \tilde{T}_3 &= \mathrm{tr}\left[\hat{\mathcal{S}}_H\left\{\frac{1}{n}\hat{\mathcal{S}}_E + \frac{1}{n}\hat{\mathcal{S}}_H\right\}^{-1}\right] \\ &= \mathrm{tr}\left[\hat{\mathcal{S}}_H\left\{\boldsymbol{I}_p + \frac{1}{\sqrt{n}}\boldsymbol{V} + \frac{1}{n}(\hat{\mathcal{S}}_H - q\boldsymbol{I}_p) + \boldsymbol{O}_{\mathrm{P}}^U\left(n^{-3/2}\right)\right\}^{-1}\right] \\ &= \mathrm{tr}\left[\hat{\mathcal{S}}_H\sum_{k=0}^{\infty}\left\{-\frac{1}{\sqrt{n}}\boldsymbol{V} - \frac{1}{n}(\hat{\mathcal{S}}_H - q\boldsymbol{I}_p) + \boldsymbol{O}_{\mathrm{P}}^U\left(n^{-3/2}\right)\right\}^k\right]. \end{aligned}$$

From (16),

$$\tilde{T}_3 = \mathrm{tr}\hat{\mathcal{S}}_H - \frac{1}{\sqrt{n}}\mathrm{tr}\{\hat{\mathcal{S}}_H\boldsymbol{V}\} + \frac{1}{n}[\mathrm{tr}\{\hat{\mathcal{S}}_H(\boldsymbol{V}^2 + q\boldsymbol{I}_p)\} - \mathrm{tr}\{\hat{\mathcal{S}}_H^2\}] + \boldsymbol{O}_{\mathrm{P}}\left(\frac{p^{3/2}}{n}\right) \tag{25}$$

(for the multivariate i.i.d. case, e.g., [8, p.164]).

Lemma 3 *Suppose Assumptions 1-3. Then, under the null hypothesis H, it holds that*

$$cum^{(J)}(\overbrace{\frac{1}{\sqrt{p}}\mathrm{tr}\hat{\mathcal{S}}_H,\cdots}^{K},\overbrace{-\frac{1}{\sqrt{pn}}\mathrm{tr}\{\hat{\mathcal{S}}_H\boldsymbol{V}\},\cdots}^{L},\overbrace{\frac{1}{\sqrt{pn}}\mathrm{tr}\{\hat{\mathcal{S}}_H\boldsymbol{V}^2\},\cdots}^{M},\overbrace{\frac{q}{\sqrt{pn}}\mathrm{tr}\hat{\mathcal{S}}_H,\cdots}^{M_0},\overbrace{\frac{\beta_i}{\sqrt{pn}}\mathrm{tr}\hat{\mathcal{S}}_H^2,\cdots}^{N})$$

$$= \boldsymbol{O}\left(p^{1-J/2+N}\cdot n^{-2L-4M-M_0-N}\right) \tag{26}$$

$$= \boldsymbol{o}\left(p^{1-J/2-6L-12M-3M_0-2N}\right), \tag{27}$$

where $K, L, M, M_0, N \geq 0$, $J = K + L + M + +M_0 + N \geq 1$ and

$$(\beta_1,\beta_2,\beta_3) = \left(0, -\frac{1}{2}, -1\right).$$

Proof (Lemma 3) *First, under $\boldsymbol{\mu} = \boldsymbol{0}$ and null hypothesis H, we prepare S_{jk} and V_{jk} as (j,k)th components of $\hat{\mathcal{S}}_H$ and $\boldsymbol{V}$, respectively:*

$$S_{jk} = \sum_{i_1=1}^{q}\frac{1}{n_{i_1}}\sum_{r=1}^{n_{i_1}}\sum_{s=1}^{n_{i_1}}\epsilon_{i_1 r}^{(j)}\epsilon_{i_1 s}^{(k)} - \frac{1}{n}\sum_{i_2=1}^{q}\sum_{i_3=1}^{q}\sum_{t=1}^{n_{i_2}}\sum_{u=1}^{n_{i_3}}\epsilon_{i_2 t}^{(j)}\epsilon_{i_3 u}^{(k)}, \tag{28}$$

$$V_{jk} = \frac{1}{\sqrt{n}}\sum_{i_4=1}^{q}\frac{n_{i_4}}{n_{i_4}-1}\sum_{r=1}^{n_{i_4}}\epsilon_{i_4 r}^{(j)}\epsilon_{i_4 r}^{(k)} - \frac{1}{\sqrt{n}}\sum_{i_4=1}^{q}\frac{1}{n_{i_4}-1}\sum_{s=1}^{n_{i_4}}\sum_{t=1}^{n_{i_4}}\epsilon_{i_4 s}^{(j)}\epsilon_{i_4 t}^{(k)} - \sqrt{n}\delta_{jk}. \tag{29}$$

Here, we can write

$$\begin{aligned}
&cum^{(J)}(\overbrace{\frac{1}{\sqrt{p}}\mathrm{tr}\hat{\mathcal{S}}_H,\cdots}^{K},\overbrace{-\frac{1}{\sqrt{pn}}\mathrm{tr}\{\hat{\mathcal{S}}_H\boldsymbol{V}\},\cdots}^{L},\overbrace{\frac{1}{\sqrt{pn}}\mathrm{tr}\{\hat{\mathcal{S}}_H\boldsymbol{V}^2\},\cdots}^{M},\overbrace{\frac{q}{\sqrt{pn}}\mathrm{tr}\hat{\mathcal{S}}_H,\cdots}^{M_0},\overbrace{\frac{\beta_i}{\sqrt{pn}}\mathrm{tr}\hat{\mathcal{S}}_H^2,\cdots}^{N})\\
=\ &(-1)^L q^{M_0}\beta_i^N\cdot p^{-J/2}n^{-L/2-M-M_0-N}\\
&\times\ cum^{(J)}(\overbrace{\mathrm{tr}\hat{\mathcal{S}}_H,\cdots}^{K+M_0},\overbrace{\mathrm{tr}\{\hat{\mathcal{S}}_H\boldsymbol{V}\},\cdots}^{L},\overbrace{\mathrm{tr}\{\hat{\mathcal{S}}_H\boldsymbol{V}^2\},\cdots}^{M},\overbrace{\mathrm{tr}\hat{\mathcal{S}}_H^2,\cdots}^{N}).
\end{aligned} \tag{30}$$

By (28) and (29), a typical term of the cumulant in (30) is

$$\begin{aligned}
&\sum_{j_{1,1}}^{p}\cdots\sum_{j_{1,K+M_0}}^{p}\sum_{j_{2,1}}^{p}\cdots\sum_{j_{2,L}}^{p}\sum_{j_{3,1}}^{p}\cdots\sum_{j_{3,M}}^{p}\sum_{j_{4,1}}^{p}\sum_{k_{4,1}}^{p}\cdots\sum_{j_{4,N}}^{p}\sum_{k_{4,N}}^{p}\\
&\sum_{r_{1,1}}^{n_i}\sum_{s_{1,1}}^{n_i}\cdots\sum_{r_{1,K+M_0}}^{n_i}\sum_{s_{1,K+M_0}}^{n_i}\sum_{r_{2,1}}^{n_i}\sum_{s_{2,1}}^{n_i}\cdots\sum_{r_{2,L}}^{n_i}\sum_{s_{2,L}}^{n_i}\sum_{r_{3,1}}^{n_i}\sum_{s_{3,1}}^{n_i}\cdots\sum_{r_{3,M}}^{n_i}\sum_{s_{3,M}}^{n_i}\\
&\sum_{r_{4,1}}^{n_i}\sum_{s_{4,1}}^{n_i}\sum_{t_{4,1}}^{n_i}\sum_{u_{4,1}}^{n_i}\cdots\sum_{r_{4,N}}^{n_i}\sum_{s_{4,N}}^{n_i}\sum_{t_{4,N}}^{n_i}\sum_{u_{4,N}}^{n_i}\boldsymbol{O}\left(n^{-K-5L/2-4M-M_0-2N}\right)\\
&\times\ cum^{(J)}[\epsilon_{ir_{1,1}}^{(j_{1,1})}\epsilon_{is_{1,1}}^{(j_{1,1})},\cdots,\epsilon_{ir_{2,1}}^{(j_{2,1})}\epsilon_{is_{2,1}}^{(j_{2,1})},\cdots,\epsilon_{ir_{3,1}}^{(j_{3,1})}\epsilon_{is_{3,1}}^{(j_{3,1})},\cdots,\\
&\qquad\epsilon_{ir_{4,1}}^{(j_{4,1})}\epsilon_{is_{4,1}}^{(k_{4,1})}\epsilon_{it_{4,1}}^{(k_{4,1})}\epsilon_{iu_{4,1}}^{(j_{4,1})},\cdots].
\end{aligned} \tag{31}$$

By using the properties of the cumulant and Theorem 2.3.2 in [5, p.19-21], the cumulant appearing in (31) has a typical main-order term

$$
\begin{aligned}
& \boldsymbol{O}\left(n^{-K-5L/2-4M-M_0-2N}\right)\ n_i^{K+L+M+M_0+2N} \\
& \times \sum_{j_{1,1}}^{p} \cdots \sum_{j_{1,K+M_0}}^{p} \sum_{j_{2,1}}^{p} \cdots \sum_{j_{2,L}}^{p} \sum_{j_{3,1}}^{p} \cdots \sum_{j_{3,M}}^{p} \sum_{j_{4,1}}^{p} \cdots \sum_{j_{4,N}}^{p} c_{j_{1,1}j_{1,2}}(0) \cdots c_{j_{1,K+M_0}j_{2,1}}(0) \\
& \quad \times\ c_{j_{2,1}j_{2,2}}(0) \cdots c_{j_{2,L}j_{3,1}}(0)\ c_{j_{3,1}j_{3,2}}(0) \cdots c_{j_{3,M}j_{4,1}}(0)\ c_{j_{4,1}j_{4,2}}(0) \cdots c_{j_{4,N}j_{1,1}}(0) \\
& \times \sum_{k_{4,1}}^{p} \cdots \sum_{k_{4,N}}^{p} c_{k_{4,1}k_{4,1}}(0) \cdots c_{k_{4,N}k_{4,N}}(0) \\
= {} & \boldsymbol{O}\left(n^{-K-5L/2-4M-M_0-2N}\right)\ n_i^{K+L+M+M_0+2N} \qquad (\textit{By Assumption 3 and } \boldsymbol{\Gamma}(0) = \boldsymbol{I}_p) \\
& \times \sum_{j}^{p} c_{jj}(0) \cdots c_{jj}(0) \times \sum_{k_{4,1}}^{p} \cdots \sum_{k_{4,N}}^{p} c_{k_{4,1}k_{4,1}}(0) \cdots c_{k_{4,N}k_{4,N}}(0) \\
= {} & \boldsymbol{O}\left(p^{1+N} \cdot n^{-3L/2-3M}\right). \qquad (32)
\end{aligned}
$$

Thus, from (32), we rewrite a typical term of (30) as

$$
\begin{aligned}
& cum^{(J)}(\overbrace{\frac{1}{\sqrt{p}}\mathrm{tr}\hat{\mathcal{S}}_H, \cdots,}^{K} \overbrace{-\frac{1}{\sqrt{pn}}\mathrm{tr}\{\hat{\mathcal{S}}_H \boldsymbol{V}\}, \cdots,}^{L} \overbrace{\frac{1}{\sqrt{pn}}\mathrm{tr}\{\hat{\mathcal{S}}_H \boldsymbol{V}^2\}, \cdots,}^{M} \overbrace{\frac{q}{\sqrt{pn}}\mathrm{tr}\hat{\mathcal{S}}_H, \cdots,}^{M_0} \overbrace{\frac{\beta_i}{\sqrt{pn}}\mathrm{tr}\hat{\mathcal{S}}_H^2, \cdots}^{N}), \\
= {} & p^{-J/2} n^{-L/2-M-M_0-N} \boldsymbol{O}\left(p^{1+N} \cdot n^{-3L/2-3M}\right) \\
= {} & \boldsymbol{O}\left(p^{1-J/2+N} \cdot n^{-2L-4M-M_0-N}\right) \\
= {} & \boldsymbol{o}\left(p^{1-J/2-6L-12M-3M_0-2N}\right). \quad (\textit{By Assumption 1})
\end{aligned}
$$

Hence, we showed (26) and (27).

Lemma 4 *Suppose Assumputions 1-3. Define W_i for every $i = 1, 2, 3$ by*

$$
\begin{aligned}
W_i &= \frac{1}{\sqrt{2(q-1)}}\left\{\frac{1}{\sqrt{p}}U^{(0)} + \frac{1}{\sqrt{pn}}U^{(1)} + \frac{1}{\sqrt{pn}}\left(U^{(2)} + \beta_i R^{(2)}\right) - \sqrt{p}(q-1)\right\} \qquad (33) \\
&= \frac{1}{\sqrt{2(q-1)}}\left\{\frac{1}{\sqrt{p}}\mathrm{tr}\hat{\mathcal{S}}_H - \frac{1}{\sqrt{pn}}\mathrm{tr}\{\hat{\mathcal{S}}_H \boldsymbol{V}\} + \frac{1}{\sqrt{pn}}\mathrm{tr}\{\hat{\mathcal{S}}_H \boldsymbol{V}^2\} + \frac{q}{\sqrt{pn}}\mathrm{tr}\hat{\mathcal{S}}_H \right. \\
&\qquad \left. + \frac{\beta_i}{\sqrt{pn}}\mathrm{tr}\{\hat{\mathcal{S}}_H^2\} - \sqrt{p}(q-1)\right\}, \qquad (34) \\
(\beta_1, \beta_2, \beta_3) &= \left(0, -\frac{1}{2}, -1\right).
\end{aligned}
$$

Then, under the null hypothesis H, the following (35)-(39) hold that

$$cum(W_i) = 0 + \boldsymbol{o}\left(p^{-1/2}\right), \tag{35}$$

$$cum(W_i, W_i) = 1 + \boldsymbol{o}\left(p^{-1/2}\right), \tag{36}$$

$$cum(W_i, W_i, W_i) = p^{-1/2}\left(\frac{2}{q-1}\right)^{3/2} \times \left\{q - 3 + 3\sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^2 - \sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^3\right\} + \boldsymbol{o}\left(p^{-1/2}\right), \tag{37}$$

$$cum^{(4)}(W_i, \cdots, W_i) = p^{-1}\left(\frac{2}{q-1}\right)^{2} \times \left\{q - 4 + 6\sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^2 - 4\sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^3 - \sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^4\right\} + \boldsymbol{o}\left(p^{-1}\right), \tag{38}$$

$$cum^{(J)}(W_i, \cdots, W_i) = \boldsymbol{O}\left(p^{1-J/2}\right), \quad (J \geq 5) \tag{39}$$

where (39) contains $K, L, M, M_0, N (\geq 0)$ of the first, second, third, fourth, and fifth terms of (34), respectively.

Proof (Lemma 4) *Now, from Lemma 3, we obtain from (33)*

$$cum(W_i) = \frac{1}{\sqrt{2(q-1)}}\left\{\frac{1}{\sqrt{p}}\{E[U^{(0)}] - p(q-1)\}\right\} + \boldsymbol{o}\left(p^{-1/2}\right).$$

Here, under Assumptions 2 and 3, from (28), we get

$$\begin{aligned} E[U^{(0)}] &= \sum_{j=1}^{p} E[S_{jj}] \\ &= \sum_{j=1}^{p}\sum_{i_1=1}^{q}\sum_{s=-n_{i_1}+1}^{n_{i_1}-1}\left(1 - \frac{|s|}{n_{i_1}}\right)c_{jj}(s) - \sum_{j=1}^{p}\sum_{i_2=1}^{q}\frac{n_{i_2}}{n}\sum_{r=-n_{i_2}+1}^{n_{i_2}-1}\left(1 - \frac{|r|}{n_{i_2}}\right)c_{jj}(r) \\ &= p(q-1). \end{aligned} \tag{40}$$

Then, we can obtain

$$cum(W_i) = 0 + \boldsymbol{o}\left(p^{-1/2}\right). \quad (\textit{By Assumption 1}) \tag{41}$$

Similarly, the main-order terms of $cum(W_i, W_i)$ and $cum(W_i, W_i, W_i)$ can be computed as follows. From (16) and (20),

$$\begin{aligned} cum(W_i, W_i) &= \frac{1}{2p(q-1)}cum(U^{(0)}, U^{(0)}) + \boldsymbol{o}\left(p^{-1/2}\right) \quad (\textit{By Lemma 3}) \\ &= \frac{1}{2p(q-1)}\sum_{j=1}^{p}\sum_{k=1}^{p}cum(S_{jj}, S_{kk}) + \boldsymbol{o}\left(p^{-1/2}\right) \\ &= 1 + \boldsymbol{o}\left(p^{-1/2}\right). \end{aligned} \tag{42}$$

In addition, we can obtain

$$cum(W_i, W_i, W_i) = \{2p(q-1)\}^{-3/2} cum(U^{(0)}, U^{(0)}, U^{(0)}) + \boldsymbol{o}\left(p^{-1/2}\right), \quad (\textit{By Lemma 3})$$

and

$$\begin{aligned}
& cum(U^{(0)}, U^{(0)}, U^{(0)}) \\
= & \sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p} cum(S_{jj}, S_{kk}, S_{ll}) \\
= & \sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p}[\\
& \sum_{i_1=1}^{q}\sum_{i_2=1}^{q}\sum_{i_3=1}^{q}\frac{1}{n_{i_1}}\frac{1}{n_{i_2}}\frac{1}{n_{i_3}}\sum_{r=1}^{n_{i_1}}\sum_{s=1}^{n_{i_1}}\sum_{t=1}^{n_{i_2}}\sum_{u=1}^{n_{i_2}}\sum_{v=1}^{n_{i_3}}\sum_{w=1}^{n_{i_3}} cum\{\epsilon_{i_1 r}^{(j)}\epsilon_{i_2 s}^{(j)}, \epsilon_{i_2 t}^{(k)}\epsilon_{i_2 u}^{(k)}, \epsilon_{i_3 v}^{(l)}\epsilon_{i_3 w}^{(l)}\} \\
& -3\frac{1}{n}\sum_{i_1=1}^{q}\sum_{i_2=1}^{q}\sum_{i_3=1}^{q}\frac{1}{n_{i_1}}\frac{1}{n_{i_2}}\sum_{r=1}^{n_{i_1}}\sum_{s=1}^{n_{i_1}}\sum_{t=1}^{n_{i_2}}\sum_{u=1}^{n_{i_2}}\sum_{v=1}^{n_{i_3}}\sum_{w=1}^{n_{i_3}} cum\{\epsilon_{i_1 r}^{(j)}\epsilon_{i_2 s}^{(j)}, \epsilon_{i_2 t}^{(k)}\epsilon_{i_2 u}^{(k)}, \epsilon_{i_3 v}^{(l)}\epsilon_{i_3 w}^{(l)}\} \\
& +3\frac{1}{n^2}\sum_{i_1=1}^{q}\sum_{i_2=1}^{q}\sum_{i_3=1}^{q}\frac{1}{n_{i_1}}\sum_{r=1}^{n_{i_1}}\sum_{s=1}^{n_{i_1}}\sum_{t=1}^{n_{i_2}}\sum_{u=1}^{n_{i_2}}\sum_{v=1}^{n_{i_3}}\sum_{w=1}^{n_{i_3}} cum\{\epsilon_{i_1 r}^{(j)}\epsilon_{i_2 s}^{(j)}, \epsilon_{i_2 t}^{(k)}\epsilon_{i_2 u}^{(k)}, \epsilon_{i_3 v}^{(l)}\epsilon_{i_3 w}^{(l)}\} \\
& -\frac{1}{n^3}\sum_{i_1=1}^{q}\sum_{i_2=1}^{q}\sum_{i_3=1}^{q}\sum_{r=1}^{n_{i_1}}\sum_{s=1}^{n_{i_1}}\sum_{t=1}^{n_{i_2}}\sum_{u=1}^{n_{i_2}}\sum_{v=1}^{n_{i_3}}\sum_{w=1}^{n_{i_3}} cum\{\epsilon_{i_1 r}^{(j)}\epsilon_{i_2 s}^{(j)}, \epsilon_{i_2 t}^{(k)}\epsilon_{i_2 u}^{(k)}, \epsilon_{i_3 v}^{(l)}\epsilon_{i_3 w}^{(l)}\} \\
&] \\
= & \sum_{j=1}^{p}\left\{\sum_{i_1=1}^{q} 8c_{jj}(0)c_{jj}(0)c_{jj}(0) - 3\sum_{i_1=1}^{q}\frac{n_{i_1}}{n}\cdot 8c_{jj}(0)c_{jj}(0)c_{jj}(0)\right. \\
& \left. +3\sum_{i_1=1}^{q}\left(\frac{n_{i_1}}{n}\right)^2 8c_{jj}(0)c_{jj}(0)c_{jj}(0) - \sum_{i_1=1}^{q}\left(\frac{n_{i_1}}{n}\right)^3\cdot 8c_{jj}(0)c_{jj}(0)c_{jj}(0)\right\} \\
& +\boldsymbol{O}\left(p\cdot n^{-1}\right) \\
= & 8p\left\{q-3+3\sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^2 - \sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^3\right\} + \boldsymbol{O}\left(p\cdot n^{-1}\right). \qquad (43)
\end{aligned}$$

Therefore,

$$cum(W_i, W_i, W_i) = p^{-1/2}\left(\frac{2}{q-1}\right)^{3/2}\left\{q-3+3\sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^2 - \sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^3\right\} + \boldsymbol{o}\left(p^{-1/2}\right). \qquad (44)$$

Similarly, we can compute

$$\begin{aligned}
cum^{(4)}(W_i, \cdots, W_i) &= \{2p(q-1)\}^{-1} cum^{(4)}(U^{(0)}, \cdots, U^{(0)}) + \boldsymbol{o}\left(p^{-2}\right) \\
&= \{2p(q-1)\}^{-1}\sum_{j_1=1}^{p}\sum_{j_2=1}^{p}\sum_{j_3=1}^{p}\sum_{j_4=1}^{p} cum(S_{j_1 j_1}, \cdots, S_{j_4 j_4}) + \boldsymbol{o}\left(p^{-1}\right)
\end{aligned}$$

$$
\begin{aligned}
&= p^{-1}\left(\frac{2}{q-1}\right)^2\left\{q-4+6\sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^2-4\sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^3-\sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^4\right\} \\
&\quad +\boldsymbol{o}\left(p^{-1}\right). \qquad (45)
\end{aligned}
$$

Hence, (35), (36), (37), and (38) were shown (from (41), (42), (44), and (45)). Furthermore, we discuss the Jth order for $J \geq 5$ *cumulant* $cum^{(J)}(W_i, \cdots, W_i)$. *From Lemma 3, we obtain*

$$
\begin{aligned}
cum^{(J)}(W_i, \ldots, W_i) &= \sum_{\substack{K,L,M,M_0,N;\\ K+L+M+M_0+N=J}} \{2(q-1)\}^{-J/2} \\
&\times cum^{(J)}(\overbrace{\frac{1}{\sqrt{p}}\mathrm{tr}\hat{\mathcal{S}}_H,\cdots}^{K}, \overbrace{-\frac{1}{\sqrt{pn}}\mathrm{tr}\{\hat{\mathcal{S}}_H\boldsymbol{V}\},\cdots}^{L}, \overbrace{\frac{1}{\sqrt{pn}}\mathrm{tr}\{\hat{\mathcal{S}}_H\boldsymbol{V}^2\},\cdots}^{M}, \overbrace{\frac{q}{\sqrt{pn}}\mathrm{tr}\hat{\mathcal{S}}_H,\cdots}^{M_0}, \overbrace{\frac{\beta_i}{\sqrt{pn}}\mathrm{tr}\hat{\mathcal{S}}_H^2,\cdots}^{N}) \\
&= \sum_{\substack{K,L,M,M_0,N;\\ K+L+M+M_0+N=J}} \boldsymbol{O}\left(p^{1-J/2+N}\cdot n^{-2L-4M-M_0-N}\right) \\
&= \max_{K,L,M,M_0,N} \boldsymbol{O}\left(p^{1-J/2+N}\cdot n^{-2L-4M-M_0-N}\right) \\
&= \boldsymbol{O}\left(p^{1-J/2}\right). \quad (L=M=M_0=N=0)
\end{aligned}
$$

Then, (39) was shown.

Remark 2 *[12] also evaluated the high-order cumulants of* T_i, $i = 1, 2, 3$ *but there is a big difference between this paper and [12]. The order of the stochastic expansion in Lemma 2 is higher than that in [12], so we needed to derive asymptotics of* W_i *as in Lemmas 3 and 4.*

Proof (Theorem 1) *The Edgeworth expansion for a multivariate time series is derived by [13, p.168-170]. We extend it to the case of high-dimensional time series. First, by the Taylor expansion and Lemma 4, we write the characteristic function of* W_i $(i = 1, 2, 3)$ *in Lemma 4 as*

$$
\begin{aligned}
& E[exp\{\mathrm{i}tW_i\}] \\
&= exp\Big\{cum(W_i)(\mathrm{i}t) + \frac{1}{2}cum(W_i, W_i)(\mathrm{i}t)^2 + \frac{1}{6}cum(W_i, W_i, W_i)(\mathrm{i}t)^3 \\
&\qquad + \frac{1}{24}cum^{(4)}(W_i, \cdots, W_i)(\mathrm{i}t)^4 + \cdots\Big\} \\
&= exp\left(-\frac{t^2}{2}\right) \times \left\{1 + p^{-1/2}\cdot\frac{1}{6}cum(W_i, W_i, W_i)(\mathrm{i}t)^3 + p^{-1}\cdot\frac{1}{24}cum^{(4)}(W_i, \cdots, W_i)(\mathrm{i}t)^4\right\} \\
&\quad +\boldsymbol{o}\left(p^{-1/2}\right). \\
&= exp\left(-\frac{t^2}{2}\right) \times \left\{1 + p^{-1/2}\cdot\frac{c_3}{6}(\mathrm{i}t)^3 + p^{-1}\cdot\frac{c_4}{24}(\mathrm{i}t)^4\right\} + \boldsymbol{o}\left(p^{-1/2}\right). \qquad (46)
\end{aligned}
$$

Inverting (46) by the Fourier inverse transform, we have

$$
P(W_i < z) = \Phi(z) - \phi(z)\left\{p^{-1/2}\cdot\frac{c_3}{6}(z^2-1) + p^{-1}\cdot\frac{c_4}{24}(z^3-3z)\right\} + \boldsymbol{o}\left(p^{-1/2}\right),
$$

where

$$\Phi(z) = \int_{-\infty}^{z} \phi(y)dy, \quad \phi(y) = (2\pi)^{-1/2} exp\left(-\frac{y^2}{2}\right).$$

Here, from Lemma 2, we observe that

$$E[exp\{\mathrm{it}T_i\}] = E[exp\{\mathrm{it}W_i\}] + \boldsymbol{o}\,(1)\,.$$

This implies (13), so we complete the proof.

5 Simulation to verify the finite sample performance We simulate the Edgeworth expansions of distributions of $T_i,\ i = 1, 2, 3$, which are given by Theorem 1. In this section, our purpose is to show that their Edgeworth expansions $P(T_i < z),\ i = 1, 2, 3$ in (13) are more numerically accurate than the first-order approximation, that is, $\Phi(z)$ in (13). Specifically, in the case of an uncorrelated disturbance that is assumed by Assumption 3, $DCC\text{-}GARCH(1, 1)$ is a typical example of that process (see [7]). Therefore, we introduce the following five simulation process steps.

1 Set $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2 = \boldsymbol{\alpha}_3 = \mathbf{0}$ for the null hypothesis H.

2 Generate 20-dimensional $\{\boldsymbol{X}_{1,1}, \ldots, \boldsymbol{X}_{1,5000}\}, \{\boldsymbol{X}_{2,1}, \ldots, \boldsymbol{X}_{2,5000}\}, \{\boldsymbol{X}_{3,1}, \ldots, \boldsymbol{X}_{3,5000}\}$, with $DCC\text{-}GARCH(1, 1)$ disturbance.

3 Calculate the test statistics $T_i,\ i = 1, 2, 3$.

4 Repeat steps 2 and 3 1,000 times independently and obtain $\{T_i^{(1)}, \ldots, T_i^{(1000)};\ i = 1, 2, 3\}$.

5 Calculate $\hat{F}_{i,n}(z),\ i = 1, 2, 3$, which is the empirical distribution of $\{T_i^{(1)}, \ldots, T_i^{(1000)};\ i = 1, 2, 3\}$.

6 Write the plot of $|\hat{F}_{i,n}(z) - \Phi(z)|$ and $|\hat{F}_{i,n}(z) - P(T_i < z)|,\ i = 1, 2, 3$, which are plotted by dotted and thick lines, respectively, in Figures 1, 3, and 5.

7 Write the plot of $\{|\hat{F}_{i,n}(z) - \Phi(z)| - |\hat{F}_{i,n}(z) - P(T_i < z)|\},\ i = 1, 2, 3$, by a dotted line, in Figures 2, 4, and 6.

We set the 20-dimensional simulation from one-way MANOVA model (1) with a 20-dimensional vector $\boldsymbol{\mu}' = (1, \cdots, 1)'$ and generate the disturbance process $\{\boldsymbol{\epsilon}_{it}\}$ of observations $\{\boldsymbol{X}_{it}\}$ in (1) by using $DCC\text{-}GARCH(1, 1)$, whose innovation term is assumed to be Gaussian. The scenarios of $DCC\text{-}GARCH(q, r)$ (see (9)) in $\boldsymbol{\epsilon}_{it}$ are

$$\begin{gathered} p = 20,\ i = 1, 2, 3,\ t = 1, \cdots, 5000, \\ j = 1, \cdots, 20, \\ q = r = 1, \\ a_j = 0.2,\ b_j = 0.7,\ c_j = 0.002, \\ \alpha = 0.1,\ \beta = 0.8, \\ \tilde{Q}_{kl} = 0.7^{(|k-l|)}, \end{gathered}$$

where $\tilde{\boldsymbol{Q}}_{kl}$ is the (k, l)-element of $\tilde{\boldsymbol{Q}}$. We set the observation length $n_i = 5000,\ i = 1, 2, 3$, because Table 1 of Section 5.1 in [12] demonstrates that T_i are stable for $n_i = 2500$ or more uncorrelated observations ($i = 1, 2, 3$). The Mathematical code and the "ccgarch" package of R are used for this algorithm. We compare the numerical accuracy of $P(T_i < z)$ with $\Phi(z)$ based on $\hat{F}_n(z)$ by using $|\hat{F}_{i,n}(z) - \Phi(z)|$, $|\hat{F}_{i,n}(z) - P(T_i < z)|$ (see Figures 1, 3, and 5), and $\{|\hat{F}_{i,n}(z) - \Phi(z)| - |\hat{F}_{i,n}(z) - P(T_i < z)|\}$, $i = 1, 2, 3$ (see Figures 2, 4, and 6).

Figures 2, 4, and 6 indicate that the Edgeworth expansions $P(T_i < z)$ of T_i work better than the normal approximation $\Phi(z)$ from the perspective of numerical accuracy.

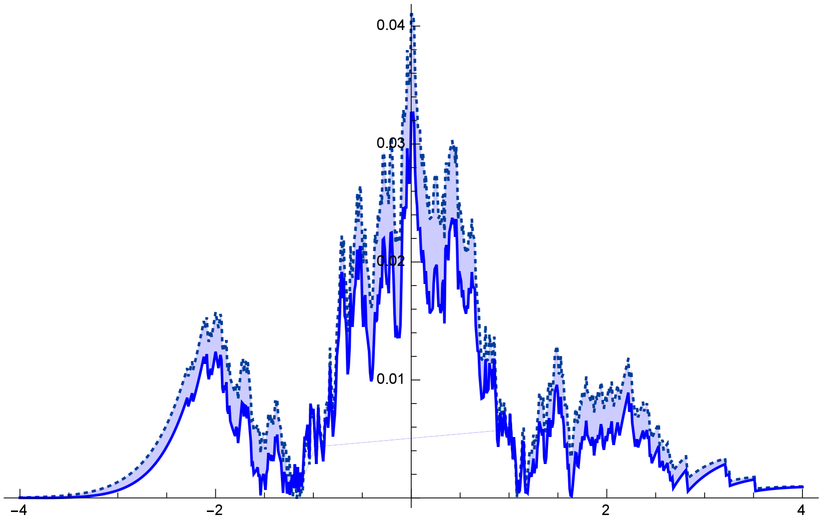

Figure 1: Plot of $|\hat{F}_{1,n}(z) - \Phi(z)|$ and $|\hat{F}_{1,n}(z) - P(T_1 < z)|$ by dotted and thick lines, respectively.

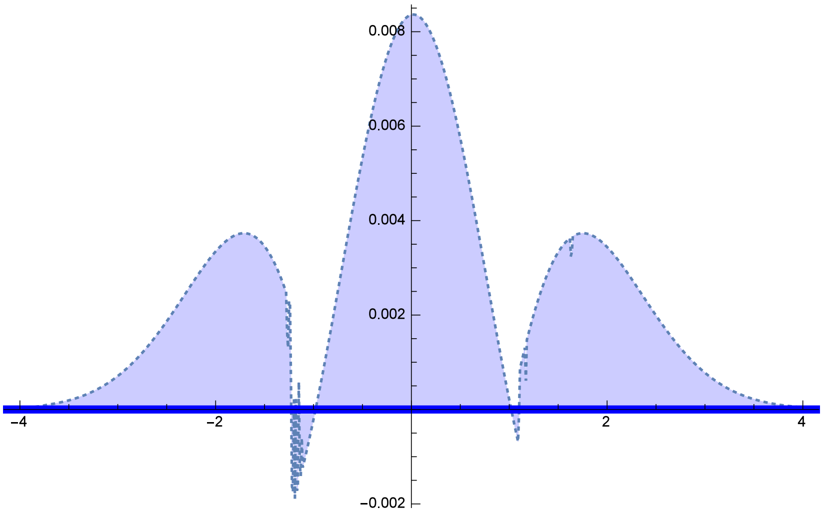

Figure 2: Plot of $\{|\hat{F}_{1,n}(z) - \Phi(z)| - |\hat{F}_{1,n}(z) - P(T_1 < z)|\}$ by a dotted line.

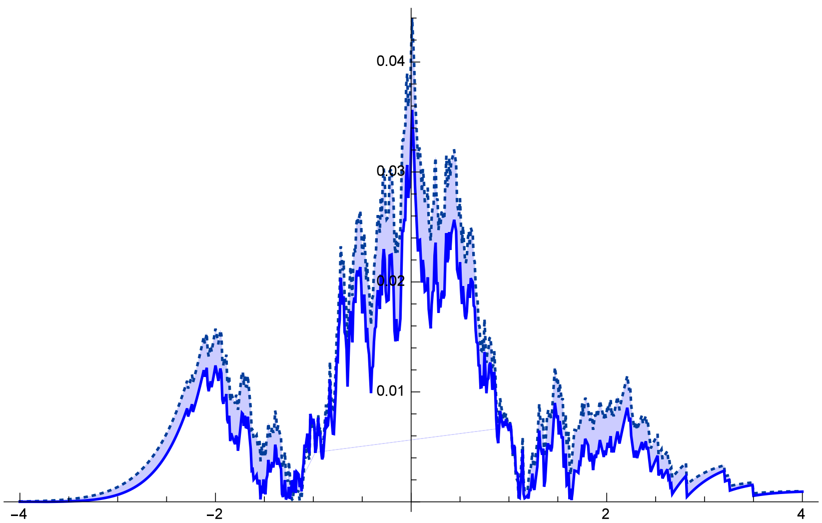

Figure 3: Plot of $|\hat{F}_{2,n}(z) - \Phi(z)|$ and $|\hat{F}_{2,n}(z) - P(T_2 < z)|$ by a dotted line and a thick one, respectively.

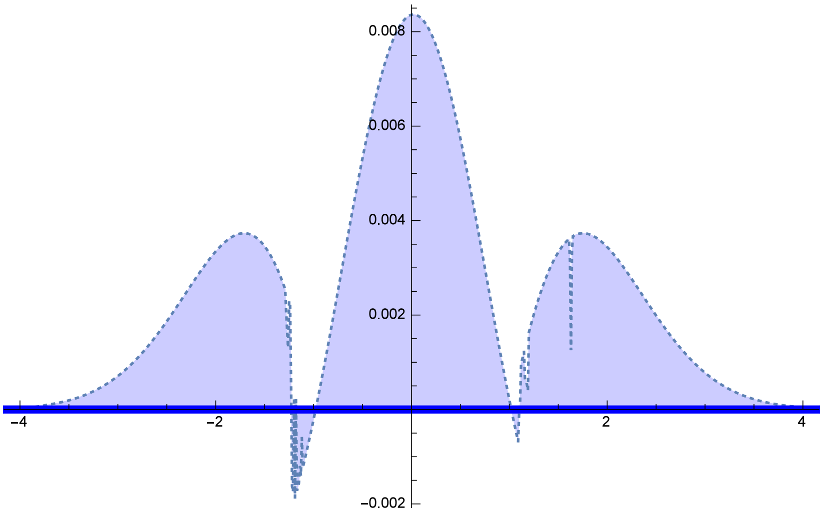

Figure 4: Plot of $\{|\hat{F}_{2,n}(z) - \Phi(z)| - |\hat{F}_{2,n}(z) - P(T_2 < z)|\}$ by a dotted line.

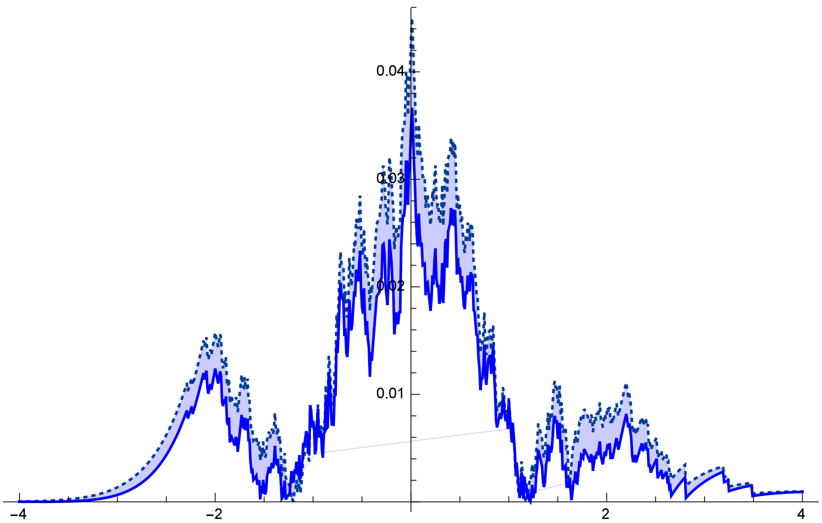

Figure 5: Plot of $|\hat{F}_{3,n}(z) - \Phi(z)|$ and $|\hat{F}_{3,n}(z) - P(T_3 < z)|$ by a dotted line and a thick one, respectively.

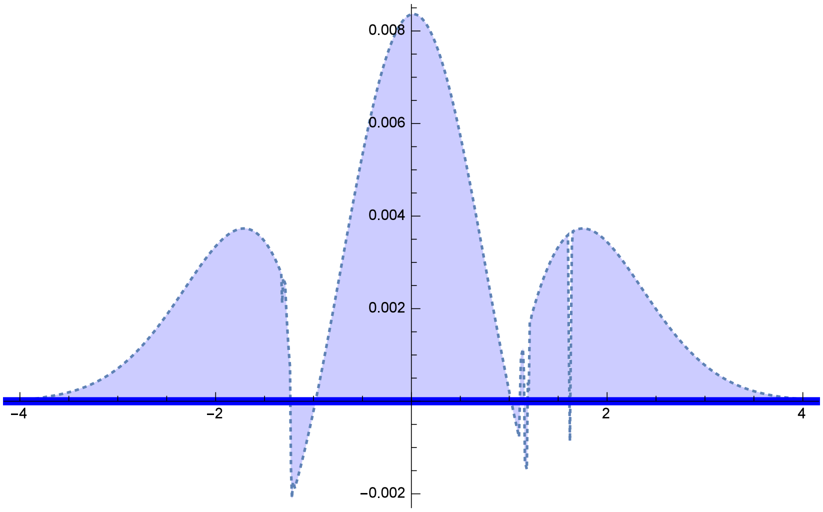

Figure 6: Plot of $\{|\hat{F}_{3,n}(z) - \Phi(z)| - |\hat{F}_{3,n}(z) - P(T_3 < z)|\}$ by a dotted line.

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MIXED SCHWARZ INEQUALITIES VIA THE MATRIX GEOMETRIC MEAN

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Abstract. In this paper, by using the Cauchy-Schwarz inequality for matrices via the matrix geometric mean due to J.I. Fujii, we show the following matrix version of a mixed Schwarz inequality for any square matrices: Let A be an n-square matrix. For any n-square matrices X, Y

$$|Y^*AX| \leq X^*|A|^{2\alpha}X \sharp U^*Y^*|A^*|^{2\beta}YU$$

holds for all $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$, where U is a unitary matrix in a polar decomposition of $Y^*AX = U|Y^*AX|$. As applications, we show matrix Parseval's equation, Lin's type extensions for a weighted version of a mixed Schwarz inequality, and a weighted version of the Wielandt inequality for matrices.

1 Introduction Let $\mathbb{M}_{m\times n} = \mathbb{M}_{m\times n}(\mathbb{C})$ be the space of $m \times n$ complex matrices and $\mathbb{M}_n = \mathbb{M}_{n\times n}(\mathbb{C})$, and denote the matrix absolute value of any $A \in \mathbb{M}_{m\times n}$ by $|A| = (A^*A)^{1/2}$. For $A \in \mathbb{M}_n$, we write $A \geq 0$ if A is positive semidefinite and $A > 0$ if A is positive definite; that is, $x^*Ax > 0$ for all nonzero column vectors $x \in \mathbb{C}^n$. For two Hermitian matrices A and B of the same size, we write $A \geq B$ if $A - B \geq 0$, and $A > B$ if $A - B > 0$. For $A \in \mathbb{M}_{m\times n}$, $\ker A$ and $\operatorname{ran} A$ mean the null space of A and the range of A, respectively.

The Cauchy-Schwarz inequality is one of the most useful and fundamental inequalities in functional analysis: For any complex n-dimensional column vectors x and y,

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle \tag{1.1}$$

and the equality holds if and only if x and y are linearly dependent. As an extension of (1.1), the following inequality holds: For any positive semidefinite matrix A in $\mathbb{M}_n$,

$$|\langle Ax, y\rangle|^2 \leq \langle Ax, x\rangle\langle Ay, y\rangle.$$

Even if A is an arbitrary matrix in $\mathbb{M}_n$, by virtue of the matrix absolute value of A, we have a mixed Schwarz inequality

$$|\langle Ax, y\rangle|^2 \leq \langle |A|x, x\rangle\langle |A^*|y, y\rangle, \tag{1.2}$$

also see [5]. In [3], Furuta showed the weighted version of (1.2) as follows: For any $A \in \mathbb{M}_n$

$$|\langle Ax, y\rangle|^2 \leq \langle |A|^{2\alpha}x, x\rangle\langle |A^*|^{2\beta}y, y\rangle \tag{1.3}$$

holds for any $x, y \in \mathbb{C}^n$ and any $\alpha, \beta \in [0,1]$ with $\alpha+\beta = 1$, and the equality in (1.3) holds if and only if $|A|^{2\alpha}x$ and A^*y are linearly dependent if and only if Ax and $|A^*|^{2\beta}y$ are linearly

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dependent. In fact, Furuta has shown the operator version of (1.3). Moreover, Kittaneh extended (1.3) for two real valued continuous functions f and g under some conditions, also see [7]. We recall the matrix Cauchy-Schwarz inequality in terms of the matrix geometric mean due to [1], also see [2]: For any $X, Y \in \mathbb{M}_n$

$$|Y^*X| \le X^*X \sharp U^*Y^*YU \tag{1.4}$$

holds, where U is a unitary matrix in a polar decomposition of $Y^*X = U|Y^*X|$ and the matrix geometric mean $A \sharp B$ is defined by

$$A \sharp B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$$

for any positive definite matrices A and B, also see [8].

In this paper, by virtue of the matrix Cauchy-Schwarz inequality (1.4) due to J.I.Fujii via the matrix geometric mean, we show the matrix version of a weighted mixed Schwarz inequality (1.3). As applications, we show matrix Parseval's equations, Lin's type extensions for a weighted version of a mixed Schwarz inequality, and a weighted version of the Wielandt inequality for matrices.

2 Weighted mixed Schwarz inequality In this section, we present a weighted version of the mixed Schwarz inequality (1.3) for matrices of the same size. As a preparation of our main assertion, we state the following matrix Cauchy-Schwarz inequality due to J.I.Fujii [2] via the matrix geometric mean:

Lemma 2.1 (Matrix Cauchy-Schwarz inequality)**.** *Let X and Y be matrices in $\mathbb{M}_n$, and $U \in \mathbb{M}_n$ a unitary matrix in a polar decomposition of $Y^*X = U|Y^*X|$. Then*

$$|Y^*X| \le X^*X \sharp U^*Y^*YU \tag{2.1}$$

and

$$|X^*Y| \le UX^*XU^* \sharp Y^*Y. \tag{2.2}$$

Under the assumption $\ker X \subseteq \ker YU$ (resp. $\ker Y \subseteq \ker XU^*$), the equality in (2.1) (resp. the equality in (2.2)) holds if and only if there exists $W \in \mathbb{M}_n$ such that $YU = XW$ (resp. $XU^* = YW$).

For any n-square matrix A, we denote the orthogonal projection on the column space of A by P_A. That is, P_A is the range projection of A. By Lemma 2.1, we have the following matrix version of the weighted Schwarz inequality (1.3) for matrices of the same size:

Theorem 2.2 (Weighted mixed Schwarz inequality)**.** *Let A, X and Y be matrices in $\mathbb{M}_n$ and $U \in \mathbb{M}_n$ a unitary matrix in a polar decomposition of $Y^*AX = U|Y^*AX|$, and $V \in \mathbb{M}_n$ a unitary matrix in a polar decomposition of $A = V|A|$. Then*

$$|Y^*AX| \le X^*|A|^{2\alpha}X \sharp U^*Y^*|A^*|^{2\beta}YU \tag{2.3}$$

and

$$|X^*A^*Y| \le UX^*|A|^{2\alpha}XU^* \sharp Y^*|A^*|^{2\beta}Y \tag{2.4}$$

hold for all $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$. Under the assumption $\ker AX \subseteq \ker A^*YU$ (resp. $\ker A^*Y \subseteq \ker AXU^*$), the equality in (2.3) (resp. the equality in (2.4)) holds if and only if *there exists* $W \in \mathbb{M}_n$ *such that* $A^*YU = |A|^{2\alpha}XW$ (*resp.* $AXU^* = |A^*|^{2\beta}YW$) *if and only if* $|A^*|^{2\beta}YU = AXW$ (*resp.* $|A|^{2\alpha}XU^* = A^*YW$).

In particular, for the case of $\alpha = 0$ *in* (2.3),

$$|Y^*AX| \leq X^*P_{|A|}X \sharp U^*Y^*|A^*|^2YU. \tag{2.5}$$

Under the assumption $\ker P_{|A|}X \subseteq \ker |A|V^*YU$, *the equality in* (2.5) *holds if and only if there exists* $W \in \mathbb{M}_n$ *such that* $|A|V^*YU = P_{|A|}XW$.

For the case of $\alpha = 1$ *in* (2.3),

$$|Y^*AX| \leq X^*|A|^2X \sharp U^*Y^*P_{|A^*|}YU. \tag{2.6}$$

Under the assumption $\ker |A|X \subseteq \ker P_{|A^*|}V^*YU$, *the equality in* (2.6) *holds if and only if there exists* $W \in \mathbb{M}_n$ *such that* $P_{|A|}V^*YU = |A|XW$.

Proof. Firstly, we show (2.3). For the case of $0 < \alpha < 1$, replacing X (resp. Y) by $|A|^\alpha X$ (resp. $|A|^\beta V^*Y$) in (2.1) of Lemma 2.1, then we obtain

$$|Y^*AX| = |(|A|^\beta V^*Y)^*|A|^\alpha X| \leq X^*|A|^{2\alpha}X \sharp U^*Y^*V|A|^{2\beta}V^*YU.$$

It follows from [3] and [4, Theorem 4 in 2.2.2] that

$$V|A|^{2\beta}V^* = (V|A|V^*)^{2\beta} = (V|A||A|V^*)^\beta = (AA^*)^\beta = |A^*|^{2\beta}$$

and we can get the desired inequality (2.3):

$$|Y^*AX| \leq X^*|A|^{2\alpha}X \sharp U^*Y^*V|A|^{2\beta}V^*YU = X^*|A|^{2\alpha}X \sharp U^*Y^*|A^*|^{2\beta}YU.$$

For the case of $\alpha = 0$, since $|Y^*AX| = |Y^*V|A|P_{|A|}X| = |(|A|V^*Y)^*P_{|A|}X|$, by replacing X (resp. Y) by $P_{|A|}X$ (resp. $|A|V^*Y$) in (2.1) of Lemma 2.1, we obtain

$$|Y^*AX| \leq X^*P_{|A|}X \sharp U^*Y^*V|A|^2V^*YU = X^*P_{|A|}X \sharp U^*Y^*|A^*|^2YU$$

and so we have (2.5). For the case of $\alpha = 1$, we have (2.6) similarly.

For the equality conditions, since $\ker AX \subseteq \ker A^*YU$ is equivalent to $\ker |A|^\alpha X \subseteq \ker |A|^\beta V^*YU$ for $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$, it follows from Lemma 2.1 that under the assumption $\ker |A|^\alpha X \subseteq \ker |A|^\beta V^*YU$, the equality in (2.3) holds if and only if there exists $W \in \mathbb{M}_n$ such that $|A|^\beta V^*YU = |A|^\alpha XW$.

By a way similar to (2.3), we can get the inequality (2.4) and the equality condition of (2.4). □

Remark 2.3. Similarly, we can consider the case of $\alpha = 0, 1$ of (2.4) in Theorem 2.2.

For the case of $\alpha = 0$, then

$$|X^*A^*Y| \leq UX^*P_{|A|}XU^* \sharp Y^*|A^*|^2Y.$$

Under the assumption $\ker |A^*|Y \subseteq \ker P_{|A^*|}VXU^*$, the equality holds if and only if there exists $W \in \mathbb{M}_n$ such that $P_{|A^*|}VXU^* = |A^*|YW$.

For the case of $\alpha = 1$, then

$$|X^*A^*Y| \leq UX^*|A|^2XU^* \sharp Y^*P_{|A^*|}Y.$$

Under the assumption $\ker P_{|A^*|}Y \subseteq \ker |A^*|VXU^*$, the equality holds if and only if there exists $W \in \mathbb{M}_n$ such that $|A^*|VXU^* = P_{|A^*|}YW$.

3 Weighted mixed Schwarz inequality for an arbitrary matrix In this section, we present the weighted version of a mixed Schwarz inequality for matrices of any different sizes. For this, we need the following lemmas, see [6, p.449].

Lemma 3.1 (Polar decomposition)**.** *Let A be an $m \times n$ matrix in $\mathbb{M}_{m\times n}$.*

(i) *If $m > n$, then $A = U|A|$, in which $U \in \mathbb{M}_{m\times n}$ consists of orthonormal columns.*

(ii) *If $m = n$, then $A = U|A|$, in which $U \in \mathbb{M}_n$ is unitary.*

(iii) *If $m < n$, then $A = |A^*|U$, in which $U \in \mathbb{M}_{n\times m}$ consists of orthonormal rows.*

The following lemma is a matrix Cauchy-Schwarz inequality for an arbitrary matrix, also see [2, Corollary 2.7].

Lemma 3.2. *Let X be a matrix in $\mathbb{M}_{k\times m}$ and Y in $\mathbb{M}_{k\times n}$.*

(i) *If $m \leq n$, then*

$$|Y^*X| \leq X^*X \sharp U^*Y^*YU, \tag{3.1}$$

in which $U \in \mathbb{M}_{n\times m}$ consists of orthonormal columns and $Y^*X = U|Y^*X|$.

(ii) *If $m > n$, then*

$$|X^*Y| \leq U^*X^*XU \sharp Y^*Y, \tag{3.2}$$

in which $U \in \mathbb{M}_{m\times n}$ consists of orthonormal columns and $X^*Y = U|X^*Y|$.

Under the assumption $\ker X \subseteq \ker YU$ (resp. $\ker Y \subseteq \ker XU$), the equality in (3.1) (resp. the equality in (3.2)) holds if and only if there exists $W \in \mathbb{M}_m$ (resp. $W \in \mathbb{M}_n$) such that $YU = XW$ (resp. $XU^* = YW$).

By using a polar decomposition for an arbitrary matrix, we have the following theorem, whose proof is similar to that of Theorem 2.2.

Theorem 3.3. *Let A be a matrix in $\mathbb{M}_{p\times m}$, X in $\mathbb{M}_{m\times n}$, Y in $\mathbb{M}_{p\times q}$. For all $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$, the following inequalities hold.*

(i) *If $q \geq n$, then*

$$|Y^*AX| \leq X^*|A|^{2\alpha}X \sharp U_1^*Y^*|A^*|^{2\beta}YU_1, \tag{3.3}$$

in which $U_1 \in \mathbb{M}_{q\times n}$ consists of orthonormal columns and $Y^*AX = U_1|Y^*AX|$.

(ii) *If $q < n$, then*

$$|X^*A^*Y| \leq U_2^*X^*|A|^{2\alpha}XU_2 \sharp Y^*|A^*|^{2\beta}Y, \tag{3.4}$$

in which $U_2 \in \mathbb{M}_{n\times q}$ consists of orthonormal columns and $X^*A^*Y = U_2|X^*A^*Y|$.

Under the assumption $\ker AX \subseteq \ker A^*YU_1$ (resp. $\ker A^*Y \subseteq \ker AXU_2$), the equality in (3.3) (resp. the equality in (3.4)) holds if and only if there exists $W \in \mathbb{M}_n$ (resp. $W \in \mathbb{M}_q$) such that $|A^*|^{2\beta}YU_1 = AXW$ (resp. $AXU_2 = |A^*|^{2\beta}YW$).

Proof. We show (3.3) only. If $p \geq m$, then by Lemma 3.1 we have $A = V_1|A|$, in which $V_1 \in \mathbb{M}_{p\times m}$ consists of orthonormal columns. In this case, we replace X (resp. Y) by $|A|^\alpha X$ (resp. $|A|^\beta V_1^* Y$) in (3.1) of Lemma 3.2, and we have $|A^*|^{2\beta} = V_1|A|^{2\beta}V_1^*$. If $p < m$, then we have $A = |A^*|V_2$, in which $V_2 \in \mathbb{M}_{m\times p}$ consists of orthonormal rows. In this case, we replace X (resp. Y) by $|A^*|^\alpha V_2 X$ (resp. $|A^*|^\beta Y$) in (3.1) of Lemma 3.2, and we have $|A|^{2\alpha} = V_2^*|A^*|^{2\alpha}V_2$. Hence we obtain (3.3) and the equality condition. □

Inspired by Kittaneh's result [7, Theorem 1], we show an extension of Theorem 3.3, which is a generalization of Schwarz inequality for two nonnegative functions f and g.

Theorem 3.4. *Let A be in $\mathbb{M}_{p\times m}$, X in $\mathbb{M}_{m\times n}$, Y in $\mathbb{M}_{p\times q}$ and f, g real valued continuous functions on $[0,\infty)$ which are nonnegative and satisfying the relation $f(t)g(t) = t$ for all $t \in [0,\infty)$. If $q \geq n$ and $p \geq m$, then*

$$|Y^*AX| \leq X^*f(|A|)^2X \,\sharp\, U^*Y^*g(|A^*|)^2YU, \tag{3.5}$$

in which $U \in \mathbb{M}_{q\times n}$ consists of orthonormal columns and $Y^*AX = U|Y^*AX|$.

Under the assumption $\ker f(|A|)X \subseteq \ker g(|A|)V^*YU$ where $V \in \mathbb{M}_{p\times m}$ consists of orthonormal columns and $A = V|A|$, the equality in (3.5) holds if and only if there exists $W \in \mathbb{M}_n$ such that $g(|A|)V^*YU = f(|A|)XW$.

Proof. Replacing X and Y by $f(|A|)X$ and $g(|A|)V^*Y$ respectively in (3.1) of Lemma 3.2, we obtain (3.5). In fact, we have $|A^*| = V|A|V^*$ and $VV^* \leq I$, and so $Vg(|A|)^2V^* \leq g(V|A|V^*)^2 = g(|A^*|)^2$. Therefore it follows that

$$\begin{aligned} |Y^*AX| &= |Y^*V|A|X| = |Y^*Vg(|A|)f(|A|)X| \\ &\leq X^*f(|A|)^2X \,\sharp\, U^*Y^*Vg(|A|)^2V^*YU \\ &\leq X^*f(|A|)^2X \,\sharp\, U^*Y^*g(V|A|V^*)^2YU \\ &= X^*f(|A|)^2X \,\sharp\, U^*Y^*g(|A^*|)^2YU \end{aligned}$$

and the equality condition holds. □

4 Lin's type extensions We consider further extensions of the weighted version of the mixed Schwarz inequality for matrices. Firstly, inspired by Lin [9], we show that some orthogonal conditions imply an improvement of the Cauchy-Schwarz inequality for matrices of any different sizes in Lemma 3.2. For this, we recall the result due to Lin [9], which is the sharpen (1.1) as follows: If $y, z \in \mathbb{C}^n$ and y is orthogonal to z, then

$$(|\langle x,y\rangle|^2 \leq) \quad |\langle x,y\rangle|^2 + \frac{\langle y,y\rangle|\langle x,z\rangle|^2}{\langle z,z\rangle} \leq \langle x,x\rangle\langle y,y\rangle \tag{4.1}$$

for all $x \in \mathbb{C}^n$. We show the matrix version of (4.1). For any matrix A, we denote by $P_A^\perp$ $(= I - P_A)$ the orthogonal projection on the orthogonal complement of the column space of A.

Lemma 4.1. *Let X be in $\mathbb{M}_{k\times m}$, Y in $\mathbb{M}_{k\times n}$, Z_X in $\mathbb{M}_{k\times l_X}$ and Z_Y in $\mathbb{M}_{k\times l_Y}$. Suppose that $X^*Z_X = 0$, $Y^*Z_Y = 0$ and $Z_Y^*Z_X = 0$.*

(i) *If $n \geq m$, then*

$$|Y^*X| \leq X^*P_{Z_Y}^\perp X \,\sharp\, U^*Y^*P_{Z_X}^\perp YU \quad (\leq X^*X \,\sharp\, U^*Y^*YU), \tag{4.2}$$

in which $U \in \mathbb{M}_{n\times m}$ consists of orthonormal columns and $Y^*X = U|Y^*X|$.

Under the assumption $\ker P_{Z_Y}^\perp X \subseteq \ker P_{Z_X}^\perp YU$, the equality in (4.2) holds if and only if there exists $W \in \mathbb{M}_m$ such that $P_{Z_X}^\perp YU = P_{Z_Y}^\perp XW$.

(ii) *If $n < m$, then*

$$|X^*Y| \le U^*X^*P_{Z_Y}^{\perp}XU \,\sharp\, Y^*P_{Z_X}^{\perp}Y \quad (\le U^*X^*XU \,\sharp\, Y^*Y),$$

in which $U \in \mathbb{M}_{m\times n}$ consists of orthonormal columns and $X^*Y = U|X^*Y|$.

Under the assumption $\ker P_{Z_X}^{\perp}Y \subseteq \ker P_{Z_Y}^{\perp}XU$, the equality holds if and only if there exists $W \in \mathbb{M}_n$ such that $P_{Z_Y}^{\perp}XU = P_{Z_X}^{\perp}YW$.

Proof. We only show (4.2). Put $X_1 = P_{Z_Y}^{\perp}X$ and $Y_1 = P_{Z_X}^{\perp}Y$. Since $X^*Z_X = Y^*Z_Y = Z_Y^*Z_X = 0$, we have $P_{Z_X}X = Y^*P_{Z_Y} = P_{Z_X}P_{Z_Y} = 0$ and it follows that

$$Y_1^*X_1 = Y^*P_{Z_X}^{\perp}P_{Z_Y}^{\perp}X = Y^*(I - P_{Z_X})(I - P_{Z_Y})X = Y^*X.$$

Hence it follows from Lemma 3.2 that

$$|Y^*X| = |Y_1^*X_1| \le X_1^*X_1 \,\sharp\, U^*Y_1^*Y_1U = X^*P_{Z_Y}^{\perp}X \,\sharp\, U^*Y^*P_{Z_X}^{\perp}YU,$$

and so we have the desired inequality (4.2) and the equality condition holds. □

Nextly, we focus on Parseval's equation: Let x, y be in $\mathbb{C}^k$ and $\{e_i\}_{i=1}^k$ a complete orthonormal system in $\mathbb{C}^k$. Then

$$\|x\|^2 = \sum_{i=1}^{k} |\langle x, e_i\rangle|^2 \tag{4.3}$$

and

$$\langle x, y\rangle = \sum_{i=1}^{k} \langle x, e_i\rangle\langle e_i, y\rangle. \tag{4.4}$$

The next result is a matrix generalization of Parseval's equation (4.3). It follows from a way similar to Gram-Schmidt orthogonalization.

Lemma 4.2. *Let X be in $\mathbb{M}_{k\times m}$, Y in $\mathbb{M}_{k\times n}$, $Z(X,1),\dots,Z(X,x)$ in $\mathbb{M}_{k\times l_X}$ and $Z(Y,1),\dots,Z(Y,y)$ in $\mathbb{M}_{k\times l_Y}$. Then*

$$X^*X = \sum_{j=0}^{y} S_j^* P_{Z(Y,j+1)} S_j \tag{4.5}$$

and

$$Y^*Y = \sum_{i=0}^{x} T_i^* P_{Z(X,i+1)} T_i,$$

where $S_0 = X$, $S_j = P_{Z(Y,j)}^{\perp}S_{j-1}$ for $j = 1, 2, \dots, y$, $T_0 = Y$, $T_i = P_{Z(X,i)}^{\perp}T_{i-1}$ for $i = 1, 2, \dots, x$ and $Z(Y, y+1)$ (resp. $Z(X, x+1)$) satisfies $\operatorname{ran} S_y \subseteq \operatorname{ran} Z(Y, y+1)$ (resp. $\operatorname{ran} T_x \subseteq \operatorname{ran} Z(X, x+1)$).

Proof. We only show (4.5). The following equation holds by induction:

$$\begin{aligned}
S_y^*S_y &= (S_{y-1}^* - S_{y-1}^*P_{Z(Y,y)})(S_{y-1} - P_{Z(Y,y)}S_{y-1}) \\
&= S_{y-1}^*S_{y-1} - S_{y-1}^*P_{Z(Y,y)}S_{y-1} \\
&\ \vdots \\
&= S_0^*S_0 - S_0^*P_{Z(Y,1)}S_0 - \cdots - S_{y-1}^*P_{Z(Y,y)}S_{y-1} \\
&= X^*X - \sum_{j=0}^{y-1} S_j^*P_{Z(Y,j+1)}S_j.
\end{aligned}$$

Since the assumption $\operatorname{ran} S_y \subseteq \operatorname{ran} Z(Y, y+1)$ implies $P_{Z(Y,y+1)}S_y = S_y$, we have

$$S_y^* P_{Z(Y,y+1)} S_y = X^*X - \sum_{j=0}^{y-1} S_j^* P_{Z(Y,j+1)} S_j$$

and so we have the desired equation (4.5). □

The following remak is a vector version of Lemma 4.2, see [9].

Remark 4.3. Let $x, z_1, \ldots, z_n \in \mathbb{C}^k$. Then we have a generalization of Parseval's equation (4.3):

$$\langle x, x\rangle = \sum_{i=0}^{n} \frac{|\langle u_i, z_{i+1}\rangle|^2}{\langle z_{i+1}, z_{i+1}\rangle},$$

where $u_0 = x$, $u_i = u_{i-1} - \dfrac{\langle u_{i-1}, z_i\rangle}{\langle z_i, z_i\rangle} z_i$ for $i = 1, 2, \ldots, n$ and $z_{n+1} = \dfrac{1}{\|u_n\|} u_n$. If $\{z_1, \ldots, z_k\}$ is a complete orthonormal system in $\mathbb{C}^k$, then we can just get Parseval's equation (4.3).

Under orthogonal conditions, we have the following matrix version of Parseval's equation (4.4).

Theorem 4.4. *Let X be in $\mathbb{M}_{k\times m}$, Y in $\mathbb{M}_{k\times n}$, $Z_1, \ldots, Z_p$ in $\mathbb{M}_{k\times l}$ and $Z_i^* Z_j = 0$ for all $i \neq j$, $i, j \in \{1, \ldots, p\}$. Then*

$$Y^*X = \sum_{q=0}^{p-1} Y^* P_{Z_{q+1}} X + T_p^* S_p \tag{4.6}$$

where $S_0 = X$, $S_j = P_{Z_j}^{\perp} S_{j-1}$ for $j = 1, \ldots, p$, $T_0 = Y$ and $T_i = P_{Z_i}^{\perp} T_{i-1}$ for $i = 1, \ldots, p$.

Proof. Since $Z_i^* Z_j = 0$, we have $P_{Z_i} P_{Z_j} = 0$ for all $i \neq j$, $i, j \in \{1, \ldots, p\}$ and it follows that

$$\begin{aligned} P_{Z_j} S_{j-1} &= P_{Z_j}(I - P_{Z_{j-1}}) S_{j-2} \\ &= P_{Z_j} S_{j-2} \\ &= \cdots = P_{Z_j} X \end{aligned}$$

and similarly we have

$$P_{Z_j} T_{j-1} = P_{Z_j} Y.$$

Hence it follows that

$$\begin{aligned} T_p^* S_p &= T_{p-1}^*(I - P_{Z_p})(I - P_{Z_p}) S_{p-1} \\ &= T_{p-1}^* S_{p-1} - T_{p-1}^* P_{Z_p} S_{p-1} \\ &\vdots \\ &= T_0^* S_0 - T_0^* P_{Z_1} S_0 - \cdots - T_{p-1}^* P_{Z_p} S_{p-1} \\ &= Y^*X - \sum_{q=0}^{p-1} T_q^* P_{Z_{q+1}} S_q \\ &= Y^*X - \sum_{q=0}^{p-1} Y^* P_{Z_{q+1}} X \end{aligned}$$

and hence we have the desired equality (4.6). □

Remark 4.5. We can consider a vector version of Theorem 4.4. Let $x, y, z_1, \ldots, z_p$ be in $\mathbb{C}^k$ and $\langle z_i, z_j\rangle = 0$ for all $i \neq j$, $i, j \in \{1, \ldots, p\}$. Then we have a generalziation of Parseval's equation (4.4):

$$\langle x, y\rangle = \sum_{i=0}^{p-1} \frac{\langle x, z_{i+1}\rangle\langle z_{i+1}, y\rangle}{\langle z_{i+1}, z_{i+1}\rangle} + \langle u_p, v_p\rangle,$$

where $u_0 = x$, $u_j = u_{j-1} - \dfrac{\langle u_{j-1}, z_j\rangle}{\langle z_j, z_j\rangle} z_j$ for $j = 1, \ldots, p$, $v_0 = y$ and $v_i = v_{i-1} - \dfrac{\langle v_{i-1}, z_i\rangle}{\langle z_i, z_i\rangle} z_i$ for $i = 1, \ldots, p$.

In [9], Lin showed the following refinement of a weighted mixed Schwarz inequality (1.3): Let A be a bounded linear operator on a complex Hilbert space $\mathcal{H}$ and $0 \neq y \in \mathcal{H}$. If A^*y is orthogonal to a vector $z \in \mathcal{H}$ with $Az \neq 0$, then

$$|\langle Ax, y\rangle|^2 + \frac{\langle |A^*|^{2\beta} y, y\rangle |\langle |A|^{2\alpha} x, z\rangle|^2}{\langle |A|^{2\alpha} z, z\rangle} \leq \langle |A|^{2\alpha} x, x\rangle\langle |A^*|^{2\beta} y, y\rangle \tag{4.7}$$

for all $x \in \mathcal{H}$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$. The next theorem is a matrix version of (4.7).

Theorem 4.6. *Let X be in $\mathbb{M}_{m\times n}$, Y in $\mathbb{M}_{p\times q}$, Z_X in $\mathbb{M}_{m\times l_X}$, Z_Y in $\mathbb{M}_{p\times l_Y}$ and A in $\mathbb{M}_{p\times m}$. Suppose that $X^*|A|^{2\alpha} Z_X = 0$, $Y^*|A^*|^{2\beta} Z_Y = 0$ and $Z_Y^* A Z_X = 0$ for given $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$. If $q \geq n$ and $p \geq m$, then*

$$\begin{aligned} |Y^*AX| \quad &\leq \quad X^*|A|^{\alpha} P^{\perp}_{|A^*|Z_Y} |A|^{\alpha} X \sharp U^*Y^*V|A|^{\beta} P^{\perp}_{|A|Z_X} |A|^{\beta} V^*YU \\ (&\leq \quad X^*|A|^{2\alpha} X \sharp U^*Y^*|A^*|^{2\beta} YU), \end{aligned}$$

in which $U \in \mathbb{M}_{q\times n}$ consists of orthonormal columns and $Y^*AX = U|Y^*AX|$, and $V \in \mathbb{M}_{p\times m}$ consists of orthonormal columns and $A = V|A|$.

Under the assumption $\ker P^{\perp}_{|A^*|Z_Y} |A|^{\alpha} X \subseteq \ker P^{\perp}_{|A|Z_X} |A|^{\beta} V^*YU$, the equality holds if and only if there exists $W \in \mathbb{M}_n$ such that $P^{\perp}_{|A|Z_X} |A|^{\beta} V^*YU = P^{\perp}_{|A^*|Z_Y} |A|^{\alpha} XW$.

Proof. Replacing X by $|A|^{\alpha} X$, Y by $|A|^{\beta} V^*Y$, Z_X by $|A|^{\alpha} Z_X$ and Z_Y by $|A|^{\beta} V^* Z_Y$ in (4.2) of Lemma 4.1, then we obtain the desired inequality and the equality condition. □

The next result is a multivariate extension of Lemma 4.1, which is a refinement of matrix Cauchy-Schwarz inequality (2.1) of Lemma 2.1:

Lemma 4.7. *Let X be in $\mathbb{M}_{k\times m}$, Y in $\mathbb{M}_{k\times n}$, $Z(X, 1), \ldots, Z(X, x)$ in $\mathbb{M}_{k\times l_X}$ and $Z(Y, 1)$, $\ldots, Z(Y, y)$ in $\mathbb{M}_{k\times l_Y}$. Suppose that $X^*Z(X, i) = 0$, $Y^*Z(Y, j) = 0$ and $Z(Y, j)^*Z(X, i) = 0$ for $i = 1, 2, \ldots, x$, $j = 1, 2, \ldots, y$ If $n \geq m$, then*

$$\begin{aligned} |Y^*X| \quad &\leq \quad (X^*X - \sum_{j=0}^{y-1} S_j^* P_{Z(Y,j+1)} S_j) \sharp U^*(Y^*Y - \sum_{i=0}^{x-1} T_i^* P_{Z(X,i+1)} T_i) U \\ (&\leq \quad X^*X \sharp U^*Y^*YU), \end{aligned}$$

in which $U \in \mathbb{M}_{n\times m}$ consists of orthonormal columns and $Y^*X = U|Y^*X|$, where $S_0 = X$, $S_j = P^{\perp}_{Z(Y,j)} S_{j-1}$ for $j = 1, 2, \ldots, y$, $T_0 = Y$ and $T_i = P^{\perp}_{Z(X,i)} T_{i-1}$ for $i = 1, 2, \ldots, x$.

Under the assumption $\ker(\prod_{b=1}^{y} P^{\perp}_{(Y,y-b+1)}) X \subseteq \ker(\prod_{a=1}^{x} P^{\perp}_{(X,x-a+1)}) YU$, the equality holds if and only if there exists $W \in \mathbb{M}_m$ such that $(\prod_{a=1}^{x} P^{\perp}_{(X,x-a+1)}) YU = (\prod_{b=1}^{y} P^{\perp}_{(Y,y-b+1)}) XW$.

Proof. By Lemma 4.2, the following equations hold:

$$S_y^* S_y = X^* X - \sum_{j=0}^{y-1} S_j^* P_{Z(Y,j+1)} S_j$$

and

$$T_x^* T_x = Y^* Y - \sum_{i=0}^{x-1} T_i^* P_{Z(X,i+1)} T_i.$$

Since $X^* Z(X,i) = 0$, $Y^* Z(Y,j) = 0$ and $Z(Y,j)^* Z(X,i) = 0$, we have $P_{Z(X,i)} X = Y^* P_{Z(Y,j)} = P_{Z(X,i)} P_{Z(Y,j)} = 0$ for $i = 1, 2, \ldots, x$ and $j = 1, 2, \ldots, y$. Then it follows that

$$\begin{aligned} T_x^* S_y &= T_{x-1}^* P_{Z(X,x)}^{\perp} P_{Z(Y,y)}^{\perp} S_{y-1} \\ &= Y^* \left(I + \sum_{s=1}^{x} \left(\sum_{1 \le c_1 < \cdots < c_s \le x} \prod_{p=1}^{s} (-1)^s P_{Z(X,c_p)} \right) \right. \\ &\qquad \left. + \sum_{t=1}^{y} \left(\sum_{1 \le d_1 < \cdots < d_t \le y} \prod_{q=1}^{t} (-1)^t P_{Z(Y,d_{t+1-q})} \right) \right) X \\ &= Y^* X. \end{aligned}$$

So, we can get the desired inequality by Lemma 3.2:

$$\begin{aligned} |Y^* X| &= |T_x^* S_y| \\ &\le S_y^* S_y \sharp U^* T_x^* T_x U \\ &= (X^* X - \sum_{j=0}^{y-1} S_j^* P_{Z(Y,j+1)} S_j) \sharp U^* (Y^* Y - \sum_{i=0}^{x-1} T_i^* P_{Z(X,i+1)} T_i) U. \end{aligned}$$

Since $S_y = (\prod_{b=1}^{y} P_{(Y,y-b+1)}^{\perp}) X$ and $T_x = (\prod_{a=1}^{x} P_{(X,x-a+1)}^{\perp}) Y$, we have the equality condition by Lemma 3.2. $\square$

Moreover, Lin showed the following multivariate extension of (4.7): Under the hypotheses of (4.7), if $A^* y$ is orthogonal to a set of vectors $\{z_1, \ldots, z_n\} \subseteq \mathcal{H}$ with $A z_i \neq 0$, $i = 1, \ldots, n$, then

$$|\langle Ax, y \rangle|^2 + \langle |A^*|^{2\beta} y, y \rangle \sum_{i=1}^{n} \frac{|\langle |A|^{2\alpha} u_{i-1}, z_i \rangle|^2}{\langle |A|^{2\alpha} z_i, z_i \rangle} \le \langle |A|^{2\alpha} x, x \rangle \langle |A^*|^{2\beta} y, y \rangle \tag{4.8}$$

for every $x \in \mathcal{H}$, where $u_i = u_{i-1} - \dfrac{\langle |A|^{2\alpha} u_{i-1}, z_i \rangle}{\langle |A|^{2\alpha} z_i, z_i \rangle} z_i$, $i = 1, \ldots, n$ with $u_0 = x$. The next result is a multivariate extension of Theorem 4.6 and a matrix version of (4.8).

Theorem 4.8. *Let X be in $\mathbb{M}_{m \times n}$, Y in $\mathbb{M}_{p \times q}$, $Z(X,1), \ldots, Z(X,x)$ in $\mathbb{M}_{m \times l_X}$, $Z(Y,1)$, $\ldots, Z(Y,y)$ in $\mathbb{M}_{p \times l_Y}$, and A in $\mathbb{M}_{p \times m}$. Suppose that $X^* |A|^{2\alpha} Z(X,i) = 0$, $Y^* |A^*|^{2\beta} Z(Y,j)$*

$= 0$, $Z(Y,j)^*AZ(X,i) = 0$ *for* $i = 1,2,\ldots,x$, $j = 1,2,\ldots,y$ *for given* $\alpha,\beta \in [0,1]$ *with* $\alpha+\beta = 1$. *If* $q \geq n$ *and* $p \geq m$, *then*

$$\begin{aligned}|Y^*AX| \quad &\leq \quad (X^*|A|^{2\alpha}X - \sum_{j=1}^{y-1} S_j^* P_{|A^*|Z(Y,j+1)} S_j) \sharp U^*(Y^*|A^*|^{2\beta}Y - \sum_{i=1}^{x-1} T_i^* P_{|A|Z(X,i+1)} T_i)U \\ &(\leq \quad X^*|A|^{2\alpha}X \sharp U^*Y^*|A^*|^{2\beta}YU),\end{aligned}$$

in which $U \in \mathbb{M}_{q\times n}$ *consists of orthonormal columns and* $Y^*AX = U|Y^*AX|$, *and* $V \in \mathbb{M}_{p\times m}$ *consists of orthonormal columns and* $A = V|A|$, *where* $S_0 = |A|^\alpha X$, $S_j = P^\perp_{|A^*|Z(Y,j)} S_{j-1}$ *for* $j = 1,2,\ldots,y$, $T_0 = |A|^\beta VY$ *and* $T_i = P^\perp_{|A|Z(X,i)} T_{i-1}$ *for* $i = 1,2,\ldots,x$.

Under the assumption $\ker(\prod_{b=1}^{y} P^\perp_{|A^*|Z(Y,y-b+1)})|A|^\alpha X \subseteq \ker(\prod_{a=1}^{x} P^\perp_{|A|Z(X,x-a+1)})|A|^\beta V^*YU$, *the equality holds if and only if there exists* $W \in \mathbb{M}_n$ *such that* $(\prod_{a=1}^{x} P^\perp_{|A|Z(X,x-a+1)})|A|^\beta V^*YU = (\prod_{b=1}^{y} P^\perp_{|A^*|Z(Y,y-b+1)})|A|^\alpha XW$, *where* $V \in \mathbb{M}_{p\times m}$ *consists of orthonormal columns and* $A = V|A|$.

Proof. Replacing X by $|A|^\alpha X$, Y by $|A|^\beta V^*Y$, $Z(X,i)$ by $|A|^\alpha Z(X,i)$ and $Z(Y,j)$ by $|A|^\beta V^*Z(Y,j)$ in Lemma 4.7 for all $i = 1,2,\ldots,x$ and $j = 1,2,\ldots,y$, then we obtain the desired inequality and the equality condition. $\square$

We note that the vector version of Theorem 4.8 is a matrix version of Theorem 4 in [9]: Let x be in $\mathbb{C}^m$, y in $\mathbb{C}^p$, $z(x,1),\ldots,z(x,a)$ in $\mathbb{C}^m$, $z(y,1),\ldots,z(y,b)$ in $\mathbb{C}^p$, and A in $\mathbb{M}_{p\times m}$. Suppose that $\langle |A|^{2\alpha}z(x,i),x\rangle = 0$, $\langle |A^*|^{2\beta}z(y,j),y\rangle = 0$, $\langle Az(x,i),z(y,j)\rangle = 0$ for $i = 1,2,\ldots,a$, $j = 1,2,\ldots,b$ for given $\alpha,\beta \in [0,1]$ with $\alpha+\beta = 1$. If $p \geq m$, then

$$\begin{aligned}|\langle Ax,y\rangle|^2 \quad &\leq \quad \left(\langle |A|^{2\alpha}x,x\rangle - \sum_{j=1}^{b-1} \frac{|\langle |A^*|z(y,j+1),s_j\rangle|^2}{\langle |A^*|^2 z(y,j+1), z(y,j+1)\rangle}\right) \\ &\qquad \times \left(\langle |A^*|^{2\beta}y,y\rangle - \sum_{i=1}^{a-1} \frac{|\langle |A|z(x,i+1),t_i\rangle|^2}{\langle |A|^2 z(x,i+1), z(x,i+1)\rangle}\right),\end{aligned}$$

in which $V \in \mathbb{M}_{p\times m}$ consists of orthonormal columns and $A = V|A|$, $s_0 = |A|^\alpha x$, $s_j = P^\perp_{|A^*|z(y,j)} s_{j-1}$ for $j = 1,2,\ldots,b$, $t_0 = |A|^\beta Vy$ and $t_i = P^\perp_{|A|z(x,i)} t_{i-1}$ for $i = 1,2,\ldots,a$.

5 Weighted Wielandt inequality We consider a different way of a refinement of a weighted Schwarz inequality in §4. We show a weighted version of matrix Wielandt inequality. We proved a matrix version of Wielandt inequality, see [2]: Let A be a positive semidefinite matrix in $\mathbb{M}_k$, with $\operatorname{rank}(A) = r$, $\lambda_1 \geq \cdots \geq \lambda_r > 0$ eigenvalues of A, and X, Y in $\mathbb{M}_{k\times n}$ such that $Y^*P_AX = 0$ where P_A is the orthogonal projection on the column space of A. Then

$$|Y^*AX| \leq \left(\frac{\lambda_1 - \lambda_r}{\lambda_1 + \lambda_r}\right)(X^*AX \sharp U^*Y^*AYU), \tag{5.1}$$

in which $U \in \mathbb{M}_n$ is a unitary matrix such that $Y^*AX = U|Y^*AX|$. The following theorem is a weighted version of (5.1).

Theorem 5.1. *Let A be a matrix in $\mathbb{M}_{p\times m}$, with rank(A)=r, $\sigma_1 \geq \cdots \geq \sigma_r > 0$ singular values of A, $X \in \mathbb{M}_{m\times n}$ and $Y \in \mathbb{M}_{p\times q}$. For all $\alpha, \beta \in [0,1]$ with $\alpha+\beta = 1$, if $p \geq m$, $q \geq n$ and $Y^*VP_{|A|}X = 0$, then*

$$|Y^*AX| \leq \left(\frac{\sigma_1 - \sigma_r}{\sigma_1 + \sigma_r}\right)(X^*|A|^{2\alpha}X \sharp U^*Y^*|A^*|^{2\beta}YU),$$

in which $U \in \mathbb{M}_{q\times n}$ consists of orthonormal columns and $Y^*AX = U|Y^*AX|$, and $V \in \mathbb{M}_{p\times m}$ consists of orthonormal columns and $A = V|A|$.

Proof. Let $c = \dfrac{2\sigma_1\sigma_r}{\sigma_1+\sigma_r}$. Since $\sigma_1 P_{|A|} - |A|$ and $|A| - \sigma_r P_{|A|}$ are positive semidefinite and they commute, it follows that $(\sigma_1 P_{|A|} - |A|)(|A| - \sigma_r P_{|A|}) \geq 0$ and hence

$$(P_{|A|} - c|A|^{\dagger})^2 \leq \left(\frac{\sigma_1 - \sigma_r}{\sigma_1 + \sigma_r}\right)^2 I, \tag{5.2}$$

where $|A|^{\dagger}$ means the Moore-Penrose generalized inverse of $|A|$. So, we can get the desired inequality:

$$\begin{aligned}
|Y^*AX| &= |Y^*AX - cY^*VP_{|A|}X| = |Y^*V|A|^{\beta}(P_{|A|} - c|A|^{\dagger})|A|^{\alpha}X| \\
&= |(P_{|A|} - c|A|^{\dagger})|A|^{\beta}V^*Y)^*(|A|^{\alpha}X)| \\
&\leq X^*|A|^{2\alpha}X \sharp U^*Y^*V|A|^{\beta}(P_{|A|} - c|A|^{\dagger})^2|A|^{\beta}V^*YU \quad \text{by Lemma 3.2} \\
&\leq X^*|A|^{2\alpha}X \sharp U^*Y^*V|A|^{\beta}\left(\frac{\sigma_1 - \sigma_r}{\sigma_1 + \sigma_r}\right)^2 |A|^{\beta}V^*YU \quad \text{by (5.2)} \\
&= \left(\frac{\sigma_1 - \sigma_r}{\sigma_1 + \sigma_r}\right)(X^*|A|^{2\alpha}X \sharp U^*Y^*|A^*|^{2\beta}YU).
\end{aligned}$$

□

Lastly, we consider a Wielandt version of Theorem 3.4 by a way similar to the proof of Theorem 5.1.

Theorem 5.2. *Let A be a matrix in $\mathbb{M}_{p\times m}$, with rank(A)=r, $\sigma_1 \geq \cdots \geq \sigma_r > 0$ singular values of A, $X \in \mathbb{M}_{m\times n}$, $Y \in \mathbb{M}_{p\times q}$ and f, g complex functions on $[0,\infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0,\infty)$ For all $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$, if $p \geq m$, $q \geq n$ and $Y^*VP_{|A|}X = 0$, then*

$$|Y^*AX| \leq \left(\frac{\sigma_1 - \sigma_r}{\sigma_1 + \sigma_r}\right)(X^*|f(|A|)|^2X \sharp U^*Y^*|g(|A^*|)|^2YU),$$

in which $U \in \mathbb{M}_{q\times n}$ consists of orthonormal columns and $Y^*AX = U|Y^*AX|$, and $V \in \mathbb{M}_{p\times m}$ consists of orthonormal columns and $A = V|A|$.

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ON A DUALITY BETWEEN THE OPERATORS AND THE SPACE OF SEQUENCES

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Abstract. On a space of sequences, the multiplication operator and the Hankel operator are defined and investigated. On the other hand, the concept of a space of sequences is basic, but its properties are not well known nevertheless. In this paper, we prove some properties of the space of sequences, and by means of this, we show certain modification of H^1-$BMOA$ duality and L^1-L^∞ duality (Theorem 5.2) from the viewpoint of theory of these operators.

1 Introduction. The multiplication operator is naturally defined on the Lebesgue space L^p as well as on the space ℓ^p. The Hankel operator is also defined on the Hardy space H^p as well as on the space ℓ_+^p. These operators are well investigated, but properties of a space of sequences are not well known nevertheless. In this paper, we shall prove some properties of the space obtained from these operators (Section 3), of the space of sequences (Section 4), and show certain modification of H^1-$BMOA$ duality and L^1-L^∞ duality (Theorem 5.2) from the viewpoint of theory of these operators.

Let T be the unit circle in the complex plane and L^p be the L^p space of functions on T with respect to Lebesgue measure. We denote by H^p the Hardy space defined by

$$H^p := \{f \in L^p \mid (f)_n = 0 \quad \text{for } n < 0\},$$

where $(f)_n$ means the n-th Fourier coefficient of f. We also denote by H_0^p the space of functions in H^p whose zeroth Fourier coefficient is zero, and by $BMOA$ the set of all analytic functions of bounded mean oscillation on T.

Let $1 < p < \infty$. It is known that for a in L^1, a function a is in L^∞ if and only if the multiplication operator $M(a)$ is defined on L^p, and L^∞ is isomorphic to $(L^1)^*$. It is also known that for a in H^2, a function a is in $BMOA$ if the Hankel operator $H(\chi_1 a)$ is defined on H^p, where $\chi_j(\theta) := e^{\sqrt{-1}j\theta}$ $(0 \le \theta \le 2\pi)$, and $BMOA$ is isomorphic to $(H^1)^*$ (cf. [1], [5]).

Now we consider the discrete versions of these topics. Let ℓ^p be the Banach space of sequences of complex numbers defined by

$$\ell^p := \left\{\varphi = \{\varphi_n\}_{n\in\mathbb{Z}} \mid \|\varphi\|_{\ell^p} := (\sum_{n\in\mathbb{Z}} |\varphi_n|^p)^{\frac{1}{p}} < \infty\right\},$$

and ℓ_+^p be the space defined by

$$\ell_+^p := \{\varphi \in \ell^p \mid \varphi_n = 0 \quad \text{for } n < 0\}.$$

Let $1 \le p < \infty$. For $a \in L^1$, a function a is in a subspace $M^p \subset L^1$ given in Section 2 if and only if the multiplication operator $M(a)$ is defined on ℓ^p. For $a \in H^2$, a function a

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is in a subspace $M^p_+ \cap H^2_0$ given in Section 3 if and only if the Hankel operator $H(\chi_1 a)$ is defined on ℓ^p_+.

Therefore it is a natural question whether there are normed spaces V^p and V^p_+ such that M^p and $M^p_+ \cap H^2_0$ are isomorphic to $(V^p)^*$ and $(V^p_+)^*$, respectively. We will show such spaces V^p and V^p_+ exist by construction. These are certain modification of H^1-$BMOA$ duality and L^1-L^∞ duality.

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2 Preliminaries. In this section, we shall give some basic facts on the multiplication operators and the Hankel operators.

We denote by $\mathfrak{B}(X)$ the set of all bounded linear operators on a Banach space X to itself, and by $(a)_n$ the n-th Fourier coefficient of a. Let $e_j := \{\delta_{j,n}\}_{n\in\mathbb{Z}}$ (δ : Kronecker's delta).

For $1 < p < \infty$ and $a \in L^\infty$, the multiplication operator $M(a)$ on L^p is defined by

$$M(a) \,:\, L^p \longrightarrow L^p \,:\, f \longmapsto a \cdot f,$$

and it is easy to see that $\|a\|_{L^\infty} = \|M(a)\|_{\mathfrak{B}(L^p)}$. Note that the j-th Fourier coefficient $(a\cdot f)_j$ of $a \cdot f$ is equal to $\sum_{k\in\mathbb{Z}}(a)_{j-k}(f)_k$ for all $j \in \mathbb{Z}$.

For a function a in L^1 and a sequence φ, we put

$$a * \varphi := \left\{ \sum_{k\in\mathbb{Z}} (a)_{j-k}\varphi_k \right\}_{j\in\mathbb{Z}},$$

whenever the sequence $a * \varphi$ can be defined. For $1 \le p < \infty$, a vector space M^p is defined by

$$M^p := \left\{ a \in L^1 \mid \|a\|_{M^p} := \sup\left\{ \|a*\varphi\|_{\ell^p} \mid \|\varphi\|_{\ell^p} \le 1 \right\} < \infty \right\}.$$

It is obvious that $\|\cdot\|_{M^p}$ is a norm on M^p. For $a \in M^p$, the multiplication operator $M(a)$ on ℓ^p is defined by

$$M(a) \,:\, \ell^p \longrightarrow \ell^p \,:\, \varphi \longmapsto a * \varphi,$$

and $\|a\|_{M^p} = \|M(a)\|_{\mathfrak{B}(\ell^p)}$.

The following properties of M^p are basic to our argument (cf. [1]).

Proposition 2.1. *1. For $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, $\|\cdot\|_{M^p} = \|\cdot\|_{M^q}$ and $M^p = M^q$.*

2. $\|\cdot\|_{M^2} = \|\cdot\|_{L^\infty}$ and $M^2 = L^\infty$.

3. $M^1 = \left\{ a \in L^1 \mid \sum_{n\in\mathbb{Z}} |(a)_n| < \infty \right\}$ and $\|a\|_{M^1} = \sum_{n\in\mathbb{Z}} |(a)_n|$.

4. For $1 \le p \le r \le 2$, $\|\cdot\|_{M^2} \le \|\cdot\|_{M^r} \le \|\cdot\|_{M^p} \le \|\cdot\|_{M^1}$ and $M^1 \subset M^p \subset M^r \subset M^2$.

5. For $1 \le p < \infty$, M^p is a Banach algebra with respect to $\|\cdot\|_{M^p}$.

Now we define the Hankel operators. Let $1 < p < \infty$. The flip operator J on L^p is defined by

$$J \,:\, L^p \longrightarrow L^p \,:\, \sum_{n\in\mathbb{Z}} (f)_n \chi_n \longmapsto \sum_{n\in\mathbb{Z}} (f)_n \chi_{-n-1},$$

the Riesz projection P is defined by

$$P \,:\, L^p \longrightarrow H^p \,:\, \sum_{n\in\mathbb{Z}}(f)_n\chi_n \longmapsto \sum_{n\geq 0}(f)_n\chi_n,$$

and it is well known that

$$c_p := \sup\{\|P(f)\|_p \mid \|f\|_p \leq 1\} < \infty$$

by the M. Riesz theorem (cf. [2]). Let I be the identity operator on L^p, and $Q := I - P$. For $a \in L^\infty$, the Hankel operator $H(a)$ on H^p is defined by

$$H(a) \,:\, H^p \longrightarrow H^p \,:\, f \longmapsto PM(a)QJf.$$

The discrete versions of these operators are similarly defined. Let $1 \leq p < \infty$. The flip operator J on ℓ^p is given by

$$J \,:\, \ell^p \longrightarrow \ell^p \,:\, \{\varphi_n\}_{n\in\mathbb{Z}} \longmapsto \{\varphi_{-n-1}\}_{n\in\mathbb{Z}}\,,$$

the Riesz projection P is

$$P \,:\, \ell^p \longrightarrow \ell^p_+ \,:\, \{\varphi_n\}_{n\in\mathbb{Z}} \longmapsto \{\varphi_n\}_{n\geq 0}\,,$$

and $Q := I - P$ (I : the identity operator on ℓ^p). For $a \in M^p$, the Hankel operator $H(a)$ on ℓ^p_+ is defined by

$$H(a) \,:\, \ell^p_+ \longrightarrow \ell^p_+ \,:\, \varphi \longmapsto PM(a)QJ\varphi.$$

3 New classes M^p_+ and N^p_+.

Note that $H(a)\{\varphi_j\}_{j\geq 0}$ is equal to $\left\{\sum_{k\geq 0}(a)_{j+k+1}\varphi_k\right\}_{j\geq 0}$. We consider a new class M^p_+ to extend the domain of the Hankel operator. For a function a in L^1 and a sequence φ, we define a sequence $a \odot \varphi$ by

$$a \odot \varphi := \left\{\sum_{k\geq 0}(a)_{j+k+1}\varphi_k\right\}_{j\geq 0},$$

whenever the sequence $a \odot \varphi$ can be defined.

Definition 3.1. *For $1 \leq p < \infty$, we define a vector space M^p_+ as*

$$M^p_+ := \left\{a \in L^1 \mid \|a\|_{M^p_+} := \sup\left\{\|a\odot\varphi\|_{\ell^p_+} \mid \|\varphi\|_{\ell^p_+} \leq 1\right\} < \infty\right\}.$$

For $a \in M^p_+$, we define the Hankel operator $H(a)$ on ℓ^p_+ as

$$H(a) \,:\, \ell^p_+ \longrightarrow \ell^p_+ \,:\, \varphi \longmapsto a \odot \varphi,$$

and $\|a\|_{M^p_+} = \|H(a)\|_{\mathfrak{B}(\ell^p_+)}$.

It is easy to see that $\|\cdot\|_{M^p_+} \leq \|\cdot\|_{M^p}$ and $M^p \subset M^p_+$. Indeed, let ℓ^p_- be the space defined by

$$\ell^p_- := \{\varphi \in \ell^p \mid \varphi_n = 0 \quad \text{for } n \geq 0\}\,,$$

and

$$\begin{aligned}
\|a\|_{M_+^p} &= \sup\left\{\left(\sum_{j\geq 0}|\sum_{k\leq -1}(a)_{j-k}\varphi_k|^p\right)^{\frac{1}{p}} \mid \|\varphi\|_{\ell_-^p}\leq 1\right\} \\
&= \sup\left\{\|P(a*\varphi)\|_{\ell_+^p} \mid \|\varphi\|_{\ell_-^p}\leq 1\right\} \\
&\leq \sup\left\{\|a*\varphi\|_{\ell^p} \mid \|\varphi\|_{\ell_-^p}\leq 1\right\} \\
&\leq \sup\{\|a*\varphi\|_{\ell^p} \mid \|\varphi\|_{\ell^p}\leq 1\} \\
&= \|a\|_{M^p}.
\end{aligned}$$

Hence, we extended the domain of the Hankel operator to M_+^p.

Here, $\|\cdot\|_{M_+^p}$ is actually a norm on $M_+^p\cap H_0^2$. In fact, it is a semi-norm and we see that

$$\begin{aligned}
\left(\sum_{n\geq 1}|(a)_n|^p\right)^{\frac{1}{p}} &= \left(\sum_{n\geq 0}|(a)_{n+1}|^p\right)^{\frac{1}{p}} \\
&\leq \sup\left\{\left(\sum_{j\geq 0}|\sum_{k\geq 0}(a)_{j+k+1}\varphi_k|^p\right)^{\frac{1}{p}} \mid \|\varphi\|_{\ell_+^p}\leq 1\right\} \\
&= \|a\|_{M_+^p}
\end{aligned}$$

for $a\in M_+^p$. Thus $\|\cdot\|_{M_+^p}$ is a norm on $M_+^p\cap H_0^2$. The real inner product $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ is defined by

$$\langle\{a_n\}_{n\in\mathbb{Z}},\{b_n\}_{n\in\mathbb{Z}}\rangle_{\mathbb{R}} := \sum_{n\in\mathbb{Z}} a_n b_n.$$

The space M_+^p has the following properties like M^p.

Proposition 3.2. *1.* $M_+^1\cap H_0^2 = M^1\cap H_0^2$.

2. For $1<p<\infty$ *and* $\frac{1}{p}+\frac{1}{q}=1$, $\|\cdot\|_{M_+^p}=\|\cdot\|_{M_+^q}$ *and* $M_+^p=M_+^q$.

3. For $1<p\leq r\leq 2$, $\|\cdot\|_{M_+^2}\leq\|\cdot\|_{M_+^r}\leq\|\cdot\|_{M_+^p}$ *and* $M_+^p\subset M_+^r\subset M_+^2$.

Proof. 1. We show

$$M_+^1\cap L^2 = \left\{a\in L^2 \mid \sum_{n\geq 1}|(a)_n|<\infty\right\}.$$

It is already proved that $\sum_{n\geq 1}|(a)_n|\leq\|a\|_{M_+^1}$ for $a\in M_+^1$. For $a\in L^2$ with $\sum_{n\geq 1}|(a)_n|<\infty$, we have

$$H(a)=H(\sum_{n\in\mathbb{Z}}(a)_n\chi_n)=H(\sum_{n\geq 1}(a)_n\chi_n)=\sum_{n\geq 1}(a)_nH(\chi_n).$$

Therefore

$$\|a\|_{M_+^1}\leq\sum_{n\geq 1}|(a)_n|\|\chi_n\|_{M_+^1}\leq\sum_{n\geq 1}|(a)_n|,$$

and we conclude

$$M^1_+ \cap L^2 = \left\{ a \in L^2 \mid \sum_{n \geq 1} |(a)_n| < \infty \right\}.$$

This implies the conclusion.

2. Let $a \in M^p_+$. Since $\langle e_j, H(a)e_k \rangle_{\mathbb{R}} = (a)_{j+k+1} = \langle H(a)e_j, e_k \rangle_{\mathbb{R}}$ holds for $j, k \geq 0$, $\langle \varphi, H(a)\psi \rangle_{\mathbb{R}} = \langle H(a)\varphi, \psi \rangle_{\mathbb{R}}$ holds for $\varphi, \psi \in \ell^0_+$. Thus

$$\begin{aligned}
\|a\|_{M^p_+} &= \sup \left\{ \|H(a)\varphi\|_{\ell^p_+} \mid \|\varphi\|_{\ell^p_+} \leq 1 \right\} \\
&= \sup \left\{ \langle H(a)\varphi, \psi \rangle_{\mathbb{R}} \mid \|\varphi\|_{\ell^p_+}, \|\psi\|_{\ell^q_+} \leq 1 \right\} \\
&= \sup \left\{ \langle \varphi, H(a)\psi \rangle_{\mathbb{R}} \mid \|\varphi\|_{\ell^p_+}, \|\psi\|_{\ell^q_+} \leq 1 \right\} \\
&= \sup \left\{ \|H(a)\psi\|_{\ell^q_+} \mid \|\psi\|_{\ell^q_+} \leq 1 \right\} \\
&= \|a\|_{M^q_+}
\end{aligned}$$

and $a \in M^q_+$. This implies the conclusion.

3. Let $a \in M^p_+$ and $1 < p \leq r \leq 2$. Since $H(a) \in \mathfrak{B}(\ell^p_+) \cap \mathfrak{B}(\ell^q_+)$, $\|H(a)\varphi\|_{\ell^r_+} \leq \|H(a)\|^{1-t}_{\mathfrak{B}(\ell^p_+)} \|H(a)\varphi\|^t_{\mathfrak{B}(\ell^q_+)}$ $(0 \leq t \leq 1)$ by the Riesz-Thorin interpolation theorem. Hence $\|a\|_{M^2_+} \leq \|a\|_{M^r_+} \leq \|a\|_{M^p_+}$. □

We also consider another new class N^p_+ to extend the domain of the Hankel operator on the Hardy space. Note that the j-th Fourier coefficient $(H(a)f)_j$ of $H(a)f$ is equal to $\sum_{k \geq 0} (a)_{j+k+1}(f)_k$ for all $j \geq 0$.

Definition 3.3. *For $1 < p < \infty$, we define a vector space N^p_+ as*

$$N^p_+ := \left\{ a \in L^1 \mid \|a\|_{N^p_+} := \sup \left\{ \| \sum_{j \geq 0} \sum_{k \geq 0} (a)_{j+k+1}(f)_k \chi_j \|_{H^p} \mid \|f\|_{H^p} \leq 1 \right\} < \infty \right\}.$$

For $a \in N^p_+$, we define the Hankel operator $H(a)$ on H^p as

$$H(a) \ : \ H^p \ \longrightarrow \ H^p \ : \ f \ \longmapsto \ \sum_{j \geq 0} \sum_{k \geq 0} (a)_{j+k+1}(f)_k \chi_j,$$

and $\|a\|_{N^p_+} = \|H(a)\|_{\mathfrak{B}(H^p)}$.

It is easy to see that $\| \cdot \|_{N^p_+} \leq c_p^2 \| \cdot \|_{L^\infty}$ and $L^\infty \subset N^p_+$. Indeed,

$$\begin{aligned}
\|a\|_{N^p_+} &= \sup \{ \|P(a \cdot (QJf))\|_{H^p} \mid \|f\|_{H^p} \leq 1 \} \\
&\leq \sup \{ c_p \|a\|_{L^\infty} c_q \|f\|_{H^p} \mid \|f\|_{H^p} \leq 1 \} \\
&\leq c_p^2 \|a\|_{L^\infty}.
\end{aligned}$$

Hence, the domain of the Hankel operator is extended to N^p_+.

Here, $\| \cdot \|_{N^p_+}$ is actually a norm on $N^p_+ \cap H^2_0$. In fact, it is a semi-norm and we see that

$$\begin{aligned}
\|a\|_{H^p} &= \|P(a \cdot \chi_{-1})\|_{H^p} \\
&= \|P(a \cdot (QJ\chi_0))\|_{H^p} \\
&\leq \|a\|_{N^p_+}
\end{aligned}$$

for $a \in H_0^2$. Thus $\|\cdot\|_{N_+^p}$ is a norm on $N_+^p \cap H_0^2$.

The space N_+^p has the following properties like M^p too.

Proposition 3.4. *1. For $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, $\|\cdot\|_{N_+^p} = \|\cdot\|_{N_+^q}$ and $N_+^p = N_+^q$.*

2. For $1 < p \leq r \leq 2$, $\|\cdot\|_{N_+^2} \leq \|\cdot\|_{N_+^r} \leq \|\cdot\|_{N_+^p}$ and $N_+^p \subset N_+^r \subset N_+^2$.

3. For $1 < p < \infty$, $N_+^p \cap H_0^2$ is isomorphic to the subspace of $BMOA$ as normed spaces, via the isomorphism :

$$a \longmapsto \chi_{-1} a$$

.

4. $\|\cdot\|_{M_+^2} = \|\cdot\|_{N_+^2}$ and $M_+^2 = N_+^2$.

5. $M_+^2 \cap H_0^2$ is isomorphic to the subspace of $BMOA$ as normed spaces.

6. For $1 \leq p < \infty$, $M_+^p \cap H_0^2$ is a Banach space with respect to $\|\cdot\|_{M_+^p}$.

Proof. 1. The proof is the same as that of Proposition 3.2.2.

2. The proof is the same as that of Proposition 3.2.3.

3. The statement is the known fact by the proof of the Nehari Theorem (cf. [1], [5]).

4. A unitary operator

$$U \ : \ H^2 \longrightarrow \ell_+^2 \ : \ \sum_{n \geq 0} \varphi_n \chi_n \longmapsto \{\varphi_n\}_{n \geq 0}$$

implies the conclusion.

5. 3 and 4 imply the conclusion.

6. For $1 < p < \infty$, 3, 4 and Proposition 3.2.3 show the statement. For $p = 1$, Proposition 2.1.5 and Proposition 3.2.1 show the statement too. $\square$

In Section 5, we will prove that M^p and $M_+^p \cap H_0^2$ are not only Banach spaces but also dual spaces of some spaces.

Let c^0 and c_+^0 be subspaces of ℓ^∞ given by

$$c^0 := \left\{ \{\varphi_n\}_{n \in \mathbb{Z}} \mid \|\varphi\|_{c^0} := \sup_{n \in \mathbb{Z}} |\varphi_n| < \infty, \quad \lim_{n \to \pm\infty} \varphi_n = 0 \right\}$$

and

$$c_+^0 := \left\{ \varphi \in c^0 \mid \varphi_n = 0 \quad \text{for } n < 0 \right\},$$

respectively. For two sequences φ and ψ, we define a sequence $\varphi * \psi$ by

$$\varphi * \psi := \left\{ \sum_{k \in \mathbb{Z}} \varphi_{j-k} \psi_k \right\}_{j \in \mathbb{Z}},$$

whenever the sequence $\varphi * \psi$ can be defined.

We show the following norm estimates of $M_+^p \cap H_0^2$ and M^p.

Proposition 3.5. *Let $1 \leq p < \infty$, and $a \in L^1$.*

1. $\|a\|_{M_+^p} = \sup \left\{ |\langle \{(a)_{n+1}\}_{n \geq 0}, \varphi * \psi \rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell_+^p}, \ \|\psi\|_{\ell_+^q} \leq 1 \right\}$ holds. When $p = 1$, we replace ℓ_+^q with c_+^0 in the right-hand side.

2. $\|a\|_{M^p} = \sup\left\{|\langle\{(a)_n\}_{n\in\mathbb{Z}}, \varphi * \psi\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^p},\ \|\psi\|_{\ell^q} \le 1\right\}$ *holds. When $p = 1$, we replace ℓ^q with c^0 in the right-hand side.*

Proof. 1. Let $1 < p \le 2$. For $\varphi \in \ell^p_+$ with $\|\varphi\|_{\ell^p_+} \le 1$, a linear mapping

$$D_\varphi \;:\; \ell^q_+ \longrightarrow \mathbb{C} \;:\; \psi \longmapsto \langle \psi, a \odot \varphi\rangle_{\mathbb{R}}$$

satisfies

$$\begin{aligned}
\|a \odot \varphi\|_{\ell^p_+} &= \|D_\varphi\|_{(\ell^q_+)^*} \\
&= \sup\left\{|\langle\psi, a\odot\varphi\rangle_{\mathbb{R}}| \mid \|\psi\|_{\ell^q_+} \le 1\right\} \\
&\le \sup\left\{|\langle a\odot\varphi, \psi\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^p_+},\ \|\psi\|_{\ell^q_+} \le 1\right\} \\
&= \sup\left\{|\langle\{(a)_{n+1}\}_{n\ge 0}, \varphi * \psi\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^p_+},\ \|\psi\|_{\ell^q_+} \le 1\right\}.
\end{aligned}$$

It implies

$$\begin{aligned}
\|a\|_{M^p_+} &= \sup\left\{\|a\odot\varphi\|_{\ell^p_+} \mid \|\varphi\|_{\ell^p_+} \le 1\right\} \\
&\le \sup\left\{|\langle\{(a)_{n+1}\}_{n\ge 0}, \varphi * \psi\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^p_+},\ \|\psi\|_{\ell^q_+} \le 1\right\}.
\end{aligned}$$

Conversely,

$$\begin{aligned}
\|a\|_{M^p_+} &= \|H(a)\|_{\mathfrak{B}(\ell^p_+)} \\
&\ge \sup\left\{|G(H(a)\varphi)| \mid \|\varphi\|_{\ell^p_+},\ \|G\|_{(\ell^p_+)^*} \le 1\right\} \\
&= \sup\left\{|\langle H(a)\varphi, g\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^p_+},\ \|g\|_{\ell^q_+} \le 1\right\} \\
&= \sup\left\{|\langle\{(a)_{n+1}\}_{n\ge 0}, \varphi * g\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^p_+},\ \|g\|_{\ell^q_+} \le 1\right\}.
\end{aligned}$$

Now let $p = 1$. $\|a\|_{M^1_+} \le \sup\left\{|\langle\{(a)_{n+1}\}_{n\ge 0}, \varphi * \psi\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^1_+},\ \|\psi\|_{c^0_+} \le 1\right\}$ holds from as above. Conversely,

$$\begin{aligned}
\|a\|_{M^1_+} &= \|H(a)\|_{\mathfrak{B}(\ell^1_+)} \\
&\ge \sup\left\{|G(H(a)\varphi)| \mid \|\varphi\|_{\ell^1_+},\ \|G\|_{(\ell^1_+)^*} \le 1\right\} \\
&= \sup\left\{|\langle H(a)\varphi, g\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^1_+},\ \|g\|_{\ell^\infty_+} \le 1\right\} \\
&\ge \sup\left\{|\langle\{(a)_{n+1}\}_{n\ge 0}, \varphi * g\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^1_+},\ \|g\|_{c^0_+} \le 1\right\}.
\end{aligned}$$

When $2 < p < \infty$, Proposition 3.2.2 leads the conclusion.

2. For $\varphi = \{\varphi_n\}_{n\in\mathbb{Z}} \in \ell^\infty$, we define φ^b as $\varphi^b := \{\varphi_{-n}\}_{n\in\mathbb{Z}}$.

Let $1 < p \le 2$. For $\varphi \in \ell^p$ with $\|\varphi\|_{\ell^p} \le 1$, a linear mapping

$$D_\varphi \;:\; \ell^q \longrightarrow \mathbb{C} \;:\; \psi \longmapsto \langle\psi, a * \varphi\rangle_{\mathbb{R}}$$

satisfies

$$\begin{aligned}
\|a * \varphi\|_{\ell^p} &= \|D_\varphi\|_{(\ell^q)^*} \\
&= \sup\left\{|\langle\psi, a*\varphi\rangle_{\mathbb{R}}| \mid \|\psi\|_{\ell^q} \le 1\right\} \\
&\le \sup\left\{|\langle a*\varphi, \psi\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^p},\ \|\psi\|_{\ell^q} \le 1\right\} \\
&= \sup\left\{|\langle\{(a)_n\}_{n\in\mathbb{Z}}, \varphi^b * \psi\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^p},\ \|\psi\|_{\ell^q} \le 1\right\} \\
&= \sup\left\{|\langle\{(a)_n\}_{n\in\mathbb{Z}}, \varphi * \psi\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^p},\ \|\psi\|_{\ell^q} \le 1\right\}.
\end{aligned}$$

It implies

$$\begin{aligned}\|a\|_{M^p} &= \sup\{\|a*\varphi\|_{\ell^p} \mid \|\varphi\|_{\ell^p} \le 1\} \\ &\le \sup\left\{|\langle\{(a)_n\}_{n\in\mathbb{Z}}, \varphi*\psi\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^p},\ \|\psi\|_{\ell^q} \le 1\right\}.\end{aligned}$$

Conversely,

$$\begin{aligned}\|a\|_{M^p} &= \|M(a)\|_{\mathfrak{B}(\ell^p)} \\ &\ge \sup\left\{|G(M(a)\varphi)| \mid \|\varphi\|_{\ell^p},\ \|G\|_{(\ell^p)^*} \le 1\right\} \\ &= \sup\{|\langle M(a)\varphi, g\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^p},\ \|g\|_{\ell^q} \le 1\} \\ &= \sup\left\{|\langle\{(a)_n\}_{n\in\mathbb{Z}}, \varphi*g\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^p},\ \|g\|_{\ell^q} \le 1\right\}.\end{aligned}$$

Hence $\|a\|_{M^p} = \sup\left\{|\langle\{(a)_n\}_{n\in\mathbb{Z}}, \varphi*\psi\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^p},\ \|\psi\|_{\ell^q} \le 1\right\}$.

Now let $p=1$. $\|a\|_{M^1} \le \sup\left\{|\langle\{(a)_n\}_{n\in\mathbb{Z}}, \varphi*\psi\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^1},\ \|\psi\|_{c^0} \le 1\right\}$ from as above. Conversely,

$$\begin{aligned}\|a\|_{M^1} &= \|M(a)\|_{\mathfrak{B}(\ell^1)} \\ &\ge \sup\left\{|G(M(a)\varphi)| \mid \|\varphi\|_{\ell^1},\ \|G\|_{(\ell^1)^*} \le 1\right\} \\ &= \sup\{|\langle M(a)\varphi, g\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^1},\ \|g\|_{\ell^\infty} \le 1\} \\ &\ge \sup\left\{|\langle\{(a)_n\}_{n\in\mathbb{Z}}, \varphi*g\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^1},\ \|g\|_{c^0} \le 1\right\}.\end{aligned}$$

When $2<p<\infty$, Proposition 2.1.1 leads the conclusion. □

4 Some spaces of sequences. In this section, we show some properties of spaces which are linearization of sets of all $\varphi*\psi$. Namely,

Definition 4.1. *For $1\le p\le\infty$, we define V_+^p and V^p as*

$$V_+^p := \left\{\varphi^1*\psi^1+\cdots+\varphi^n*\psi^n \mid n\in\mathbb{N},\ \varphi^1,\cdots,\varphi^n\in\ell_+^p,\ \psi^1,\cdots,\psi^n\in\ell_+^q\right\},$$

and

$$V^p := \left\{\varphi^1*\psi^1+\cdots+\varphi^n*\psi^n \mid n\in\mathbb{N},\ \varphi^1,\cdots,\varphi^n\in\ell^p,\ \psi^1,\cdots,\psi^n\in\ell^q\right\}.$$

When $p=1$, we replace ℓ_+^q and ℓ^q with c_+^0 and c^0 in the right-hand side, respectively. When $p=\infty$, we also replace ℓ_+^p and ℓ^p with c_+^0 and c^0 in the right-hand side, respectively.

If $f=\varphi^1*\psi^1+\cdots+\varphi^n*\psi^n$ and $g=\Phi^1*\Psi^1+\cdots+\Phi^m*\Psi^m$ belong to each of above sets, then

$$f+g=\varphi^1*\psi^1+\cdots+\varphi^n*\psi^n+\Phi^1*\Psi^1+\cdots+\Phi^m*\Psi^m$$

belongs to the same sets. Thus it is easy to see that V_+^p and V^p are vector spaces, respectively.

Definition 4.2. *For $f=\varphi^1*\psi^1+\cdots+\varphi^n*\psi^n\in V_+^p$, we define $\|f\|_{V_+^p}$ as*

$$\|f\|_{V_+^p} := \inf\left\{\sum_{1\le j\le n}\|\varphi^j\|_{\ell_+^p}\|\psi^j\|_{\ell_+^q} \mid \textit{representations of } f \textit{ in } V_+^p\right\}.$$

For $f=\varphi^1*\psi^1+\cdots+\varphi^n*\psi^n\in V^p$, we define $\|f\|_{V^p}$ as

$$\|f\|_{V^p} := \inf\left\{\sum_{1\le j\le n}\|\varphi^j\|_{\ell^p}\|\psi^j\|_{\ell^q} \mid \textit{representations of } f \textit{ in } V^p\right\}.$$

These spaces have the following properties.

Proposition 4.3. *1. For $\varphi \in \ell^p_+$ and $\psi \in \ell^q_+$, there are $\Phi \in \ell^p_+$ and $\Psi \in \ell^q_+$ with $\varphi * \psi = \Phi * \Psi$ and $\|\Phi\|_{\ell^p_+} = \|\Psi\|_{\ell^q_+}$.*

2. For $f \in V^p_+$, $\sup_{j\geq 0}|f_j| \leq \|f\|_{V^p_+}$.

3. $V^p_+ \subset c^0_+$.

4. For $f \in \ell^q_+$, $\|f\|_{V^p_+} \leq \|f\|_{\ell^q_+}$.

5. $\|\cdot\|_{V^1_+} = \|\cdot\|_{c^0_+}$ and $V^1_+ = c^0_+$.

6. $\|\cdot\|_{V^p_+} = \|\cdot\|_{V^q_+}$ and $V^p_+ = V^q_+$.

7. V^2_+ and H^1 are isometrically isomorphic via the isomorphism :

$$\{(f)_j\}_{j\geq 0} \longleftrightarrow f \text{ whose Fourier coefficients are } \{(f)_j\}_{j\geq 0}.$$

Proof. 1. If we set $\Phi := \sqrt{\frac{\|\psi\|_q}{\|\varphi\|_p}}\varphi$ and $\Psi := \sqrt{\frac{\|\varphi\|_p}{\|\psi\|_q}}\psi$, then $\varphi * \psi = \Phi * \Psi$ and $\|\Phi\|_p = \sqrt{\|\varphi\|_p\|\psi\|_q} = \|\Psi\|_q$.

2. Take $f = \varphi^1 * \psi^1 + \cdots + \varphi^n * \psi^n \in V^p_+$. By the Hölder's inequality,

$$\begin{aligned} |f_j| &\leq |(\varphi^1 * \psi^1)_j| + \cdots + |(\varphi^n * \psi^n)_j| \\ &\leq \|\varphi^1\|_{\ell^p_+}\|\psi^1\|_{\ell^q_+} + \cdots + \|\varphi^n\|_{\ell^p_+}\|\psi^n\|_{\ell^q_+} \end{aligned}$$

for $j \geq 0$. Thus $\sup_{j\geq 0}|f_j| \leq \|f\|_{V^p_+}$.

3. For any $\epsilon > 0$, $\varphi = \{\varphi_j\}_{j\geq 0} \in \ell^p_+$, $\psi = \{\psi_j\}_{j\geq 0} \in \ell^q_+$, there is an $N \in \mathbb{N}$ such that $\|\{\varphi_j\}_{j\geq N}\|_p < \frac{\epsilon}{2\|\psi\|_q}$ and $\|\{\psi_j\}_{j\geq N}\|_q < \frac{\epsilon}{2\|\varphi\|_p}$. By the Hölder's inequality,

$$\begin{aligned} |(\varphi * \psi)_j| &= |\sum_{k\geq 0}\varphi_{j-N+k}\psi_{N-k} + \sum_{k\geq 1}\varphi_{j-N-k}\psi_{N+k}| \\ &\leq |\sum_{k\geq 0}\varphi_{j-N+k}\psi_{N-k}| + |\sum_{k\geq 1}\varphi_{j-N-k}\psi_{N+k}| \\ &\leq \|\{\varphi_{j-N+k}\}_{k\geq 0}\|_p\|\{\psi_{N-k}\}_{k\geq 0}\|_q + \|\{\varphi_{j-N-k}\}_{k\geq 1}\|_p\|\{\psi_{N+k}\}_{k\geq 1}\|_q \\ &< \frac{\epsilon}{2\|\psi\|_q}\|\psi\|_q + \|\varphi\|_p\frac{\epsilon}{2\|\varphi\|_p} = \epsilon \end{aligned}$$

hold for any $j \geq 2N$.

Thus, for any $\epsilon > 0$, $f = \varphi^1*\psi^1+\cdots+\varphi^m*\psi^m \in V^p_+$, if we put $\epsilon_n := \frac{\epsilon}{(n+1)^2}$ $(1 \leq n \leq m)$, then there exist $N_1, \cdots, N_m \in \mathbb{N}$ such that

$$\begin{aligned} |f_j| &\leq |(\varphi^1 * \psi^1)_j| + \cdots + |(\varphi^n * \psi^n)_j| \\ &< \epsilon_1 + \cdots + \epsilon_m < \epsilon \end{aligned}$$

for any $j \geq 2\max_{1\leq n\leq m} N_n$. This and 2 mean the conclusion.

4. Let $f \in \ell^q_+$. We regard f as $e_0 * f$, and therefore $\|f\|_{V^p_+} \leq \|e_0\|_{\ell^p_+}\|f\|_{\ell^q_+} = \|f\|_{\ell^q_+}$.

5. By 3 and 4, it is immediately seen.

6. By $\varphi * \psi = \psi * \varphi$, it is easy to see.

7. Since ℓ^2_+ and H^2 are isometrically isomorphic via the isomorphism : $\{(\varphi)_j\}_{j\geq 0} \longleftrightarrow \varphi = \sum_{j\geq 0}(\varphi)_j\chi_j$, we can see easily that V^2_+ and

$$\left\{\varphi^1\psi^1 + \cdots + \varphi^n\psi^n \mid n \in \mathbb{N},\ \varphi^1, \cdots, \varphi^n \in H^2,\ \psi^1, \cdots, \psi^n \in H^2\right\}$$

are isometrically isomorphic, whenever a norm of the space of the right-hand side is defined by

$$\|f\| := \inf\left\{\|\varphi^1\|_{H^2}\|\psi^1\|_{H^2} + \cdots + \|\varphi^n\|_{H^2}\|\psi^n\|_{H^2} \mid \text{representations of } f\right\}.$$

We show that this normed space is equal to H^1. By the Hölder's inequality,

$$\begin{aligned}\int |\varphi^1\psi^1 + \cdots + \varphi^n\psi^n|\frac{d\theta}{2\pi} &\leq \int |\varphi^1\psi^1|\frac{d\theta}{2\pi} + \cdots + \int |\varphi^n\psi^n|\frac{d\theta}{2\pi} \\ &\leq \|\varphi^1\|_{H^2}\|\psi^1\|_{H^2} + \cdots + \|\varphi^n\|_{H^2}\|\psi^n\|_{H^2},\end{aligned}$$

therefore $\|\cdot\|_{H^1} \leq \|\cdot\|$. Conversely, let $f \in H^1$. By the inner-outer factorization theorem, there are an inner function $g \in H^\infty$ and an outer function $h \in H^1$ satisfying $f = gh$. If we set $\varphi := gh^{\frac{1}{2}} \in H^2$ and $\psi := h^{\frac{1}{2}} \in H^2$, then $f = gh = \varphi\psi$ and $\|f\| \leq \|\varphi\|_{H^2}\|\psi\|_{H^2} = \|f\|_{H^1}$. Consequently, V^2_+ and H^1 are isometrically isomorphic. □

Proposition 4.4. *1. For $\varphi \in \ell^p$ and $\psi \in \ell^q$, there are $\Phi \in \ell^p$ and $\Psi \in \ell^q$ with $\varphi * \psi = \Phi * \Psi$ and $\|\Phi\|_{\ell^p} = \|\Psi\|_{\ell^q}$.*

2. *For $f \in V^p$, $\sup_{j\in\mathbb{Z}}|f_j| \leq \|f\|_{V^p}$.*
3. *$V^p \subset c^0$.*
4. *For $f \in \ell^q$, $\|f\|_{V^p} \leq \|f\|_{\ell^q}$.*
5. *$\|\cdot\|_{V^1} = \|\cdot\|_{c^0}$ and $V^1 = c^0$.*
6. *$\|\cdot\|_{V^p} = \|\cdot\|_{V^q}$ and $V^p = V^q$.*
7. *V^2 and L^1 are isometrically isomorphic via the isomorphism :*

$$\{(f)_j\}_{j\in\mathbb{Z}} \longleftrightarrow f \text{ whose Fourier coefficients are } \{(f)_j\}_{j\in\mathbb{Z}}.$$

Proof. The proof is the same as that of Proposition 4.3. □

Remark 4.5. *By Proposition 4.3.2 and Proposition 4.4.2, it is seen that $\|\cdot\|_{V^p_+}$ and $\|\cdot\|_{V^p}$ are norms on V^p_+ and V^p, respectively.*

Now, we consider what representations of an element of these spaces we can take. In general, it doesn't say that a representation $\varphi^1 * \psi^1 + \cdots + \varphi^n * \psi^n$ of an element f satisfies

$$\|\varphi^1\|_p\|\psi^1\|_q \fallingdotseq \cdots \fallingdotseq \|\varphi^n\|_p\|\psi^n\|_q.$$

However, the following result says that there is such a representation for all f.

Theorem 4.6. *Let $\epsilon, \epsilon' > 0$.*

1. *For $f \in V^p_+$, there is a representation of f, $f = \Phi^1 * \Psi^1 + \cdots + \Phi^N * \Psi^N$ such that $\frac{\|f\|_{V^p_+} - \epsilon'}{N} < \|\Phi^j\|^2_{\ell^p_+} < \frac{\|f\|_{V^p_+} + \epsilon}{N}$ and $\|\Phi^j\|_{\ell^p_+} = \|\Psi^j\|_{\ell^q_+}$ for $1 \leq j \leq N$.*

2. For $f \in V^p$, there is a representation of f, $f = \Phi^1 * \Psi^1 + \cdots + \Phi^N * \Psi^N$ such that $\frac{\|f\|_{V^p} - \epsilon'}{N} < \|\Phi^j\|_{\ell^p}^2 < \frac{\|f\|_{V^p} + \epsilon}{N}$ and $\|\Phi^j\|_{\ell^p} = \|\Psi^j\|_{\ell^q}$ for $1 \leq j \leq N$.

Proof. 1. If we take an $f \in V_+^p$, then by Proposition 4.3.1, there is a representation of f, $f = \varphi^1 * \psi^1 + \cdots + \varphi^n * \psi^n$ such that

$$\|f\|_{V_+^p} \leq \|\varphi^1\|_p^2 + \cdots + \|\varphi^n\|_p^2 < \|f\|_{V_+^p} + \epsilon$$

and $\|\varphi^j\|_p = \|\psi^j\|_q$ for $1 \leq j \leq n$.

Assume $n \geq 2$ and $\|\varphi^1\|_p^2 \leq \cdots \leq \|\varphi^n\|_p^2$.

We show that it may assume $\|\varphi^1\|_p^2 > 0$ without loss of generality. Indeed, assume $\|\varphi^n\|_p^2 > 0$ and $\|\varphi^1\|_p^2 = 0$. We take $2 \leq k \leq n$ with $\|\varphi^k\|_p^2 > 0$ and $\|\varphi^{k-1}\|_p^2 = 0$. If we set $\Phi^i := \begin{cases} \frac{\varphi^k}{\sqrt{k}}, & 1 \leq i \leq k \\ \varphi^i, & k+1 \leq i \leq n \end{cases}$ and $\Psi^i := \begin{cases} \frac{\psi^k}{\sqrt{k}}, & 1 \leq i \leq k \\ \psi^i, & k+1 \leq i \leq n, \end{cases}$
then $f = \varphi^1 * \psi^1 + \cdots + \varphi^n * \psi^n = \Phi^1 * \Psi^1 + \cdots + \Phi^n * \Psi^n$ and $0 < \|\Phi^1\|_p^2 \leq \cdots \leq \|\Phi^n\|_p^2$.

When $\|f\|_{V_+^p} > \epsilon'$, if we take an ϵ'' with $0 < \epsilon'' \leq \frac{\|f\|_{V_+^p}}{\|f\|_{V_+^p} - \epsilon'} - 1$, then we can take $r_j \in \mathbb{Q}$ satisfying

$$\|\varphi^j\|_p^2 \leq r_j < \min\left\{ \frac{\|\varphi^j\|_p^2(\|f\|_{V_+^p} + \epsilon)}{\|\varphi^1\|_p^2 + \cdots + \|\varphi^n\|_p^2}, \|\varphi^j\|_p^2(1 + \epsilon'') \right\}$$

for $1 \leq j \leq n$, and take $k_j \in \mathbb{N}$ satisfying the ratio

$$r_1 : \cdots : r_n = k_1 : \cdots : k_n$$

for $1 \leq j \leq n$.

Let $k_0 := 0$ and $N := \sum_{0 \leq \ell \leq n} k_\ell$. If we set $\Phi^j := \frac{\varphi^i}{\sqrt{k_i}}$ and $\Psi^j := \frac{\psi^i}{\sqrt{k_i}}$ for $1 \leq i \leq n$ and $\sum_{0 \leq \ell \leq i-1} k_\ell + 1 \leq j \leq \sum_{0 \leq \ell \leq i-1} k_\ell + k_i$, then we see that

$$f = \varphi^1 * \psi^1 + \cdots + \varphi^n * \psi^n = \Phi^1 * \Psi^1 + \cdots + \Phi^N * \Psi^N$$

and $\frac{\|f\|_{V_+^p} - \epsilon'}{N} < \|\Phi^j\|_p^2 = \|\Psi^j\|_q^2 < \frac{\|f\|_{V_+^p} + \epsilon}{N}$ for $1 \leq j \leq N$. Indeed, for $1 \leq i \leq n$ and $\sum_{0 \leq \ell \leq i-1} k_\ell + 1 \leq j \leq \sum_{0 \leq \ell \leq i-1} k_\ell + k_i$,

$$\|\Phi^j\|_p^2 = \|\frac{\varphi^i}{\sqrt{k_i}}\|_p^2 = \frac{\|\varphi^i\|_p^2}{r_i} \cdot \frac{r_i}{k_i} = \frac{\|\varphi^i\|_p^2}{r_i} \cdot \frac{r_1 + \cdots + r_n}{k_1 + \cdots + k_n} = \frac{\|\varphi^i\|_p^2}{r_i} \cdot \frac{r_1 + \cdots + r_n}{N},$$

and

$$\begin{aligned}
\frac{\|f\|_{V_+^p} - \epsilon'}{N} &\leq \frac{\|f\|_{V_+^p}}{N(1 + \epsilon'')} \\
&\leq \frac{\|\varphi^1\|_p^2 + \cdots + \|\varphi^n\|_p^2}{N(1 + \epsilon'')} \\
&\leq \frac{r_1 + \cdots + r_n}{N(1 + \epsilon'')} \\
&< \frac{\|\varphi^i\|_p^2}{r_i} \cdot \frac{r_1 + \cdots + r_n}{N} \\
&\leq \frac{r_1 + \cdots + r_n}{N} \\
&< \frac{\|f\|_{V_+^p} + \epsilon}{N}
\end{aligned}$$

hold.

When $\|f\|_{V_+^p} \leq \epsilon'$, we can take $r_j \in \mathbb{Q}$ satisfying

$$\|\varphi^j\|_p^2 \leq r_j < \frac{\|\varphi^j\|_p^2(\|f\|_{V_+^p} + \epsilon)}{\|\varphi^1\|_p^2 + \cdots + \|\varphi^n\|_p^2}$$

for $1 \leq j \leq n$, and take $k_j \in \mathbb{N}$ satisfying the ratio

$$r_1 : \cdots : r_n = k_1 : \cdots : k_n$$

for $1 \leq j \leq n$.

Let $k_0 := 0$ and $N := \sum_{0 \leq \ell \leq n} k_\ell$. If we set $\Phi^j := \frac{\varphi^i}{\sqrt{k_i}}$ and $\Psi^j := \frac{\psi^i}{\sqrt{k_i}}$, then we see that

$$\begin{aligned}
\frac{\|f\|_{V_+^p} - \epsilon'}{N} &\leq 0 \\
&< \|\Phi^j\|_p^2 \\
&= \frac{\|\varphi^i\|_p^2}{r_i} \cdot \frac{r_1 + \cdots + r_n}{N} \\
&\leq \frac{r_1 + \cdots + r_n}{N} \\
&< \frac{\|f\|_{V_+^p} + \epsilon}{N}
\end{aligned}$$

hold for $1 \leq i \leq n$ and $\sum_{0 \leq \ell \leq i-1} k_\ell + 1 \leq j \leq \sum_{0 \leq \ell \leq i-1} k_\ell + k_i$. This leads the conclusion.

2. The proof is the same as that of 1. $\square$

Remark 4.7. *The auther generalized Theorem 4.6 to Theorem 2.2 of [4] after this paper submitted.*

5 Duality theorems.

Here, we extend Proposition 3.5 to V_+^p and V^p.

Proposition 5.1. *Let $1 \leq p < \infty$, and $a \in L^1$.*

1. $\|a\|_{M_+^p} = \sup\left\{|\langle\{(a)_{k+1}\}_{k\geq 0}, f\rangle_{\mathbb{R}}| \mid \|f\|_{V_+^p} \leq 1\right\}$ *holds.*

2. $\|a\|_{M^p} = \sup\left\{|\langle\{(a)_k\}_{k\in\mathbb{Z}}, f\rangle_{\mathbb{R}}| \mid \|f\|_{V^p} \leq 1\right\}$ *holds.*

Proof. 1. Since $\left\{\varphi * \psi \mid \|\varphi\|_{\ell_+^p},\ \|\psi\|_{\ell_+^q} \leq 1\right\} \subset \left\{f \in V_+^p \mid \|f\|_{V_+^p} \leq 1\right\}$,

$$\begin{aligned}
\|a\|_{M_+^p} &= \sup\left\{|\langle\{(a)_{k+1}\}_{k\geq 0}, \varphi * \psi\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell_+^p},\ \|\psi\|_{\ell_+^q} \leq 1\right\} \\
&\leq \sup\left\{|\langle\{(a)_{k+1}\}_{k\geq 0}, f\rangle_{\mathbb{R}}| \mid \|f\|_{V_+^p} \leq 1\right\}
\end{aligned}$$

hold by Proposition 3.5.1.

Conversely, for any $\epsilon, \epsilon' > 0$, $f \in V_+^p$, there is a representation of f in V_+^p, $f = \sum_{1\leq j\leq n} \phi^j * \psi^j$ such that $\frac{\|f\|_{V_+^p} - \epsilon'}{n} < \|\phi^j\|_{\ell^p}^2 = \|\psi^j\|_{\ell^q}^2 < \frac{\|f\|_{V_+^p} + \epsilon}{n}$ for all $1 \leq j \leq n$ by

Theorem 4.6.1. Thus

$$
\begin{aligned}
&|\langle \{(a)_{k+1}\}_{k\geq 0}, f\rangle_{\mathbb{R}}| \leq \sum_{1\leq j\leq n} |\langle \{(a)_{k+1}\}_{k\geq 0}, \varphi^j * \psi^j\rangle_{\mathbb{R}}| \\
&\leq \sum_{1\leq j\leq n} \sup\left\{ |\langle \{(a)_{n+1}\}_{n\geq 0}, \varphi * \psi\rangle_{\mathbb{R}}| \mid (\varphi,\psi)\in E^p_+, \|\varphi\|_{\ell^p},\ \|\psi\|_{\ell^q} \leq \frac{\|f\|_{V^p_+}+\epsilon}{n} \right\} \\
&= n\cdot \frac{\|f\|_{V^p_+}+\epsilon}{n} \sup\left\{ |\langle \{(a)_{n+1}\}_{n\geq 0}, \varphi * \psi\rangle_{\mathbb{R}}| \mid (\varphi,\psi)\in E^p_+, \|\varphi\|_{\ell^p},\ \|\psi\|_{\ell^q} \leq 1 \right\} \\
&= (\|f\|_{V^p_+}+\epsilon)\|a\|_{M^p_+}
\end{aligned}
$$

hold by Proposition 3.5.1. Hence,

$$\sup\left\{ |\langle \{(a)_{k+1}\}_{k\geq 0}, f\rangle_{\mathbb{R}}| \mid \|f\|_{V^p_+} \leq 1 \right\} \leq \|a\|_{M^p_+}.$$

2. The proof is the same as that of 1. □

Finally, we show the main results.

Theorem 5.2. *Let* $1 \leq p < \infty$*.*

1. $M^p_+ \cap H^2_0$ *and* $(V^p_+)^*$ *are isometrically isomorphic as normed spaces.*
2. M^p *and* $(V^p)^*$ *are isometrically isomorphic as normed spaces.*

Proof. 1. Let $a \in M^p_+ \cap H^2_0$. By Proposition 5.1.1,

$$D\ :\ V^p_+ \longrightarrow \mathbb{C}\ :\ f \longmapsto \langle f, \{(a)_{n+1}\}_{n\geq 0}\rangle_{\mathbb{R}}$$

satisfies $\|D\|_{(V^p_+)^*} = \sup\left\{ |\langle \{(a)_{k+1}\}_{k\geq 0}, f\rangle_{\mathbb{R}}| \mid \|f\|_{V^p_+} \leq 1 \right\} = \|a\|_{M^p_+}$ and $D \in (V^p_+)^*$.

Coversely, let $D \in (V^p_+)^*$ and let $a_{n+1} := D(e_n)$ $n \geq 0$.

When $1 < p \leq 2$, since

$$D(f) = \sum_{n\geq 0} f_n D(e_n) = \sum_{n\geq 0} f_n a_{n+1} = \langle f, \{a_{n+1}\}_{n\geq 0}\rangle_{\mathbb{R}}$$

holds for $f = \sum_{n\geq 0} f_n e_n \in V^p_+$,

$$
\begin{aligned}
\infty > \|D\|_{(V^p_+)^*} &= \sup\left\{ |\langle f, \{a_{n+1}\}_{n\geq 0}\rangle_{\mathbb{R}}| \mid \|f\|_{V^p_+} \leq 1 \right\} \\
&\geq \sup\left\{ |\langle f, \{a_{n+1}\}_{n\geq 0}\rangle_{\mathbb{R}}| \mid \|f\|_{\ell^q_+} \leq 1 \right\}.
\end{aligned}
$$

This implies $\{a_{n+1}\}_{n\geq 0} \in (\ell^q_+)^* \cong \ell^p_+ \subset \ell^2_+ \cong H^2$ and $a := \sum_{n\geq 1} a_n \chi_n \in H^2_0$. Thus, by Proposition 5.1.1, $\|a\|_{M^p_+} = \sup\left\{ |\langle \{(a)_{k+1}\}_{k\geq 0}, f\rangle_{\mathbb{R}}| \mid \|f\|_{V^p_+} \leq 1 \right\} = \|D\|_{(V^p_+)^*}$ and $a \in M^p_+$.

When $p = 1$, we replace ℓ^q_+ with c^0_+. When $2 < p < \infty$, it is soon from $M^p_+ \cap H^2_0 = M^q_+ \cap H^2_0$ and $V^p_+ = V^q_+$.

2. Let $a \in M^p$. By Proposition 5.1.2,

$$D\ :\ V^p \longrightarrow \mathbb{C}\ :\ f \longmapsto \langle f, \{(a)_n\}_{n\in\mathbb{Z}}\rangle_{\mathbb{R}}$$

satisfies $\|D\|_{(V^p)^*} = \sup\left\{|\langle\{(a)_k\}_{k\in\mathbb{Z}}, f\rangle_{\mathbb{R}}| \mid \|f\|_{V^p} \le 1\right\} = \|a\|_{M^p}$ and $D \in (V^p)^*$.

Coversely, let $D \in (V^p)^*$ and let $a_n := D(e_n)$ $n \in \mathbb{Z}$.

When $1 < p \le 2$, since

$$D(f) = \sum_{n\in\mathbb{Z}} f_n D(e_n) = \sum_{n\in\mathbb{Z}} f_n a_n = \langle f, \{a_n\}_{n\in\mathbb{Z}}\rangle_{\mathbb{R}}$$

holds for $f = \sum_{n\in\mathbb{Z}} f_n e_n \in V^p$,

$$\begin{aligned} \infty > \|D\|_{(V^p)^*} &= \sup\left\{|\langle f, \{a_n\}_{n\in\mathbb{Z}}\rangle_{\mathbb{R}}| \mid \|f\|_{V^p} \le 1\right\} \\ &\ge \sup\left\{|\langle f, \{a_n\}_{n\in\mathbb{Z}}\rangle_{\mathbb{R}}| \mid \|f\|_{\ell^q} \le 1\right\}. \end{aligned}$$

This implies $\{a_n\}_{n\in\mathbb{Z}} \in (\ell^q)^* \cong \ell^p \subset \ell^2 \cong L^2$ and $a := \sum_{n\in\mathbb{Z}} a_n\chi_n \in L^2$. Thus, by Proposition 5.1.2, $\|a\|_{M^p} = \sup\left\{|\langle\{(a)_k\}_{k\in\mathbb{Z}}, f\rangle_{\mathbb{R}}| \mid \|f\|_{V^p} \le 1\right\} = \|D\|_{(V^p)^*}$ and $a \in M^p$.

When $p = 1$, we replace ℓ^q with c^0. When $2 < p < \infty$, it is soon from $M^p = M^q$ and $V^p = V^q$. □

Remark 5.3. *Theorem 5.2.1 is the modification of the H^1-$BMOA$ duality because of Proposition 3.4.5 and Proposition 4.3.7. Also Theorem 5.2.2 is the modification of the classical L^1-L^∞ duality because of Proposition 2.1.2 and Proposition 4.4.7.*

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Submission to the SCMJ

In September 2012, the way of submission to Scientiae Mathematicae Japonicae (SCMJ) was changed. Submissions should be sent electronically (in PDF file) to the editorial office of International Society for Mathematical Sciences (ISMS).

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ISMS Publications: We published **Mathematica Japonica (M.J.)** in print, which was first published in 1948 and has gained an international reputation in about sixty years, and its offshoot **Scientiae Mathematicae (SCM)** both online and in print. In January 2001, the two publications were unified and changed to **Scientiae Mathematicae Japonicae (SCMJ)**, which is the "21st Century New Unified Series of Mathematica Japonica and Scientiae Mathematicae" and published both online and in print. Ahead of this, the online version of SCMJ was first published in September 2000. The whole number of SCMJ exceeds 270, which is the largest amount in the publications of mathematical sciences in Japan. The features of SCMJ are:

1) About 80 eminent professors and researchers of not only Japan but also 20 foreign countries join the Editorial Board. The accepted papers are published both online and in print. SCMJ is reviewed by Mathematical Review and Zentralblatt from cover to cover.
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Table 1: Membership Dues for 2019

Categories	**Domestic**	**Overseas**	**Developing countries**
1-year Regular member	￥8,000	US$80， Euro75	US$50, Euro47
1-year Students member	￥4,000	US$50， Euro47	US$30， Euro28
Life member*	Calculated as below*	US$750， Euro710	US$440, Euro416
Honorary member	Free	Free	Free

(Regarding submitted papers,we apply above presented new fee after April 15 in 2015 on registoration date.) * Regular member between 63 - 73 years old can apply the category.

(73 − age) × ￥3,000

Regular member over 73 years old can maintain the qualification and the privileges of the ISMS members, if they wish.

Categories of 3-year members were abolished.

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